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NUCLEI FOR TOTALLY POINT DETERMINING GRAPHS

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The *nucleus* (*edge nucleus*) of a point determining graph is defined by Geoffroy and Sumner to be the set of all points (edges) whose removal leaves the graph point determining. It is the purpose of this paper to develop the analogous concepts for totally point determining graphs, that is, graphs in which distinct points have distinct neighborhoods and closed neighborhoods.

1. Introduction

It will be assumed throughout this paper that all graphs are finite, undirected, and without loops or multiple edges. All undefined terminology shall conform to that of Behzad and Chartrand [1]. However, we shall use the term "point" instead of "vertex".

In [3, 4, 5, 7, 9, 10], the idea of the nucleus and the edge nucleus of a point determining graph were introduced and subsequently developed. It is the purpose of this paper to develop the analogous concepts for totally point determining graphs. The basic definitions follow.

Let G be a graph and S a subset of G. S is called a π-set ($\bar{\pi}$-set) of G if and only if $N(a)=N(b)$ ($\bar{N}(a)=\bar{N}(b)$) for every pair of distinct points a and b in S and S is maximal with respect to this property. If a and b belong to the same π-set ($\bar{\pi}$-set), then we refer to $\{a, b\}$ as a π-pair ($\bar{\pi}$-pair).

A graph G is said to be *point determining* (*totally point determining*) if and only if G has no π-pairs (π or $\bar{\pi}$-pairs). Note that G is totally point determining is equivalent to the condition that G and $\bar{G}$ are both point determining. For a totally point determining graph G, the *total nucleus of* G is the set G^* consisting of all the points v of G such that $G-v$ is a totally point determining graph.

The concept of the total nucleus of a totally point determining is analogous to that of the nucleus of a point determining graph. That is, if G is a point determining graph, then the nucleus of G is the set G^0 consisting of all the points v of G such that $G-v$ is point determining. Note that x is not in G^0 if and only if there exist a and b in G such that $N(a)=N(b)-x$.

If G is a totally point determining graph and v is in G, then v is in $G-G^*$ if and only if there exist a and b in $G-v$ such that in $G-v$, $N(a)=N(b)$ or $\bar{N}(a)=\bar{N}(b)$. We will adopt the convention that when we write "$N(x)=N(y)-z$"

or "$\bar{N}(x) = \bar{N}(y) - z$", we mean to imply that x, y and z are distinct and y is adjacent to z; so with this understanding, v is in $G - G^*$ if and only if there exist a and b in $G - v$ such that $N(a) = N(b) - v$ or $\bar{N}(a) = \bar{N}(b) - v$. Also, instead of writing "x is adjacent to y", we will simply write "$x \perp y$".

However, unlike G^0, G^* may be empty. For example, it is easy to see that the path on four points has no totally removable points. Hence, we would like to establish necessary and sufficient conditions under which a connected totally point determining graph has a non-empty total nucleus. To do so we need to consider the following concept introduced by Sabidussi, (see [6]).

Let $\{G_x\colon x \text{ in } X\}$ be a family of graphs indexed by another graph X. Let $\#$ denote adjacency in X and for each x in X, let $\perp_x$ denote adjacency in G_x. The *X-join* of this family of graphs is

$$G = \bigcup_{x \text{ in } X} (G_x \times \{x\})$$

with adjacency relation $\perp$ defined by the following:
For (a, r) and (b, s) in G, $(a, r) \perp (b, s)$ if and only if either $r \# s$ or, $r = s$ and $a \perp_r b$.

There is an alternate approach to the concept of X-join of a family of graphs.

Let G be a graph and K a subset of G. Then K is said to be a *partitive subset of* G if and only if for every x in $G - K$ either $N(x) \cap K = \emptyset$, or K is contained in $N(x)$.

It is easy to see that G is the X-join of $\{G_x\colon x \text{ in } X\}$ if and only if the set of points of G can be partitioned by a family of partitive subsets $\{V_x\colon x \text{ in } X\}$ of G such that the graph induced by V_x is G_x, $\langle V_x \rangle = G_x$, for each x in X.

The following lemma gives us several basic tools to be used in discussing the removal of points and edges in a totally point determining graph.

Lemma 1.1. *Let G be a totally point determining graph and let a, b, c, d, and e be points of G with $\bar{N}(a) = \bar{N}(b) - c$.*

(i) *If $\bar{N}(d) = \bar{N}(c) - e$, then $b = e$.*
(ii) *If $\bar{N}(c) = \bar{N}(d) - e$, then $a = e$.*
(iii) *If $N($[illegible] $N(b) - e$, then $a = e$.*
(iv) *If $N($[illegible] $N(e) - a$, then $b = e$ or $c = d$.*
(v) *If $N(d$[illegible] $N(e) - b$, then $a = e$ or $c = e$.*

Proof. (i) Suppose $b \neq e$. Then $b \perp c$, so b is in $\bar{N}(d)$. But then a is in $\bar{N}(d)$, since $\bar{N}(a) = \bar{N}(b) - c$ and $d \neq c$. Thus a is a member of $\bar{N}(c)$ and we have a contradiction.

(ii) $b \perp c$, so b is in $\bar{N}(d)$. But then $a \perp d$ and a is not in $\bar{N}(c)$. Thus $a = e$.

(iii) $a \perp b$, so $a = e$ or $a \perp d$. If $a \perp d$, then d is in $\bar{N}(b)$ contradicts $N(d) = N(b) - e$. Thus $a = e$.

(iv) $e \perp c$, so $e = b$ or $e \perp b$. If $e \neq b$, then $d \perp b$ but d is not in $N(a)$. Therefore, $d = c$.

(v) $e \perp b$, so $e = c$ or e is in $\bar{N}(a)$. If $e \perp a$, then $d \perp a$. But then d is not in $N(b)$ is a contradiction. Thus $e = c$ or $e = a$.

2. Properties of the total nucleus

Before proceeding to characterize the connected totally point determining graphs with nonempty total nucleus, let us prove the following lemma which will facilitate the proof of the characterization. Note first that a *book* is simply the path on four points.

Lemma 2.1. *Let G be a totally point determining graph. Let a be a point of G such that a belongs to some partitive hook K of G. If either*

(i) $N(a) = N(b) - c$ *for some* b, c,

(ii) $\bar{N}(a) = \bar{N}(b) - c$ *and* $N(d) = N(c) - b$ *for some* b, c, d, *or*

(iii) $\bar{N}(a) = \bar{N}(b) - c$ *and* $\bar{N}(d) = \bar{N}(c) - b$ *for some* b, c, d,

then b is in K.

Proof. Suppose (i) holds. Let d be in $N(a) \cap K$. Then $d \perp b$, but $a \pm b$. Since K is partitive, then b is in K.

Suppose (ii) holds and b is not in K. Then since $\bar{N}(a) = \bar{N}(b) - c$, $\deg_k a = 2$ and c is in K. Since c is in K, we can choose f in $N(c) \cap K$. Now f is in $N(c) - b$, so $f \perp d$. Note that d is not in K since $d \pm b$. But $N(d) \cap K \neq \emptyset$, so $K \cap N(d) = K$. This, however, is a contradiction since $N(d) = N(c) - b$ implies $c \pm d$.

Suppose now that (iii) holds and b is not in K. $b \perp a$ so $N(b) \cap K = K$. As above, c is in K. d is not in K since $d \pm b$. But then $d \perp a$, so $a \in \bar{N}(c)$. This, however, is a contradiction since $\bar{N}(a) = \bar{N}(b) - c$.

Theorem 2.2. *Let G be a connected totally point determining graph. G^* is empty if and only if G is an X-join of hooks.*

Proof. If G is an X-join of hooks, then it is clear that $G^* = \emptyset$. Now suppose that $G^* = \emptyset$. Then if G is not an X-join of hooks, it follows that there is some x in G which does not lie in any partitive hook. Choose such an x so that $\deg x$ is as small as possible.

Suppose x is in G^0. Then since x is not in G^*, there exist p and q in G such that $\bar{N}(p) = \bar{N}(q) - x$. Now if q is not in G^0, then there exist u and v in G such that $N(u) = N(v) - q$. By Lemma 1.1(v), either $v = p$ or $v = x$. Suppose $v = p$. Then since $\bar{N}(p) = \bar{N}(q) - x$, we have $N(u) = N(p) - q = N(q) - \{p, x\}$. Now if p is not in G^0, then for some w in G, $N(w) = N(q) - p$. But then $N(u) = N(w) - x$, contrary to x belonging to G^0. Thus we may assume p is not in G^0. But p is in G^*, so by Lemma 1.1(ii), $\bar{N}(x) = \bar{N}(a) - p$ for some a in G. Thus a is in $N(p)$. Since u is not in $N(x)$, we must have $u \pm a$. But then a is not in $N(p) - q$, so $a = q$. Thus

$\bar{N}(x)=\bar{N}(q)-p$. From $\bar{N}(p)=\bar{N}(q)-x$, it follows that $N(x)=N(p)$, a contradiction. Thus $v \neq p$. So $v=x$ and $N(u)=N(x)-q$. But now $\deg u < \deg x$, so u must belong to some partitive hook K. By Lemma 2.1, x is in K and we have a contradiction.

Hence we may assume $q \in G^0$. Then since $\bar{N}(p)=\bar{N}(q)-x$, we have $\bar{N}(r)=\bar{N}(x)-q$ for some r in G. Thus $\deg r < \deg x$, so r must belong to some partitive hook K of G. By Lemma 2.1, x is in K since $\bar{N}(r)=\bar{N}(x)-q$ and $\bar{N}(p)=\bar{N}(q)-x$. But this is a contradiction.

Thus it must be that x is not in G^0. Now there exist p and q in G such that $N(p)=N(q)-x$. If q is not in G^0, then $N(r)=N(x)-q$ for some r in G. But then again by Lemma 2.1, we have a contradiction. Thus q is in G^0-G^*, so $\bar{N}(s)=\bar{N}(r)-q$ for some r and s in G. Hence we may assume that $r \neq x$, else we again have a contradiction by Lemma 2.1. But then a routine argument will show that $N(p)=N(q)-x$, $\bar{N}(p)=\bar{N}(r)-q$ and $\bar{N}(x)=\bar{N}(q)-r$. However, these relations guarantee that $\langle\{p, q, r, x\}\rangle$ is a partitive hook of G. This contradicts the choice of x and hence completes the proof.

For any graph G and x in G, $\{x\}$ and G are clearly partitive subsets of G. By a non-trivial partitive subset of G, we mean a partitive subset K of G such that K is neither a singleton nor the entire graph.

A graph G is said to be *indecomposable* if and only if G does not contain any non-trivial partitive subsets. It is easy to see that a graph is totally point determining if and only if it has no partitive subsets of order two. Thus every indecomposable graph is totally point determining. Since any component of a graph is a partitive subset, every indecomposable graph of order at least three is connected. Therefore, as an immediate consequence of Theorem 2.2, we have the following result.

Corollary 2.3. *If G is an indecomposable graph having at least five points, then $G^* \neq \emptyset$.*

For $n \geq 1$, Sumner defined an ortho n-path to be K_2 if $n=1$ and, otherwise, to be the graph consisting of the points $p_1, p_2, \ldots, p_{2n}$ where the neighborhoods of the points are determined by

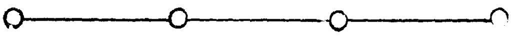
$$N(p_{2i-1})=N(p_{2i+1})-P_{2i+2}$$

and

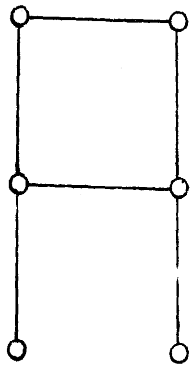
$$N(p_{2i+2})=N(p_{2i})-p_{2i-1},$$

for $i=1,2,\ldots,n-1$.

In Fig. 1, we see the ortho 2-path and ortho 3-path. The ortho 3-path shows that we cannot strengthen Corollary 2.3 to guarantee that an indecomposable graph must contain a point whose removal leaves the graph indecomposable. However, in [8], Sumner has shown that the following is true.



ortho 2-path



ortho 3-path

Fig. 1.

Theorem 2.4. *Every indecomposable graph contains either a point or an edge whose removal leaves the graph indecomposable.*

Further scrutiny of ortho n-paths establishes that every such graph is connected, point determining and bipartite. By the following result, also in [8], every ortho n-path is indecomposable.

Theorem 2.5. *For a bipartite graph G, G is indecomposable if and only if G is connected and point determining.*

By definition of the ortho n-path, p_2 and p_{2n-1} are the only points in G^*. Hence they are the only candidates for points whose removal leaves the graph indecomposable. But $G-p_2$ and $G-p_{2n-1}$ are not connected and thus, not indecomposable. Furthermore, we are willing to make the following conjecture.

Conjecture 2.6. *The only critical indecomposable graphs are the ortho n-paths.*

Unlike the nucleus of a point determining graph, there exists an infinite number of connected, totally point determining graphs G with $|G^*|=1$. For example, the graphs in Fig. 2 all have exactly one totally removable point and the graph in (*b*) yields such a graph for each value of n. However, we can show the following.

Theorem 2.7. *If G is an indecomposable graph, then $|G^*|=1$ if and only if G is the graph in* Fig. 2(a).

We shall omit the tedious but routine proof of Theorem 2.7.

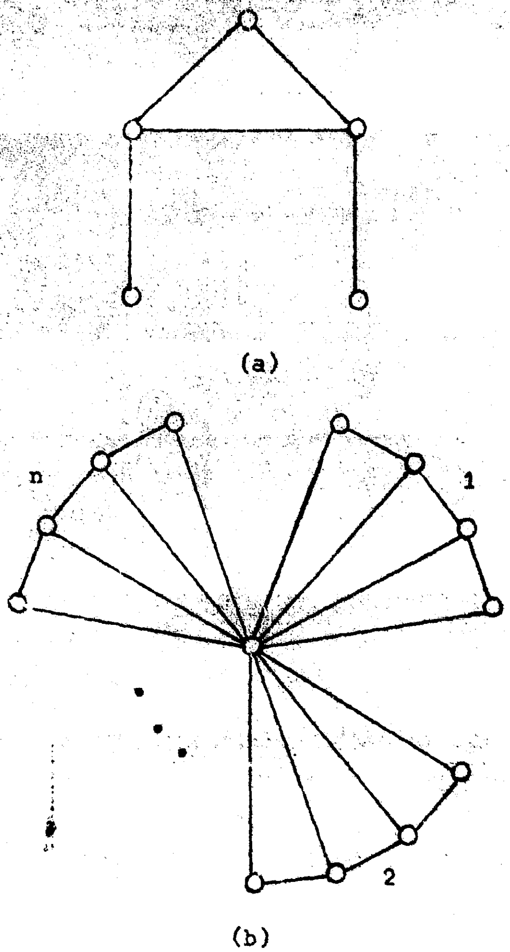


Fig. 2.

3. Possible total nuclei for connected totally point determining graphs

As in [10], we would like to consider the problem of which graphs may be the total nucleus of some totally point determining graph. To establish our results, we make only slight modifications to the technique developed there to answer the analogous question for the nucleus of a point determining graph.

Lemma 3.1. *Let H be a graph that is not totally point determining. Then there exists a graph H_1 such that*

(i) *H_1 is totally point determining,*
(ii) *H is an induced subgraph of H_1 and*
(iii) *H_1^* is contained in H.*

Moreover, if H is connected, then H_1 may be chosen to be connected.

Proof. We may obtain H_1 from H by adjoining a single endpoint to all but one element of each π-set of H and all but one element of each $\bar{\pi}$-set of H.

Lemma 3.2. *If H is any graph and G_1 is a totally point determining graph with $G_1^* \subseteq H \subseteq G_1$, then there exists a totally point determining graph G with $G^* = H$. Moreover, if G_1 is connected, then G may be chosen to be connected also.*

Proof. Let G_1 be a totally point determining graph with $G_1^* \subseteq H \subseteq G_1$ chosen so that $|H - G_1^*|$ is as small as possible. Suppose $G_1^* \neq H$ and let x be in $H - G_1^*$. Form a new graph G_2 from G_1 by adjoining a path on four points with each point on this path also adjacent to each point of $N(x)$ in G_1. It is easy to check that G_2 is totally point determining and that $G_2^* = G_1^* \cup \{x\} \subseteq H \subseteq G_2$. But this is a contradiction.

As an immediate consequence of Lemmas 3.1 and 3.2, for any graph H there exists a totally point determining graph G with $G^* = H$. Moreover, G may be chosen to be connected if H is connected.

It has been shown in [10] that not every graph is the nucleus of some connected point determining graph. However, for totally point determining graphs the result is all inclusive.

Theorem 3.3. *For any graph H, there exists a connected totally point determining graph G with $G^* = H$.*

As noted above, we only need to consider the case where H is not connected. Hence the following observation together with the next three lemmas, 3.4, 3.5, and 3.6, constitute a proof of Theorem 3.3.

If H consists solely of isolated points, then the graphs in Fig. 3 (where the

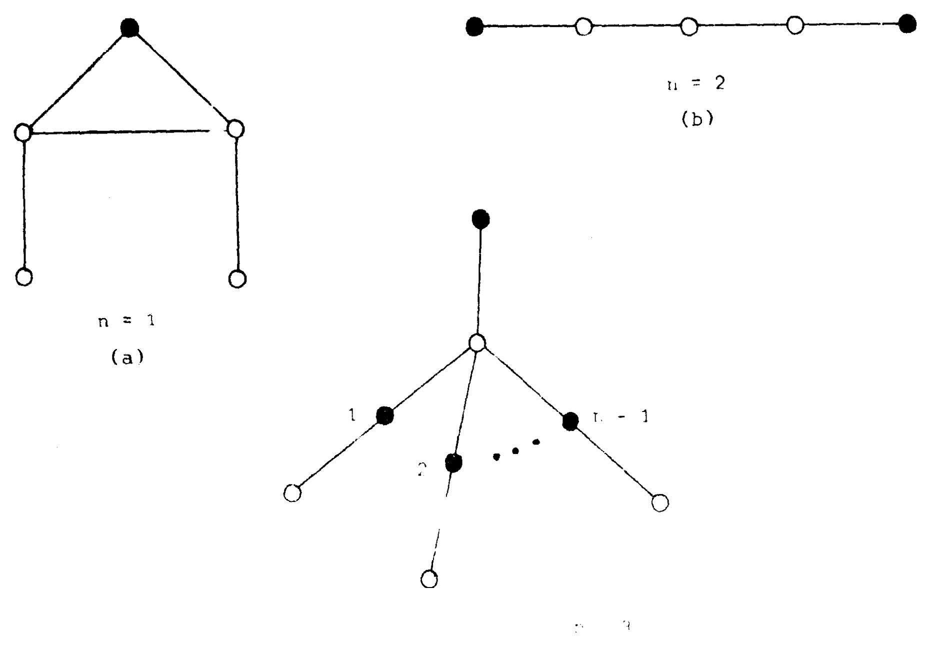


Fig. 3.

totally removable points are the shaded points in each graph) show that Theorem 3.3 is satisfied in this case.

Lemma 3.4. *If H is any graph without isolated points, then there exists a connected totally point determining graph G with $G^* = H$.*

Proof. Suppose that H consists of the non-trivial components $C_0, C_1, \ldots, C_n$. As noted above we may assume $n \geq 1$. We form the graph G_1 as follows. For each $i = 0, 1, \ldots, n$, let x_i be an element of C_i. Adjoin the new points a_i and b_i for $i = 1, 2, \ldots, n$, where the point a_i is adjacent to all of the points in $N(x_0) \cup \{b_i\}$, and b_i is adjacent to all of the points in $N(x_i) \cup \{a_i\}$ (see Fig. 4).

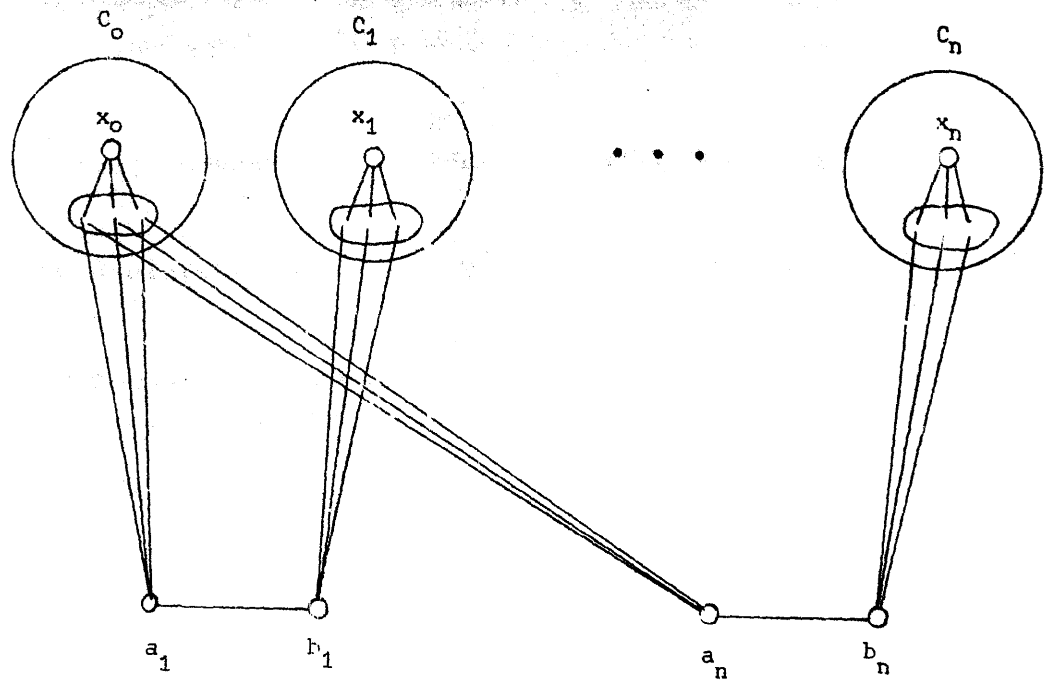


Fig. 4.

If the graph thus far obtained is not totally point determining, then any π-set or $\bar{\pi}$-set is contained in some C_i. Hence, by adjoining an end-point to all but one point of each such set, we obtain a connected totally point determining graph G_1 such that $G_1^* \subseteq H \subseteq G_1$. Thus by Lemma 3.2, there exists a connected totally point determining graph G with $G^* = H$.

Lemma 3.5. *If H is a graph with exactly one isolated point, then there exists a connected totally point determining graph G with $G^* = H$.*

Proof. Suppose first that H has only one non-trivial component C. We form G_1 as follows. Choose x in C. Adjoin the new points y, y_1, y_2, y_3, and y_4 so that the graph induced by $\{y_1, y_2, y_3, y_4\}$ is a hook; the only points of C adjacent to y_i for $i = 1, 2, 3, 4$ are precisely the points in $N(x)$; and y is adjacent to only y_1, y_2, y_3, and y_4 (see Fig. 5).

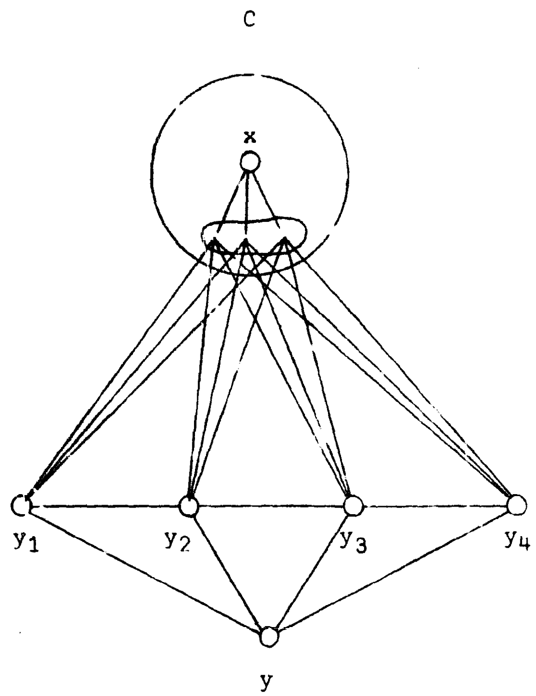


Fig. 5.

If the graph thus far obtained is not totally point determining, then any π-set or $\bar{\pi}$-set belongs to C. Adjoin an endpoint to all but one point of each such set. Then $C\cup\{y\}$ is a copy of H and, hence we obtain a connected totally point determining graph G_1 with $G_1^*\subseteq H\subseteq G_1$. Thus by Lemma 3.2, the theorem follows in this case.

Suppose now that H has at least 2 non-trivial components. Let $C_0, C_1, \ldots, C_n$ be these non-trivial components where $n\geq 2$. Let G_0 be the graph formed in the proof of Lemma 3.4 and pictured in Fig. 4. We form G_1 from G_0 as follows. Adjoin the new points y, y_1, y_2, y_3 and y_4 so that the graph induced by $\{y_1, y_2, y_3, y_4\}$ is a hook; the only points of $\bigcup_{i=0}^{n} C_i$ adjacent to y_j for $j=1,2,3,4$ are the points in $\bigcup_{i=0}^{n} N(x_i)$; and y is adjacent to only y_1, y_2, y_3 and y_4 (see Fig. 6). Any π-set or $\bar{\pi}$-set must lie in some C_i. Hence we obtain a connected totally point determining graph G_1 from G_0 as before by adjoining endpoints to all but one point of each such set. Thus $(\bigcup_{i=0}^{n} C_i)\cup\{y\}$ constitutes a copy of H in G_1 so $G_1^*\subseteq H\subseteq G_1$. By Lemma 3.2, this completes the proof.

Lemma 3.6. *If H is a graph with $n\geq 2$ isolated points, then there exists a connected totally point determining graph G with $G^*=H$.*

Proof. Let $C_1, C_2, \ldots, C_k$ be the non-trivial components of H for $k\geq 1$ and let x_i be in C_i for $i=1,2,\ldots,k$. Form the graph G_0 as follows:

Adjoin the new points $\bigcup_{i=1}^{k}\{a_i, b_i\}$, where for each $i=1,2,\ldots,k$, the point a_i is adjacent to all of the points in $N(x_i)\cup\{b_i\}$; and for $i=1,2,\ldots,k-1$, b_i is adjacent to all of the points in $N(x_{i+1})\cup\{a_i\}$ and b_k is adjacent to only a_k (see Fig. 7).

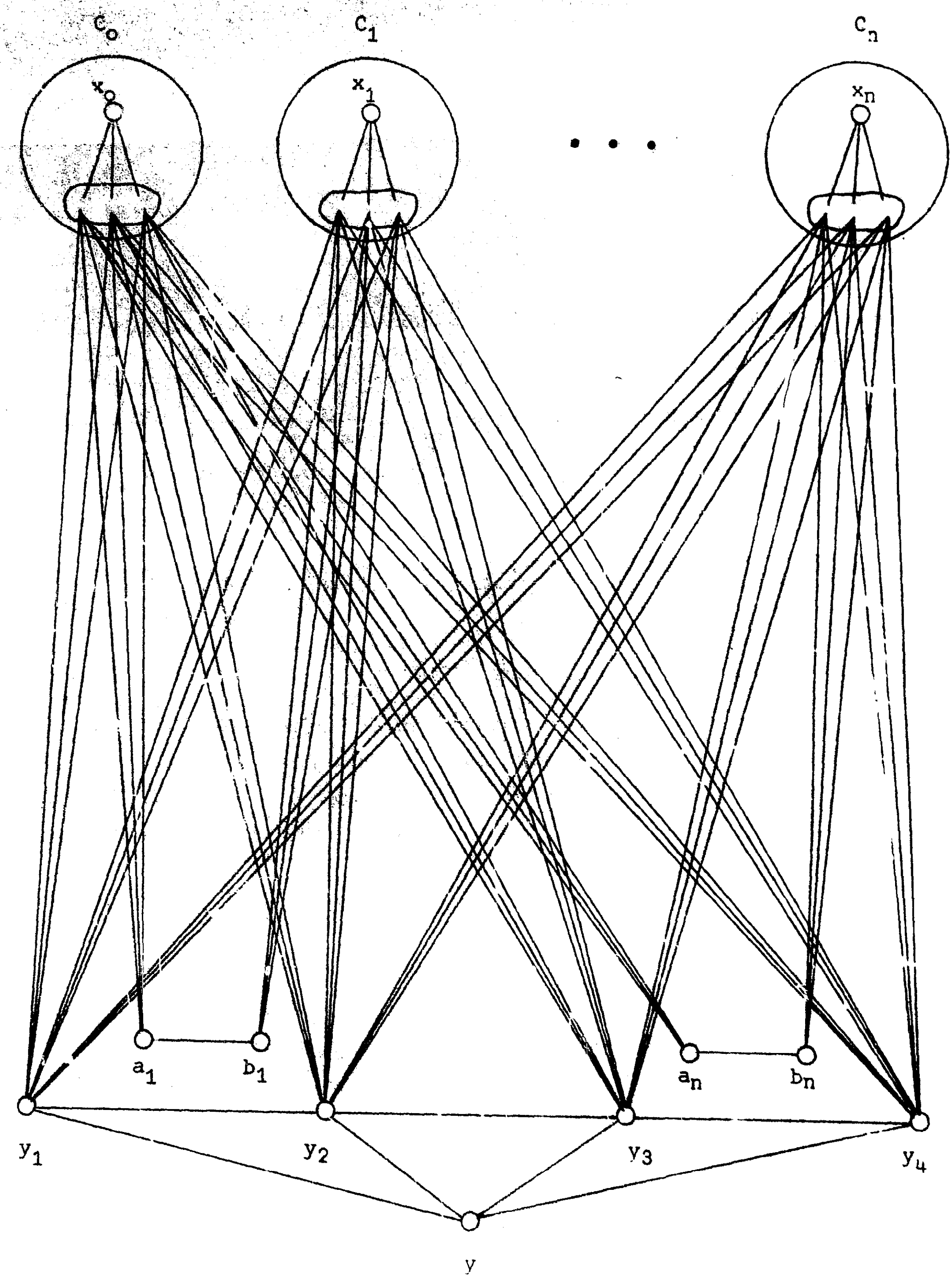


Fig. 6.

Let H_0 be the graph in Fig. 8. Attach H_0 to G_0 via the edge $b_k y$ to obtain the graph in Fig. 9. Now $(\bigcup_{i=1}^{k} C_i) \cup \{y_1, y_2, \ldots, y_n\}$ is a copy of H. If the graph obtained thus far is not totally point determining, we derive a totally point determining graph from it as before by attaching appropriate endpoints. Thus we obtain a connected totally point determining graph G_1 such that $G_1^* \subseteq H \subseteq G_1$. By Lemma 3.2 this completes the proof.

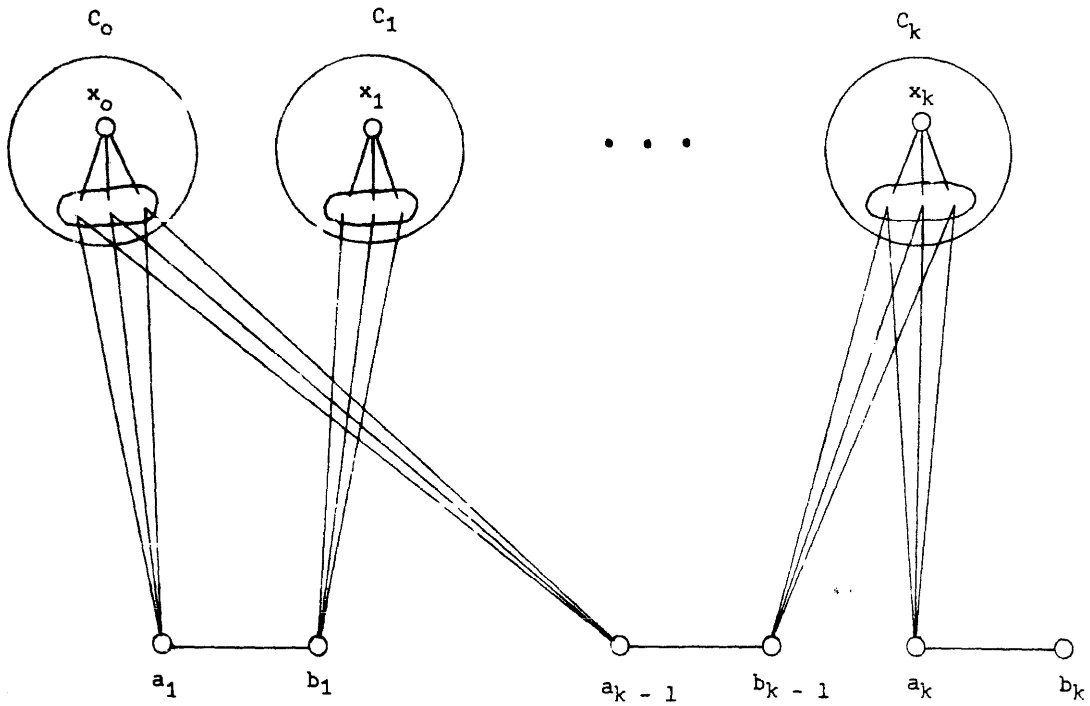


Fig. 7.

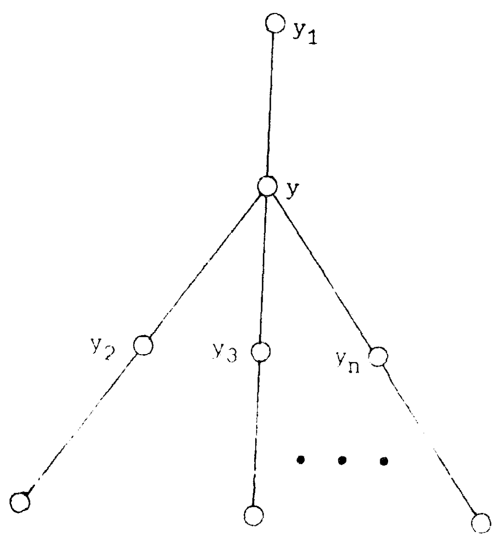


Fig. 8

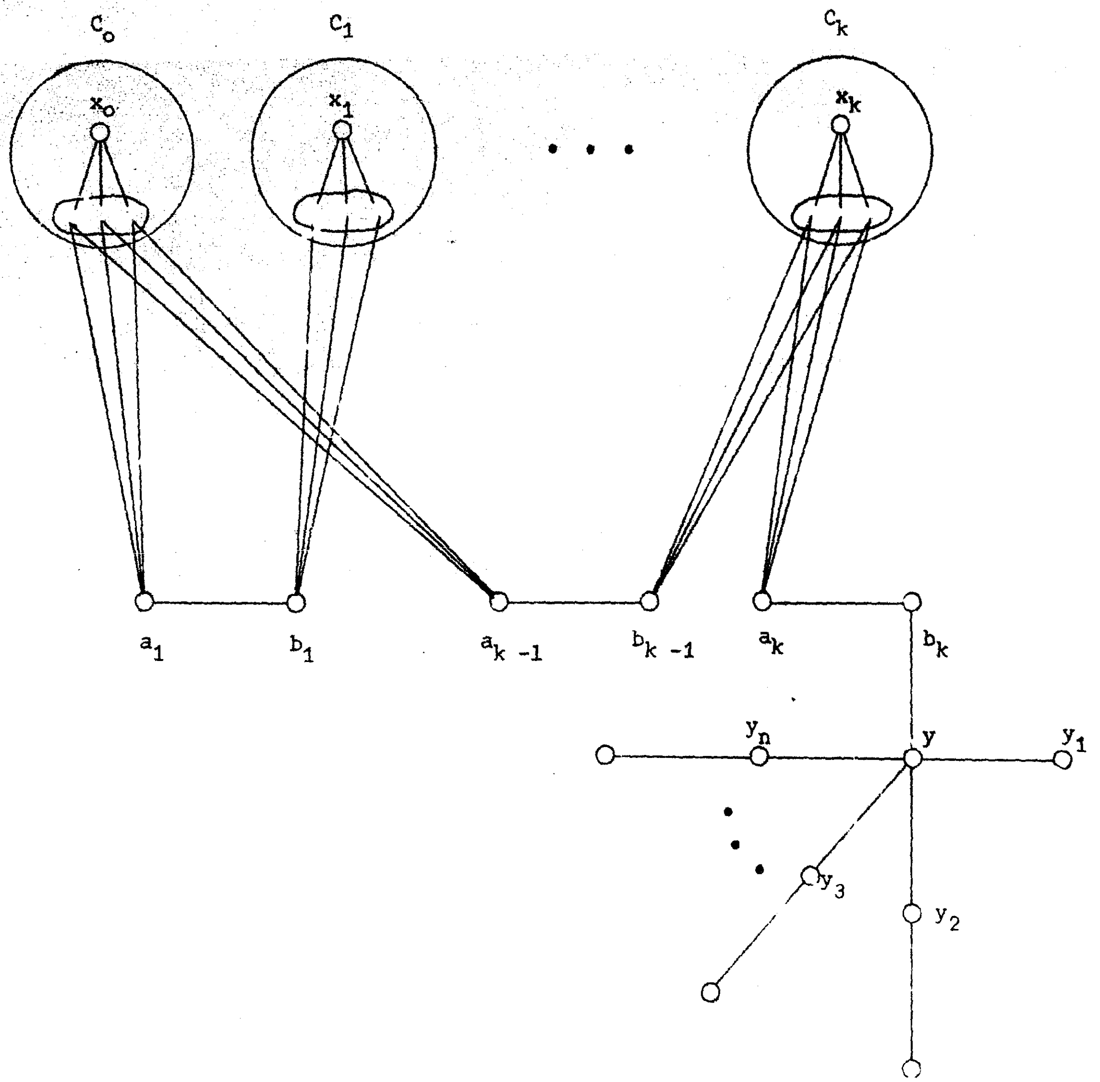


Fig. 9.

4. The total edge nucleus

In [5], we considered the problem of edge removal in a point determining graph via the edge nucleus. That is, for a point determining graph G, we investigated the properties of the edge nucleus of G, $E^0(G)$, consisting of all the edges e of G such that $G-e$ is point determining. Let us now consider the problem of edge removal in totally point determining graphs through a similar set of edges of the graph.

For a totally point determining graph G, the *total edge nucleus of* G is the set $E^*(G)$ consisting of all those edges e of G such $G-e$ is a totally point determining graph. We shall let $E^*(G)$ also represent the graph formed by the edges in $E^*(G)$ and we shall use $V(E^*(G))$ to denote the set of points in this graph.

Note that if G is a totally point determining graph and xy is in $E(G)-E^*(G)$, then there exists z in G such that either $N(z)=N(x)-y$, $N(z)=N(y)-x$, $\bar{N}(z)=\bar{N}(x)-y$ or $\bar{N}(z)=\bar{N}(y)-x$.

Similar to the result in [5] relating the intersection of the nucleus and the points in the edge nucleus, we have the following.

Theorem 4.1. *If G is a connected totally point determining graph such that G is not an X-join of hooks and G is different from the graph in* Fig. 2(a), *then* $G^* \cap V(E^*(G)) \neq \emptyset$.

Proof. Choose x in G^* such that deg x is minimal. Suppose x is not in $V(E^*(G))$. Let y be an element of $N(x)$ such that deg y is as small as possible. Since xy is not in $E^*(G)$ and x is in G^*, there exists z in G such that $N(z) = N(x) - y$ or $\bar{N}(z) = \bar{N}(x) - y$.

Suppose $\bar{N}(z) = \bar{N}(x) - y$. Then zx is not in $E^*(G)$, so $\bar{N}(y) = \bar{N}(x) - z$ or there exists w in G such that $N(w) = N(x) - z$. But $\bar{N}(y) = \bar{N}(x) - z$ implies $N(y) = N(z)$, so $N(w) = N(x) - z$. By the minimality of deg x, w is not in G^*. Thus there exist u and v in G such that $\bar{N}(u) = \bar{N}(v) - w$ or $N(z) = N(u) - w$. In the latter case, $u \perp x$ but u is not in $N(z)$, so $u = y$. But this contradicts the minimality of deg y. Hence, we may assume $\bar{N}(u) = \bar{N}(v) - w$. Now $v \perp x$ and $v \neq z$. Since vx is not in $E^*(G)$, there exists t in G such that $N(t) = N(x) - v$ or $\bar{N}(t) = \bar{N}(x) - v$. But $N(t) = N(x) - v$ is a contradiction, since $t \perp z$ and t is not in $\bar{N}(x)$. Also from $\bar{N}(t) = \bar{N}(x) - v$, we must have $x = w$, since $\bar{N}(u) = \bar{N}(v) - w$. But this impossible since $N(w) = N(x) - z$.

Therefore, we may assume that $N(z) = N(x) - y$. By the minimality of deg x, z is in G^*. Thus there exist v and w such that $\bar{N}(v) = \bar{N}(w) - z$ or $N(y) = N(w) - z$.

Suppose $N(y) = N(w) - z$. wx is not in $E^*(G)$, so there exists u in G such that $N(u) = N(x) - w$ or $\bar{N}(u) = \bar{N}(x) - w$. But if $N(u) = N(x) - w$, then $x = z$, which is a contradiction. Hence, $\bar{N}(u) = \bar{N}(x) - w$. But then $u = y$, otherwise, since u is in $N(x) - y = N(z)$, we would have z in $\bar{N}(u) \subseteq \bar{N}(x)$. Since G is connected and $|G| \geq 5$, we may choose a in $G - \{w, x, y, z\}$ such that $N(a) \cap \{x, y, w, z\} \neq \emptyset$. But then $a \perp x$ and ax is not in $E^*(G)$. Hence, there exists b in G such that $N(b) = N(x) - a$ or $\bar{N}(b) = \bar{N}(x) - a$. In the first case, $b \perp y$ and b is not in $\bar{N}(x)$ is a contradiction to $\bar{N}(y) = \bar{N}(x) - w$. In the latter, $b \perp x$ and $b \neq y$ since $a \neq w$. But then $b \perp z$ so that z is in $\bar{N}(x)$, again a contradiction.

Suppose $\bar{N}(v) = \bar{N}(w) - z$. Since $N(z) = N(x) - y$, $w \perp x$. wx is not in $E^*(G)$ so there exists u in G such that $\bar{N}(u) = \bar{N}(x) - w$ or $N(u) = N(x) - w$. The former implies $x = z$, so $N(u) = N(x) - w$. If $v \neq x$, then $v \perp x$ and $v \neq w$. But then from $v \perp u$, it follows that u is in $\bar{N}(w)$, which is a contradiction. Therefore, $v = x$ and $\bar{N}(x) = \bar{N}(w) - z$. Since $N(u) = N(x) - w$, it follows from the minimality of deg x that u is not in G^*. Thus there exist s and t in G such that $\bar{N}(s) = \bar{N}(t) - u$ or $N(w) = N(s) - u$. But $N(w) = N(s) - u$ implies $s \perp x$ and s is not in $\bar{N}(w)$, contrary to $\bar{N}(x) = \bar{N}(w) - z$. Hence $\bar{N}(s) = \bar{N}(t) - u$. We claim $s = x$. If $s \neq x$, then it would follow from $t \perp x$ that $s \perp x$. But $s \neq u$, so $s = w$. By the minimality of deg x, and since $s \perp x$ and deg $s <$ deg t, we have $t \neq y$. Note xt is not in $E^*(G)$. So there exists r in G such that $N(r) = N(x) - t$ or $\bar{N}(r) = \bar{N}(x) - t$. In the latter case we have $x = u$, which is a contradiction. If $N(r) = N(x) - t$, then since $r \perp w$ and $r \pm x$, we have $r = z$. But then from $N(z) = N(x) - t$ and $N(z) = N(x) - y$, we obtain $t = y$. But this is a contradiction.

Therefore, $s = x$ and $\bar{N}(x) = \bar{N}(t) - u$. If $t \neq y$, then $t \perp z$ and z is not in $\bar{N}(x)$ so $u = z$. Now $N(z) = N(x) - w$. But this is impossible, since it follows from $\bar{N}(x) = \bar{N}(w) - z$ that $w \perp z$. Thus $t = y$ and $N(x) = N(y) - u$. Thus we have the induced subgraph in Fig. 10. Since G is connected and this subgraph is properly contained

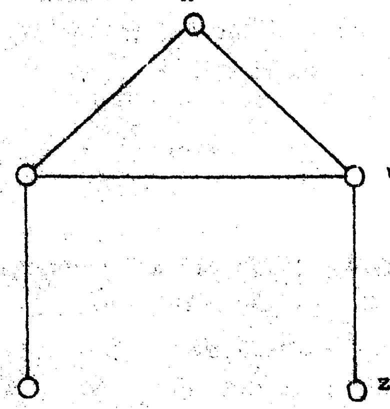


Fig. 10.

in G, we may choose a in $G - \{u, w, x, y, z\}$ such that $N(a) \cap \{u, w, x, y, z\} \neq \emptyset$. But then $a \perp x$, so there exists b in G such that $N(b) = N(x) - a$ or $\bar{N}(b) = \bar{N}(x) - a$. If $N(b) = N(x) - a$, then $b \perp w$. Hence, since b is not in $\bar{N}(x)$, we have $b = z$. But then $N(z) = N(x) - a$ and $N(z) = N(x) - y$ yields $y = a$, a contradiction. Since $\bar{N}(b) = \bar{N}(x) - a$ and $\deg b < \deg x < \deg y$, we have $b \neq y$. But then $b \perp z$, and hence z is in $N(x)$. This is impossible since $N(z) = N(x) - y$.

As a consequence of Theorem 4.1, we see that if G is a connected totally point determining graph and is not an X-join of hooks, then $E^*(G) \neq \emptyset$. However, with the next two results, Entringer and Gassman in [2] have supplied necessary and sufficient conditions for the total edge nucleus to be nonempty.

Theorem 4.2. *If G is a connected totally point determining graph, then $E^*(G) = \emptyset$ if and only if G is the path on four points.*

A *tail of length n* in a graph G is an induced subgraph T with vertex set $\{t_1, t_2, \ldots, t_n\}$ satisfying $N(t_1) = \{t_2\}$, $N(t_i) = \{t_{i-1}, t_{i+1}\}$ for $2 \leq i \leq n-2$, and $N(t_n) = \{t_{n-1}, a\}$ where a is some point of G. We say that the tail T is *adjoined at the point* a to G.

Let G be a totally point determining graph such that $E^*(G) = \emptyset$. If there exists a component C of G such that $E^*(C) \neq \emptyset$, then G must have an isolated point. Since G cannot contain more than one isolated point, the next theorem completes the characterization of the totally point determining graphs G for which $E^*(G) = \emptyset$.

Theorem 4.3. *If G is a graph with exactly one isolated point, then G is totally point determining and* $E^*(G)=\emptyset$ *if and only if each component C of G consists of a connected bipartite graph B whose vertex set* $B_1 \cup B_2$, *with* $B_1 \cap B_2 = \emptyset \neq B_2$, *satisfies* (i), (ii), *and* (iii) *together with tails adjoined at the points of* B_2 *so that* (iv), (v) *and* (vi) *are satisfied.*

(i) B_1 *and* B_2 *are both independent subsets of B.*

(ii) *Distinct points of* B_1 *have distinct neighborhoods.*

(iii) *Each non-empty subset of the neighborhood of a point of* B_1 *is the neighborhood of some point of* B_1.

(iv) *If* $B_2=\{b_2\}$ *and* $B_1=\emptyset$, *then* b_2 *has one tail of length* 3 *or at least two tails each of length* 2 *or* 3 *adjoined.*

(v) *If* $B_2=\{b_2\}$ *and* $B_1\neq\emptyset$, *then* b_2 *has at least one tail of length* 2 *or* 3 *adjoined.*

(vi) *If* B_2 *is not a singleton, then each point of* B_2 *has an arbitrary number (possibly zero) of tails of length* 2 *or* 3 *adjoined.*

When comparing $E(G^0)$ and $E^0(G)$ for a connected, non-complete, point determining graph G, we could only guarantee that $E(G^0)\cap E^0(G)\neq\emptyset$, provided G^0 has no isolated points (see [5]). However, for totally point determining graphs we obtain a much stronger relationship.

Theorem 4.4. *If G is a totally point determining graph, then* $E(G^*)\subseteq E^*(G)$.

The proof of Theorem 4.4 is trivial and hence is omitted.

Concerning the removal of edges in cycles of a totally point determining graph, in [5] we showed the following.

Theorem 4.5. *If G is a totally point determining graph and C is an odd cycle of G, then there exists an edge of C that is also in* $E^0(G)$.

The graph in Fig. 11 shows that we cannot extend Theorem 4.5 to $E^*(G)$ since the spanning cycle of this graph contains no totally removable edge. However, we can show the following.

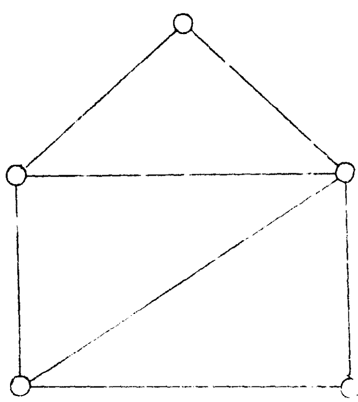


Fig. 11.

Theorem 4.6. *If G is a totally point determining graph, then every triangle contains a totally removable edge of G.*

Proof. Consider the triangle induced by $\{x, y, z\}$. By Theorem 4.5, we may assume without loss of generality that xy is in $E^0(G)$. Suppose xy is not in $E^*(G)$. Then we may assume there exists w in G such that $\bar{N}(w) = \bar{N}(x) - y$. $w \neq z$ since $w \not\perp y$. Now if xz is not in $E^*(G)$, then there exists u in G such that $\bar{N}(u) = \bar{N}(x) - z$, $\bar{N}(u) = \bar{N}(z) - x$, $N(u) = N(x) - z$ or $N(u) = N(z) - x$. $\bar{N}(u) = \bar{N}(z) - x$ is impossible since $\bar{N}(w) = \bar{N}(x) - y$ and $y \neq z$. Both $N(u) = N(z) - x$ and $N(u) = N(x) - z$ contradict $\bar{N}(w) = \bar{N}(x) - y$, since in either case u is in $\bar{N}(y) - \bar{N}(x)$. Therefore, $\bar{N}(u) = \bar{N}(x) - z$. Consider yz. If yz is not in $E^*(G)$, then there exists v in G such that $\bar{N}(v) = \bar{N}(y) - z$, $\bar{N}(v) = \bar{N}(z) - y$, $N(v) = N(y) - z$ or $N(v) = N(z) - y$. Since $\bar{N}(w) = \bar{N}(x) - y$ and $z \neq x$, we cannot have $\bar{N}(v) = \bar{N}(y) - z$. Similarly, from $\bar{N}(u) = \bar{N}(x) - z$, we cannot have $N(v) = N(z) - y$. If $N(v) = N(y) - z$, then $v \perp x$ but $v \not\perp w$. But then $v = w$ and since $w \perp z$, we have a contradiction. Thus $N(v) = N(z) - y$. From $v \perp x$, we see that v is in $\bar{N}(u)$. But $v \not\perp u$, since $z \not\perp u$, and hence $v = u$. But this is impossible since $u \perp y$. Therefore, one of xy, sz and yz is a totally removable edge.

It can be shown that if G is a connected totally point determining graph with $|G| \geq 5$, then $L(G)$ is totally point determining (see [5]). Thus we have the following result relating $(L(G))^*$ and $E^*(G)$.

Theorem 4.7. *Let G be a connected totally point determining graph. If G is not the path on five points, the* $E^*(G) \cap (L(G))^* \neq \emptyset$.

Proof. Let $e = xy$ be in $E^*(G)$ and assume e is chosen so that $\deg x + \deg y$ is minimal. Suppose e is not in $(L(G))^*$. Then there exist a and b in $L(G)$ such that $\bar{N}(a) = \bar{N}(b) - e$ or $N(a) = N(b) - e$.

Suppose $\bar{N}(a) = \bar{N}(b) - e$. Then $b = xz$, since $b \perp e$. Also, since $a \perp b$ and $a \not\perp e$, we have $a = zu$ for some u in $G - \{x, y\}$. In addition $N(u) - z \subseteq \{x\}$ and $N(x) - \{y, z\} \subseteq \{u\}$. Thus $N(u) = N(x) - y$ or $\bar{N}(u) = \bar{N}(x) - y$; but this contradicts xy being in $E^*(G)$.

Suppose $N(a) = N(b) - e$. Then $b \perp e$ so that $b = xz$. Also since $a \not\perp b$ and $a \not\perp e$, $a = uv$ for some u and v in $G - \{x, y, z\}$. Also, $N(a) = N(b) - e$ implies

$$N(u) - v \subseteq \{x, z\},$$

$$N(v) - u \subseteq \{x, z\},$$

$$N(z) - x \subseteq \{u, v\},$$

$$N(x) - \{y, z\} \subseteq \{u, v\}.$$

Suppose neither u nor v is adjacent to z. Then $N(z) = \{x\}$. Since G is point determining, $\deg y > \deg z$. Then by the minimality of $\deg x + \deg y$, xz is not in

$E^*(G)$. Thus there exists w in G such that $N(w)=N(x)-z$ or $\bar{N}(w)=\bar{N}(x)-z$. Since G is connected we may assume without loss of generality that $u \perp$ [illegible]. If $N(w)=N(x)-z$, then $w \perp u$ implies $w=v$; but this is impossible since $v \not\perp y$. If $\bar{N}(w)=\bar{N}(x)-z$, $w \neq y$ since $y \not\perp u$. But then $w \perp x$ and w is not y or z, so $w=u$ or $w=v$. In either case, $\bar{N}(u)=\bar{N}(v)$ and we have a contradiction.

Hence, we may assume without loss of generality that $u \perp z$. Also $v \not\perp x$, for otherwise, $N(u)=N(x)-y$ or $\bar{N}(u)=\bar{N}(x)-y$, and either of these contradicts xy being in $E^*(G)$. If $v \perp z$, then $u \perp x$ since $\bar{N}(v) \neq \bar{N}(u)$; but then $\bar{N}(u)=\bar{N}(z)$. Therefore, $N(v)=\{u\}$.

If $u \perp x$, then $\bar{N}(z)=\bar{N}(x)-y$, contrary to xy being in $E^*(G)$. Thus $N(u)=\{v, z\}$ and G is the graph in Fig. 12. Since G is not the path on five points, deg $y \geq 2$. But then by the minimality of deg x + deg y, uv is not in $E^*(G)$. But this is impossible.

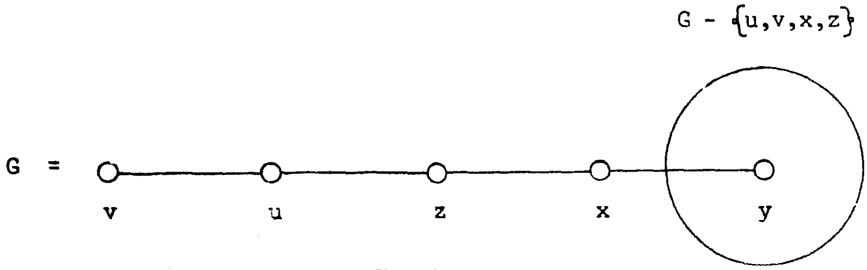


Fig. 12.

In a sense, Theorem 4.7 is best possible. For the example in Fig. 13, $E^*(G)=\{3,4,7\}$ and $(L(G))^*=\{2,3,6,7\}$. Therefore, in general we do not have either $E^*(G) \subseteq (L(G))^*$ or $(L(G))^* \subseteq E^*(G)$.

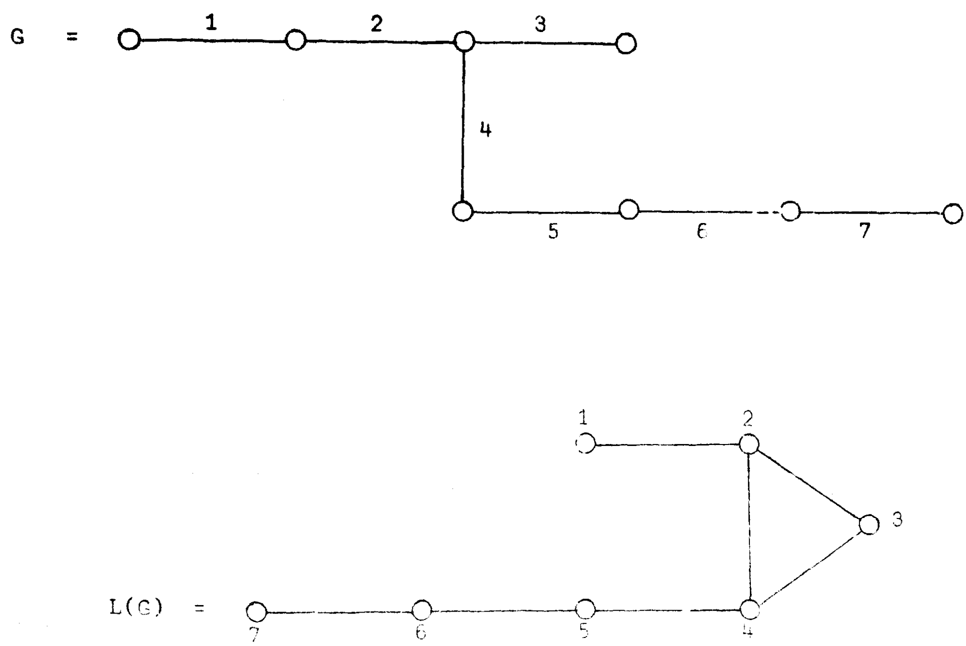


Fig. 13.

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