Discrete Mathematics 309 (2009) 5092–5105

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A non-commutative generalization of k-Schur functions

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ARTICLE INFO

Article history:
Received 5 April 2008
Received in revised form 16 March 2009
Accepted 16 March 2009
Available online 15 April 2009

keywords:
Non-commutative symmetric functions
k-Schur functions
Symmetric functions with additional parameters

ABSTRACT

We introduce non-commutative analogs of k-Schur functions of Lapointe–Lascoux and Morse. We give explicit formulas for the expansions of non-commutative functions with one and two parameters in terms of these new functions. These results are similar to the conjectures existing in the commutative case.

1. Introduction

In their approach studying Macdonald q, t-Kostka polynomials $K_{\lambda\mu}(q, t)$, L. Lapointe, A. Lascoux and J. Morse introduced k-Schur functions in [18]. Abstractly, the k-Schur functions $\{s_\lambda^{(k)}[X; t]\}$ are generalizations of the Schur functions $s_\lambda[X]$ and are the fundamental basis of the subspace of the symmetric functions linearly spanned by the Hall–Littlewood symmetric functions $Q'_\lambda[X; t] = \sum_\mu K_{\mu\lambda}(0, t)s_\mu[X]$ with all parts of the partition λ smaller than or equal to k. The motivating property for defining the k-Schur functions was that the Macdonald polynomials $H_\lambda[X; q, t] = \sum_\mu K_{\mu\lambda}(q, t)s_\mu[X]$ [24] expand positively in this basis when λ has parts bounded by k. In more recent works [19–22], Lapointe and Morse have studied in particular, properties of these symmetric functions and conjectured other definitions. Continuing research has established the importance of k-Schur functions in other areas of mathematics including connections to the geometry of the affine Grassmannian.

In this article, we introduce a basis for a subspace of the Hopf algebra of non-commutative symmetric functions studied in [5,9,16,17] which we believe is a good analog of the k-Schur functions. This basis satisfies many of the same properties of the k-Schur functions and, unlike the commutative counterparts, our analogous versions are very well behaved so that properties which are difficult to prove or are conjectural in the commutative case, can be proven for the non-commutative versions.

Several non-equivalent analogs of the Macdonald and Hall–Littlewood symmetric functions have been introduced [3,13,14,25] to model various properties of symmetric functions with extra parameters. In this article, we concentrate on the versions introduced in [3] because they have exactly the properties of the commutative counterparts which we wish to understand better. In particular, the non-commutative analogs introduced in [3] are known to have an operator $\blacktriangledown$ which is analogous to the operator ∇ introduced in [2]. The operator ∇ is defined so that the Macdonald symmetric functions $\tilde{H}_\lambda[X; q, t] = t^{n(\lambda)}H_\lambda[X; q, 1/t]$ are eigenfunctions with eigenvalues $t^{n(\lambda)}q^{n(\lambda')}$ (here $n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i$) and similarly $\blacktriangledown$ is defined so that the q, t-analogs of non-commutative symmetric functions are eigenfunctions (see Definition 17).

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0012-365X/$ – see front matter
doi:10.1016/j.disc.2009.03.022

The main results that we demonstrate here are difficult to appreciate without comparing these with the corresponding properties of the k-Schur functions. We list here some of the striking conjectures about the k-Schur functions (all of which except property (5) are from [18] or [19]) and preceding each of the statements, we list each of the corresponding theorems for the non-commutative versions. Let λ be a partition with all parts less than or equal to k.

(1) (Theorem 9) The k-Schur functions are a basis for the space of symmetric functions $\mathcal{L}\{Q'_\mu[X; t] : \mu_1 \le k\}$. In particular, the element $Q'_\lambda[X; t]$ expands positively in the k-Schur functions.
(2) (Theorem 10) $s^{(k)}_\lambda[X; t]$ expands positively in terms of the functions $s^{(k+1)}_\mu[X; t]$.
(3) (Theorem 14) The Macdonald symmetric function $H_\lambda[X; q, t] = \sum_\mu K_{\mu\lambda}(q, t)s_\mu[X]$ expands positively in terms of the k-Schur basis.
(4) (Propositions 15 and 16) The fundamental involution ω applied to $s^{(k)}_\lambda[X; t]$ is another k-Schur function with the t parameter inverted.
(5) (Theorem 26) The operator ∇ acting on $s^{(k)}_\lambda[X; 1/t]$ expands positively in the k-Schur basis also with the parameter inverted.

Part of the motivation for considering this filtration of the non-commutative symmetric functions is that it reflects all of the properties of the k-level filtration of the symmetric functions. The properties of the γ-Schur non-commutative functions have motivated us to study a k-level analog of the q, t-Catalan numbers [1] and it has also helped in making new conjectures on the commutative symmetric functions through observations of the non-commutative counterparts (see for instance Conjecture 21 which was made after observing the analogous property of the non-commutative versions of these functions).

2. Non-commutative symmetric functions

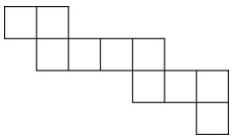
2.1. Preliminaries on compositions

The number of elements of a sequence α is called the length and is denoted by $l(\alpha)$. A sequence of positive integers $\alpha = (\alpha_1, \ldots, \alpha_{l(\alpha)})$ is a composition of size n, written $\alpha \models n$, if

$$\alpha_1 + \cdots + \alpha_{l(\alpha)} = n.$$

A composition α is usually represented by a rim-hook diagram whose rows have lengths $\alpha_1, \ldots, \alpha_{l(\alpha)}$ (read from top to bottom).

Example 1. The composition $\alpha = (2, 4, 3, 1)$ of size 10 can be represented by the diagram



In the theory of non-commutative symmetric functions, we are interested in two kinds of concatenation operations. The first operation is the usual concatenation defined, for two compositions α and β, by

$$\alpha \cdot \beta = (\alpha_1, \alpha_2, \ldots, \alpha_{l(\alpha)}, \beta_1, \beta_2, \ldots, \beta_{l(\beta)}). \tag{1}$$

The second operation is the attachment defined by

$$\alpha|\beta = (\alpha_1, \alpha_2, \ldots, \alpha_{l(\alpha)-1}, \alpha_{l(\alpha)} + \beta_1, \beta_2, \beta_3, \ldots, \beta_{l(\beta)}). \tag{2}$$

The descent set $D(\alpha)$ of a composition α is defined as the set

$$D(\alpha) = \{\alpha_1,\ \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{l(\alpha)-1}\}. \tag{3}$$

The descent set $D(\alpha)$ characterizes the composition α and is of size $l(\alpha) - 1$. It is easy to see that the compositions of n are in one-to-one correspondence with the subsets of $\{1, 2, \ldots, n-1\}$.

For any composition α, the major index statistic is defined by

$$n(\alpha) = \sum_{i \in D(\alpha)} i = \sum_{i=1}^{l(\alpha)} (i-1)\alpha_{l(\alpha)+1-i}. \tag{4}$$

For two compositions α and β, we can refine the previous statistic by defining $c(\alpha, \beta)$ as

$$c(\alpha, \beta) = \sum_{i \in D(\alpha) \cap D(\beta)} i. \tag{5}$$

There is a natural partial order $\le$ on the set of compositions of n, which is called the refinement order. We say that α is finer than β, written $\alpha \le \beta$, if

$$D(\beta) \subseteq D(\alpha). \tag{6}$$

We can also say that $\alpha \leq \beta$ if there exists a sequence of compositions $\gamma^{(1)}, \ldots, \gamma^{(k)}$ such that

$$\alpha = \gamma^{(1)} \cdot \gamma^{(2)} \cdot \ldots \cdot \gamma^{(k)} \quad \text{and} \quad \beta = \gamma^{(1)} \mid \gamma^{(2)} \mid \ldots \mid \gamma^{(k)}. \tag{7}$$

There exist three standard involutions on compositions. The first one is the reverse of a composition defined by

$$\overleftarrow{\alpha} = (\alpha_{l(\alpha)}, \alpha_{l(\alpha)-1}, \ldots, \alpha_1). \tag{8}$$

If the descent set of α is $D(\alpha) = \{i_1, \ldots, i_k\}$ then

$$D(\overleftarrow{\alpha}) = \{|\alpha| - i_1, |\alpha| - i_2, \ldots, |\alpha| - i_k\}. \tag{9}$$

The second involution is the complement of a composition. For any composition α of n, the complement α^c of α is the composition with descent set the complement of $D(\alpha)$

$$D(\alpha^c) = \{1, 2, \ldots, n-1\} \backslash D(\alpha). \tag{10}$$

The third one is the analog of the conjugate of a partition and corresponds to the flipping of the composition about the line $y = x$. It is defined by

$$\alpha' = \overleftarrow{\alpha^c} = \overleftarrow{\alpha}^c. \tag{11}$$

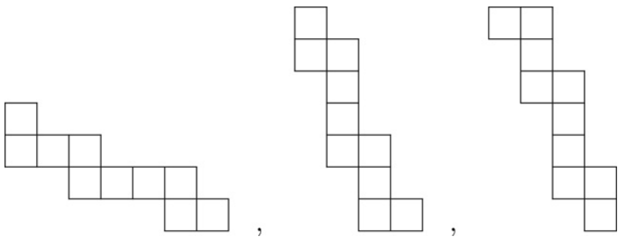
Example 2. The descent set of the composition $\alpha = (2, 4, 3, 1)$ given in Example 1 is

$$D(\alpha) = \{2, 6, 9\}.$$

The three previous involutions applied on the composition α give

$$\overleftarrow{\alpha} = (1, 3, 4, 2), \qquad \alpha^c = (1, 2, 1, 1, 2, 1, 2) \quad \text{and} \quad \alpha' = (2, 1, 2, 1, 1, 2, 1).$$

These compositions correspond respectively to the following diagrams



2.2. Non-commutative symmetric functions

For more details about non-commutative symmetric functions see [5,9,16,17]. We use the convention of bold font for writing down the non-commutative symmetric functions. Let $A = \{a_1, a_2, \ldots\}$ be a sequence of non-commutative variables and X the corresponding sequence where variables commute. For any composition α, we define the non-commutative homogeneous functions by

$$\mathbf{h}_\alpha(A) = \mathbf{h}_{\alpha_1}(A) \ldots \mathbf{h}_{\alpha_{l(\alpha)}}(A), \tag{12}$$

where $\mathbf{h}_n(A)$ is a non-commuting generator of the algebra that is analogous to the element $h_n(X)$ in the space of symmetric functions. That is

$$\mathbf{h}_n(A) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} a_{i_1} a_{i_2} \cdots a_{i_n}. \tag{13}$$

The product of two non-commutative homogeneous symmetric functions is given by

$$\mathbf{h}_\alpha(A)\mathbf{h}_\beta(A) = \mathbf{h}_{\alpha \cdot \beta}(A). \tag{14}$$

The space of non-commutative symmetric functions **Sym** over the field $\mathbb{C}(q, t)$ of rational functions in the parameters q and t is defined by

$$\mathbf{Sym} = \mathbb{C}(q, t)\langle \mathbf{h}_1, \mathbf{h}_2, \ldots \rangle. \tag{15}$$

The analogs of Schur functions are the ribbon Schur functions defined for any composition α by

$$\mathbf{R}_\alpha(A) = \sum_{\alpha \leq \beta} (-1)^{l(\alpha) - l(\beta)} \mathbf{h}_\beta(A). \tag{16}$$

The multiplication rule for two ribbon Schur functions is given by

$$\mathbf{R}_\alpha(A)\mathbf{R}_\beta(A) = \mathbf{R}_{\alpha \cdot \beta}(A) + \mathbf{R}_{\alpha | \beta}(A). \tag{17}$$

There are two involutions ω^c and $\overleftarrow{\omega}$ on the non-commutative symmetric functions which are the analogs of the involution ω in *Sym*. They are defined for any composition α by

$$\omega^c(\mathbf{R}_\alpha(A)) = \mathbf{R}_{\alpha^c}(A) \quad \text{and} \quad \overleftarrow{\omega}(\mathbf{R}_\alpha(A)) = \mathbf{R}_{\overleftarrow{\alpha}}(A). \tag{18}$$

We define the commutative evaluation of a non-commutative symmetric function through the surjective map

$$\begin{array}{rccc} \chi: & \mathbf{Sym} & \longrightarrow & \mathrm{Sym} \\ & \mathbf{h}_\alpha(A) & \longmapsto & h_\alpha(X). \end{array} \tag{19}$$

The image of the ribbon Schur function $\mathbf{R}_\alpha(A)$ by χ is the commutative skew Schur function indexed by the skew partition corresponding to the ribbon α.

2.3. Deformations of non-commutative symmetric functions

The modified Hall–Littlewood functions $Q'_\lambda(X;t)$ (resp. modified Macdonald polynomials $\tilde{H}_\lambda(X;q,t)$) are t-analogs (resp. (q,t)-analogs) of the complete functions $h_\lambda(X)$. In this section, we recall basic statements on the non-commutative analogs of these deformations defined in [3]. There exist different non-commutative analogs of Hall–Littlewood functions and Macdonald polynomials which have been considered in [13,14] by Hivert, Lascoux and Thibon and more recently in [25] by Tevlin.

2.3.1. Non-commutative Hall–Littlewood functions

In [3], the authors define non-commutative analogs of Hall–Littlewood functions by

$$\mathbf{H}_\alpha(A;t) = \sum_{\beta \geq \alpha} t^{c(\alpha,\beta^c)} \mathbf{R}_\beta(A). \tag{20}$$

The non-commutative Hall–Littlewood functions $\mathbf{H}_\alpha(A;t)$ satisfy the following specializations

$$\mathbf{H}_\alpha(A;0) = \mathbf{R}_\alpha(A) \quad \text{and} \quad \mathbf{H}_\alpha(A;1) = \mathbf{h}_\alpha(A). \tag{21}$$

Example 3. The expansion of the non-commutative Hall–Littlewood $\mathbf{H}_{121}(A;t)$ in the ribbon Schur basis is

$$\mathbf{H}_{121}(A;t) = \mathbf{R}_{121}(A) + t\,\mathbf{R}_{31}(A) + t^3\,\mathbf{R}_{13}(A) + t^4\,\mathbf{R}_4(A).$$

For any hook composition $\alpha = (1^a, b)$, the commutative image of $\mathbf{H}_\alpha(A;t)$ by χ coincides with the commutative modified Hall–Littlewood functions $Q'_{(b,1^a)}(X;t)$

$$\chi(\mathbf{H}_{(1^a,b)}(A;t)) = Q'_{(b,1^a)}(X;t). \tag{22}$$

In [3], we find more detailed statements on these non-commutative functions. For example, there are an explicit expansion of the product of two Hall–Littlewood functions in terms of Hall–Littlewood functions, a Pieri formula, some creation operators and a factorization formula at primitive roots of unity. Most of these properties also exist for the analogs considered in [13].

2.3.2. Non-commutative Macdonald polynomials

In [3], the authors also give a definition for non-commutative analogs of Macdonald polynomials in **Sym**. These functions are defined by

$$\mathbf{H}_\alpha(A;q,t) = \sum_{\beta \models |\alpha|} t^{c(\alpha,\beta^c)} q^{c(\alpha',\overleftarrow{\beta})} \mathbf{R}_\beta(A). \tag{23}$$

We define non-commutative analogs of the modified Macdonald polynomials $\widetilde{\mathbf{H}}_\lambda(X;q,t)$ by

$$\widetilde{\mathbf{H}}_\alpha(A;q,t) = t^{n(\alpha)} \mathbf{H}_\alpha\left(A;q,\frac{1}{t}\right) = \sum_{\beta \models |\alpha|} t^{c(\alpha,\beta)} q^{c(\alpha',\overleftarrow{\beta})} \mathbf{R}_\beta(A). \tag{24}$$

The right hand side of (24) comes from the following property

$$n(\alpha) - c(\alpha,\beta^c) = \sum_{i \in D(\alpha)} i - \sum_{i \in D(\alpha) \cap (\{1,\dots,n\} \setminus D(\beta))} i = \sum_{i \in D(\alpha) \cap D(\beta)} i = c(\alpha,\beta). \tag{25}$$

Example 4. The expansion of the non-commutative Macdonald polynomial $\widetilde{\mathbf{H}}_{31}(A;q,t)$ in the ribbon Schur basis is

$$\begin{aligned} \widetilde{\mathbf{H}}_{31}(A;q,t) = {} & \mathbf{R}_4(A) + q^3\,\mathbf{R}_{13}(A) + q^2\,\mathbf{R}_{22}(A) + q^5\,\mathbf{R}_{112}(A) + t^3\,\mathbf{R}_{31}(A) \\ & + q^3t^3\,\mathbf{R}_{121}(A) + q^2t^3\,\mathbf{R}_{211}(A) + q^5t^3\,\mathbf{R}_{1111}(A). \end{aligned}$$

These definitions can be expressed in terms of tensor products of 2×2 matrices (see [3] for more details). For two matrices B and $C = (c_{ij})_{1\le i,j\le m}$, we use the following convention for the definition of their tensor product

$$B \otimes C = [c_{ij}B]_{1\le i,j\le m}. \tag{26}$$

Let n be a non-negative integer. To consider column vectors of elements of degree n in **Sym**, we need a total order on the set of compositions of n. We use the total order corresponding to the rank function ϕ defined by

$$\begin{array}{cccc} \phi: & \{\alpha,\ \alpha \models n\} & \longrightarrow & \{0,\ldots,2^{n-1}-1\} \\ & \alpha & \longmapsto & \displaystyle\sum_{i\in D(\alpha)} 2^{i-1}. \end{array} \tag{27}$$

More precisely, an element of a basis indexed by a composition α appears in the row $\phi(\alpha)$ of the vector.

Example 5. Using this total order, the compositions of 4 are listed as follows

$$(4),\ (13),\ (22),\ (112),\ (31),\ (121),\ (211),\ (1111)\,.$$

We denote by $\mathbf{R}(A)$ the column vector of ribbon Schur functions $(\mathbf{R}_\alpha(A))_{\alpha\models n}$ and by $\mathbf{H}(A;q,t)$ the column vector of Macdonald polynomials $(\mathbf{H}_\alpha(A;q,t))_{\alpha\models n}$. Directly from Eq. (23), we obtain the following formula

$$\mathbf{H}(A;q,t) = \begin{bmatrix} 1 & q^{n-1} \\ t & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & q^{n-2} \\ t^2 & 1 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & q \\ t^{n-1} & 1 \end{bmatrix} \mathbf{R}(A). \tag{28}$$

The matrix expression for the column vector $\widetilde{\mathbf{H}}(A;q,t)$ of modified non-commutative Macdonald polynomials defined with Eq. (24) is given by

$$\widetilde{\mathbf{H}}(A;q,t) = \begin{bmatrix} 1 & q^{n-1} \\ 1 & t \end{bmatrix} \otimes \begin{bmatrix} 1 & q^{n-2} \\ 1 & t^2 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & q \\ 1 & t^{n-1} \end{bmatrix} \mathbf{R}(A). \tag{29}$$

Example 6. The matrix expression of Macdonald polynomials for $n = 3$ is given by

$$\widetilde{\mathbf{H}}(A;q,t) = \begin{bmatrix} 1 & q^2 \\ 1 & t \end{bmatrix} \otimes \begin{bmatrix} 1 & q \\ 1 & t^2 \end{bmatrix} \mathbf{R}(A) = \left[\begin{array}{c|cccc} (3) & 1 & q^2 & q & q^3 \\ (12) & 1 & t & q & qt \\ (21) & 1 & q^2 & t^2 & q^2t^2 \\ (111) & 1 & t & t^2 & t^3 \end{array}\right] \mathbf{R}(A).$$

3. Non-commutative analogs of k-Schur functions

We define analogs of k-Schur functions in the space of non-commutative symmetric functions **Sym**. To find k-Schur functions, Lapointe, Lascoux and Morse originally observed in [18] that certain linear combinations of Hall–Littlewood functions were Schur positive and essentially give atoms that make up the Macdonald symmetric functions.

Let n be a non-negative integer and γ a composition of n. The subspace $\mathbf{Sym}^{(\gamma)}$ of **Sym** is defined as the following homogeneous linear span of some non-commutative Hall–Littlewood functions $\mathbf{H}_\alpha(A;t)$

$$\mathbf{Sym}^{(\gamma)} = \mathcal{L}\{\mathbf{H}_\alpha(A;t) \text{ such that } \alpha \models |\gamma| \text{ and } \alpha \le \gamma\}. \tag{30}$$

This space is a natural analog of the subspace of $Sym^{(k)}$ generated by the modified Hall–Littlewood functions $Q'_\lambda(X;t)$ indexed by partitions λ with the first part being less than k.

Definition 7. Let α and γ be two compositions of n such that $\alpha \le \gamma$. The γ-ribbon Schur function $\mathbf{R}^{(\gamma)}_\alpha(A;t)$ is defined by

$$\mathbf{R}^{(\gamma)}_\alpha(A;t) = \sum_{\substack{\beta\ge\alpha \\ D(\alpha)\setminus D(\beta)\subseteq D(\gamma)}} t^{c(\alpha,\beta^c)}\mathbf{R}_\beta(A). \tag{31}$$

The compositions β which appear in the previous sum are those which appear in the interval of the composition poset for the refinement order between α and the composition with descent set $D(\alpha)\setminus D(\gamma)$.

Example 8. For $n = 5$, the expansion of the (131)-ribbon Schur function $\mathbf{R}^{(131)}_{1121}(A)$ is

$$\mathbf{R}^{(131)}_{1121}(A;t) = \mathbf{R}_{1121}(A) + t\,\mathbf{R}_{221}(A) + t^4\,\mathbf{R}_{113}(A) + t^5\,\mathbf{R}_{23}(A). \tag{32}$$

Directly from Definition 7, the γ-ribbon Schur functions reduce for special cases of the level γ to some particular non-commutative symmetric functions.

$$\mathbf{R}^{((|\alpha|))}_\alpha(A;t) = \mathbf{R}_\alpha(A) \quad \text{and} \quad \mathbf{R}^{(\alpha)}_\alpha(A;t) = \mathbf{H}_\alpha(A;t). \tag{33}$$

At this moment, it is not clear that the γ-ribbon Schur functions form a basis of the subspace $\mathbf{Sym}^{(\gamma)}$. The following theorem gives us an explicit expression for the γ-ribbon Schur functions in terms of Hall–Littlewood functions.

Theorem 9. *The set of elements* $\left\{\mathbf{R}_\alpha^{(\gamma)}(A;t)\right\}_{\alpha\le\gamma}$ *is a basis of* $\mathbf{Sym}^{(\gamma)}$. *The change of bases between Hall–Littlewood functions and* γ*-ribbon Schur functions is given by*

$$\mathbf{H}_\alpha(A;t)=\sum_{\alpha\le\beta\le\gamma} t^{c(\alpha,\beta^c)}\mathbf{R}_\beta^{(\gamma)}(A;t). \tag{34}$$

The inverse change of basis is given by

$$\mathbf{R}_\alpha^{(\gamma)}(A;t)=\sum_{\alpha\le\beta\le\gamma}(-1)^{l(\alpha)-l(\beta)}t^{c(\alpha,\beta^c)}\mathbf{H}_\beta(A;t). \tag{35}$$

Proof. In this theorem, Eq. (35) is a Möbius inversion over the Boolean lattice of Eq. (34). The proof of Eq. (34) follows from Theorem 10 which gives branching rules for γ-Schur functions and from the limit cases given in Eq. (33). □

Theorem 10. *Let* γ *and* $\widetilde{\gamma}$ *be two compositions of* n *such that* $\gamma\le\widetilde{\gamma}$. *For any composition* $\alpha\le\gamma$ *of* n, *the branching rule from the* γ*-Schur functions to the* $\widetilde{\gamma}$*-Schur functions is given by*

$$\mathbf{R}_\alpha^{(\gamma)}(A;t)=\sum_{\substack{\beta:\widetilde{\gamma}\ge\beta\ge\alpha\\ D(\alpha)\setminus D(\beta)\subseteq D(\gamma)\setminus D(\widetilde{\gamma})}} t^{c(\alpha,\beta^c)}\mathbf{R}_\beta^{(\widetilde{\gamma})}(A;t). \tag{36}$$

This theorem means that there exists a family of branching rules for a given level γ. This theorem is also an analog of the branching rules from the k-Schur functions to the $(k+1)$-Schur functions in the commutative case. They are still conjectural and seem to be related to a poset structure of the k-shapes.

Example 11. The two branchings for the (221)-Schur function $\mathbf{R}_{(1121)}^{(221)}(A;t)$ to levels (41) and (23) are

$$\mathbf{R}_{(1121)}^{(221)}(A;t)=\mathbf{R}_{(1121)}^{(41)}(A;t)+t^2\,\mathbf{R}_{(131)}^{(41)}(A;t) \tag{37}$$

$$=\mathbf{R}_{(1121)}^{(23)}(A;t)+t^4\,\mathbf{R}_{(113)}^{(23)}(A;t). \tag{38}$$

In this example

$$\mathbf{R}_{(1121)}^{(221)}(A;t)=\mathbf{R}_{(1121)}(A)+t^2\mathbf{R}_{(131)}(A)+t^4\mathbf{R}_{(113)}(A)+t^6\mathbf{R}_{(14)}(A),$$

and

$$\begin{aligned}\mathbf{R}_{(1121)}^{(41)}(A;t)&=\mathbf{R}_{(1121)}(A)+t^4\mathbf{R}_{(113)}(A)\\ \mathbf{R}_{(131)}^{(41)}(A;t)&=\mathbf{R}_{(131)}(A)+t^4\mathbf{R}_{(14)}(A),\\ \mathbf{R}_{(1121)}^{(23)}(A;t)&=\mathbf{R}_{(1121)}(A)+t^2\mathbf{R}_{(131)}(A)\\ \mathbf{R}_{(113)}^{(23)}(A;t)&=\mathbf{R}_{(113)}(A)+t^2\mathbf{R}_{(14)}(A).\end{aligned}$$

In order to prove Theorem 10, we need to prove the following two technical lemmas.

Lemma 12. *Let* β *be a composition of* n. *For two compositions* α *and* δ *of* n *such that* $\alpha\le\beta\le\delta$,

$$c(\alpha,\beta^c)+c(\beta,\delta^c)=c(\alpha,\delta^c).$$

Proof. The quantity $c(\alpha,\beta^c)+c(\beta,\delta^c)$ is the sum over all i in the set $D(\alpha)\cap D(\beta^c)$ and $D(\beta)\cap D(\delta^c)$. Since $D(\beta)\subseteq D(\alpha)$ and $D(\beta^c)\subseteq D(\delta^c)$,

$$\begin{aligned}(D(\alpha)\cap D(\beta^c))\cup(D(\beta)\cap D(\delta^c))&=(D(\alpha)\cap D(\beta^c))\cup(D(\alpha)\cap D(\beta)\cap D(\delta^c))\\ &=D(\alpha)\cap(D(\beta^c)\cup(D(\beta)\cap D(\delta^c)))\\ &=D(\alpha)\cap((D(\beta^c)\cap D(\delta^c))\cup(D(\beta)\cap D(\delta^c)))\\ &=D(\alpha)\cap D(\delta^c)\cap(D(\beta^c)\cup D(\beta))\\ &=D(\alpha)\cap D(\delta^c).\end{aligned}$$

Therefore $c(\alpha,\beta^c)+c(\beta,\delta^c)=c(\alpha,\delta^c)$. □

Lemma 13. *Let* γ *and* $\widetilde{\gamma}$ *be two compositions of* n *such that* $\gamma\le\widetilde{\gamma}$. *Let* α *be a composition of* n *such that* $\alpha\le\gamma$. *The map which sends* (δ,β) *to* δ *is a bijection between the two sets*

$$A=\{(\delta,\beta)\text{ such that }\delta\ge\beta\ge\alpha\text{ and }\beta\le\tilde{\gamma}\text{ and }D(\alpha)\setminus D(\beta)\subseteq D(\gamma)\setminus D(\tilde{\gamma})\text{ and }D(\beta)\setminus D(\delta)\subseteq D(\tilde{\gamma})\}$$

and

$$B = \{\delta \text{ such that } \delta \geq \alpha \text{ and } D(\alpha)\backslash D(\delta) \subseteq D(\gamma)\}.$$

Proof. Let α be a composition of n and assume that (δ, β) is in A. Let i be in $D(\alpha)\backslash D(\delta)$, then either $i \in D(\beta)$ or $i \notin D(\beta)$. If $i \in D(\beta)$, then since $D(\beta)\backslash D(\delta) \subseteq D(\tilde{\gamma})$,

$$i \in D(\tilde{\gamma}) \subseteq D(\gamma).$$

If $i \notin D(\beta)$, then

$$i \in D(\alpha)\backslash D(\beta) \subseteq D(\gamma)\backslash D(\tilde{\gamma}) \subseteq D(\gamma).$$

Consequently, δ is an element of B.

If (δ, β) is in the set A, then we have $D(\beta)\backslash D(\delta) \subseteq D(\tilde{\gamma})$, so therefore we can conclude that $D(\beta) \subseteq D(\tilde{\gamma}) \cup D(\delta)$. We also have the conditions that $\beta \leq \delta$ and $\beta \leq \tilde{\gamma}$. Therefore $D(\delta) \subseteq D(\beta)$ and $D(\tilde{\gamma}) \subseteq D(\beta)$. Therefore $D(\delta) \cup D(\tilde{\gamma}) \subseteq D(\beta)$. Thus for each δ, there is at most one pair (δ, β) in A that will have $D(\beta) = D(\delta) \cup D(\tilde{\gamma})$. But we need to show that this pair is in fact in the set A.

Let i be in $D(\alpha)\backslash D(\beta) = D(\alpha)\backslash(D(\delta) \cup D(\tilde{\gamma}))$ so that $i \notin D(\delta)$ and $i \notin D(\tilde{\gamma})$. Hence,

$$i \in D(\alpha)\backslash D(\delta) \subseteq D(\gamma)$$

and therefore

$$i \in D(\gamma)\backslash D(\tilde{\gamma}).$$

We conclude that $D(\alpha)\backslash D(\beta) \subseteq D(\gamma)\backslash D(\tilde{\gamma})$.

Moreover, if $i \in D(\beta)\backslash D(\delta) = (D(\delta) \cup D(\tilde{\gamma}))\backslash D(\delta)$ it must be that $i \in D(\tilde{\gamma})$. Therefore $D(\beta)\backslash D(\delta) \subseteq D(\tilde{\gamma})$. These two conditions imply that $(\delta, \beta) \in A$. □

Proof of Theorem 10. Let α be a composition of n and consider the following expression obtained using Lemma 12

$$\begin{aligned}\sum_{\substack{\beta:\tilde{\gamma}\geq\beta\geq\alpha\\ D(\alpha)\backslash D(\beta)\subseteq D(\gamma)\backslash D(\tilde{\gamma})}} t^{c(\alpha,\beta^c)}\,\mathbf{R}_\beta^{(\tilde{\gamma})}(A;t) &= \sum_{\substack{\beta:\tilde{\gamma}\geq\beta\geq\alpha\\ D(\alpha)\backslash D(\beta)\subseteq D(\gamma)\backslash D(\tilde{\gamma})}} t^{c(\alpha,\beta^c)} \left(\sum_{\substack{\delta\geq\beta\\ D(\beta)\backslash D(\delta)\subseteq D(\tilde{\gamma})}} t^{c(\beta,\delta^c)}\,\mathbf{R}_\delta(A)\right)\\ &= \sum_{\substack{\beta:\tilde{\gamma}\geq\beta\geq\alpha\\ D(\alpha)\backslash D(\beta)\subseteq D(\gamma)\backslash D(\tilde{\gamma})}} \left(\sum_{\substack{\delta\geq\beta\\ D(\beta)\backslash D(\delta)\subseteq D(\tilde{\gamma})}} t^{c(\alpha,\delta^c)}\mathbf{R}_\delta(A)\right). \end{aligned}\tag{39}$$

Lemma 13 shows that there is exactly one term in this sum for every composition in the interval between α and the composition with descent set equal to $D(\alpha)\backslash D(\gamma)$ (i.e. compositions δ such that $D(\alpha)\backslash D(\delta) \subseteq D(\gamma)$).

From Definition 7, this implies that Eq. (39) is equal to $\mathbf{R}_\alpha^{(\gamma)}(A;t)$. □

Theorem 14. *Let α and γ be two compositions of n such that $\alpha \leq \gamma$. The non-commutative Macdonald polynomials $\mathbf{H}_\alpha(A;q,t)$ and $\widetilde{\mathbf{H}}_\alpha(A;q,t)$ are γ-Schur positive. More precisely,*

$$\mathbf{H}_\alpha(A;q,t) = \sum_{\beta\leq\gamma} t^{c(\alpha,\beta^c)} q^{c(\alpha',\overleftarrow{\beta})} \mathbf{R}_\beta^{(\gamma)}(A),\tag{40}$$

and

$$\widetilde{\mathbf{H}}_\alpha(A;q,t) = \sum_{\beta\leq\gamma} t^{c(\alpha,\beta)} q^{c(\alpha',\overleftarrow{\beta})} \mathbf{R}_\beta^{(\gamma)}\left(A;\frac{1}{t}\right).\tag{41}$$

Proof.

$$\begin{aligned}\sum_{\beta\leq\gamma} t^{c(\alpha,\beta^c)} q^{c(\alpha',\overleftarrow{\beta})} \mathbf{R}_\beta^{(\gamma)}(A;t) &= \sum_{\beta\leq\gamma} t^{c(\alpha,\beta^c)} q^{c(\alpha',\overleftarrow{\beta})} \left(\sum_{\substack{\delta\geq\beta\\ D(\beta)\backslash D(\delta)\subseteq D(\gamma)}} t^{c(\beta,\delta^c)}\mathbf{R}_\delta(A)\right)\\ &= \sum_{\beta\leq\gamma}\left(\sum_{\substack{\delta\geq\beta\\ D(\beta)\backslash D(\delta)\subseteq D(\gamma)}} t^{c(\alpha,\beta^c)+c(\beta,\delta^c)} q^{c(\alpha',\overleftarrow{\beta})}\mathbf{R}_\delta(A)\right).\end{aligned}\tag{42}$$

Using Lemma 13 with $\alpha = \gamma = (1^n)$ and $\tilde{\gamma} = \gamma$, we see that there is a 1–1 correspondence between the set

$$\{(\delta, \beta) \text{ such that } \delta \geq \beta \text{ and } \beta \leq \gamma \text{ and } D(\beta)\backslash D(\delta) \subseteq D(\gamma)\}$$

and the set of all compositions of n. Consequently, each β which appears in this sum is determined from the composition δ and its descent set is given by

$$D(\beta) = D(\gamma) \cup D(\delta).$$

Now we want to show that $c(\alpha, \beta^c) + c(\beta, \delta^c) = c(\alpha, \delta^c)$ in the specific case where β does not satisfy the conditions of Lemma 12. We must come up with an independent argument. Note that $D(\gamma) \subseteq D(\alpha)$ and $D(\beta) = D(\gamma) \cup D(\delta)$, hence

$$\begin{aligned}(D(\alpha) \cap D(\beta^c)) \cup (D(\beta) \cap D(\delta^c)) &= (D(\alpha) \cap D(\gamma^c) \cap D(\delta^c)) \cup ((D(\gamma) \cup D(\delta)) \cap D(\delta^c)) \\ &= (D(\alpha) \cap D(\gamma^c) \cap D(\delta^c)) \cup (D(\gamma) \cap D(\delta^c)) \\ &= ((D(\alpha) \cap D(\gamma^c)) \cup D(\gamma)) \cap D(\delta^c) \\ &= D(\alpha) \cap D(\delta^c).\end{aligned}$$

Moreover, since $D(\gamma) \subseteq D(\alpha)$, we have

$$D(\alpha') \cap D(\overleftarrow{\gamma}) = \emptyset.$$

Therefore

$$D(\alpha') \cap D(\overleftarrow{\beta}) = D(\alpha') \cap (D(\overleftarrow{\gamma}) \cup D(\overleftarrow{\delta})) = D(\alpha') \cap D(\overleftarrow{\delta}).$$

We conclude that

$$c(\alpha', \overleftarrow{\beta}) = c(\alpha', \overleftarrow{\delta}).$$

Finally, we have that Eq. (42) is equivalent to

$$\sum_{\beta \le \gamma} t^{c(\alpha,\beta^c)} q^{c(\alpha', \overleftarrow{\beta})} \mathbf{R}_\beta^{(\gamma)}(A; t) = \sum_{\delta \models n} t^{c(\alpha,\delta^c)} q^{c(\alpha', \overleftarrow{\delta})} \mathbf{R}_\delta(A) = \mathbf{H}_\alpha(A; q, t).$$

The expansion for the modified version $\widetilde{\mathbf{H}}_\alpha(X; t)$ is obtained by using Eq. (24) in the previous equation. □

From now on, we need to use an order on compositions, different from the one used in Section 2.3.2. Given a fixed $\gamma \models n$, let

$$D(\gamma) = \{i_1, \ldots, i_k\} \quad \text{and} \quad D(\gamma^c) = \{j_1, \ldots, j_{n-k-1}\},$$

where $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_{n-k-1}$.

Given this, let σ_γ be the unique permutation of $\{1, 2, \ldots, n-1\}$ defined by

$$\begin{aligned}\sigma_\gamma(i_s) &= s && \text{for } 1 \le s \le k, \\ \sigma_\gamma(j_r) &= r + k && \text{for } 1 \le r \le n-k-1.\end{aligned}$$

We then define for all compositions α of n the rank function

$$\phi_\gamma(\alpha) = \sum_{i \in D(\alpha)} 2^{\sigma_\gamma(i)-1}. \tag{43}$$

Let us denote by $\widetilde{\mathbf{H}}|_\gamma(A; q, t)$ the column vector of the modified non-commutative Macdonald polynomials $\widetilde{\mathbf{H}}_\alpha(X; q, t)$ indexed by compositions α such that $\alpha \le \gamma$ ordered using ϕ_γ. The expression of (41) given in Theorem 14 can be expressed in terms of 1×1 and 2×2 matrices

$$\widetilde{\mathbf{H}}|_\gamma(A; q, t) = \bigotimes_{i \in D(\gamma)} \left[t^i\right] \bigotimes_{i \notin D(\gamma)} \begin{bmatrix} 1 & q^{n-i} \\ 1 & t^i \end{bmatrix} \mathbf{R}^{(\gamma)}\left(A; \frac{1}{t}\right). \tag{44}$$

Proposition 15. *Let γ be a composition of n. For any composition $\alpha \le \gamma$ of n, let us define ζ, the composition with descent set $D(\zeta) = D(\gamma^c) \setminus D(\alpha) \cup D(\gamma)$. There exists an analog of the k-conjugation given by*

$$\omega^c\left(\mathbf{R}_\alpha^{(\gamma)}(A; t)\right) = t^{n(\gamma)} \mathbf{R}_\zeta^{(\gamma)}\left(A; \frac{1}{t}\right). \tag{45}$$

Proof. By Eq. (31) we have

$$\omega^c\left(\mathbf{R}_\alpha^{(\gamma)}(A; t)\right) = \sum_{\substack{\beta \ge \alpha \\ D(\alpha)\setminus D(\beta) \subseteq D(\gamma)}} t^{c(\alpha,\beta^c)} \mathbf{R}_{\beta^c}(A).$$

To check that the exponent of t agrees with the right hand side of the equation stated in the proposition, we notice that $D(\zeta) \cap D(\beta) = (D(\gamma^c) \setminus D(\alpha)) \cap D(\beta) \cup D(\gamma) \cap D(\beta) = D(\gamma) \cap D(\beta)$, since $D(\beta) \subseteq D(\alpha)$. Therefore $n(\gamma) - c(\gamma, \beta) = c(\gamma, \beta^c)$. Since we also have that $D(\gamma) \subseteq D(\alpha)$ and $D(\alpha) \setminus D(\beta) \subseteq D(\gamma)$, then it follows that $n(\gamma) - c(\zeta, \beta) = n(\gamma) - c(\gamma, \beta) = c(\gamma, \beta^c) = c(\alpha, \beta^c)$.

This shows that our sum has reduced to

$$\omega^c\left(\mathbf{R}_\alpha^{(\gamma)}(A;t)\right)=\sum_{\substack{\beta\geq\alpha\\ D(\alpha)\setminus D(\beta)\subseteq D(\gamma)}} t^{c(\alpha,\beta^c)}\mathbf{R}_{\beta^c}(A)=\sum_{\substack{\beta\geq\alpha\\ D(\alpha)\setminus D(\beta)\subseteq D(\gamma)}} t^{n(\gamma)-c(\zeta,\beta)}\mathbf{R}_{\beta^c}(A).$$

It is also necessary to verify that the terms in the right hand side of the equation are those that also appear in $\mathbf{R}_\zeta^{(\gamma)}(A;t)$. That is, we show $D(\beta)\subseteq D(\alpha)$ and $D(\alpha)\setminus D(\beta)\subseteq D(\gamma)$, if and only if $D(\beta^c)\subseteq D(\zeta)$ and $D(\zeta)\setminus D(\beta^c)\subseteq D(\gamma)$. By noticing that $D(\beta^c)=(D(\gamma)\cap D(\beta^c))\uplus(D(\gamma^c)\cap D(\beta^c))$, then $D(\gamma)\cap D(\beta^c)\subseteq D(\gamma)$ and $D(\gamma^c)\cap D(\beta^c)=D(\gamma^c)\setminus D(\alpha)$, we see that $\beta\geq\zeta$. This also shows that $D(\zeta)\setminus D(\beta^c)=D(\gamma)\setminus(D(\gamma)\cap D(\beta^c))$ and is hence a subset of $D(\gamma)$.

We conclude that

$$\sum_{\substack{\beta\geq\alpha\\ D(\alpha)\setminus D(\beta)\subseteq D(\gamma)}} t^{n(\gamma)-c(\zeta,\beta)}\mathbf{R}_{\beta^c}(A)=t^{n(\gamma)}\sum_{\substack{\beta^c\geq\zeta\\ D(\zeta)\setminus D(\beta^c)\subseteq D(\gamma)}} t^{-c(\zeta,\beta)}\mathbf{R}_{\beta^c}(A)=t^{n(\gamma)}\mathbf{R}_\zeta^{(\gamma)}\left(A;\frac{1}{t}\right).\quad\square$$

Proposition 16. *At $t=1$, we have another analog of the k-conjugation given by*

$$\overleftarrow{\omega}\left(\mathbf{R}_\alpha^{(\gamma)}(A;1)\right)=\mathbf{R}_{\overleftarrow{\alpha}}^{(\overleftarrow{\gamma})}(A;1). \tag{46}$$

The action of $\overleftarrow{\omega}$ is a consequence of Proposition 32 which follows easily from the definitions and hence we do not provide a proof here.

4. The non-commutative nabla operator

In the theory of commutative symmetric functions, the operator ∇ is defined as the linear operator which admits the modified Macdonald polynomials $\widetilde{H}_\lambda(X;q,t)$ as eigenvectors for the eigenvalues $t^{n(\lambda)}q^{n(\lambda')}$. This operator is related to the combinatorics of Dyck paths and to the space of diagonal harmonics [2,6–8,12,10,11,23]. In [3], the authors give a non-commutative analog $\blacktriangledown$ of the operator nabla in the space **Sym**.

Definition 17. The non-commutative nabla operator $\blacktriangledown$ is the linear operator defined on the basis of non-commutative modified Macdonald polynomials by

$$\blacktriangledown\left(\widetilde{\mathbf{H}}_\alpha(A;q,t)\right)=t^{n(\alpha)}q^{n(\alpha')}\widetilde{\mathbf{H}}_\alpha(A;q,t). \tag{47}$$

This definition can be reformulated in terms of 2×2 matrices as proved in [3] by

$$\blacktriangledown\left(\widetilde{\mathbf{H}}(A;q,t)\right)=\begin{bmatrix}q^{n-1}&0\\0&t\end{bmatrix}\otimes\begin{bmatrix}q^{n-2}&0\\0&t^2\end{bmatrix}\otimes\cdots\otimes\begin{bmatrix}q&0\\0&t^{n-1}\end{bmatrix}\widetilde{\mathbf{H}}(A;q,t). \tag{48}$$

Proposition 18 ([3]). *For all compositions α, the non-commutative functions $\blacktriangledown(\mathbf{R}_\alpha(A))$ is ribbon Schur positive, up to a global sign. More precisely, in terms of matrices, we have*

$$\blacktriangledown(\mathbf{R}(A))=\begin{bmatrix}0&-q^{n-1}t\\1&(t+q^{n-1})\end{bmatrix}\otimes\begin{bmatrix}0&-q^{n-2}t^2\\1&(t^2+q^{n-2})\end{bmatrix}\otimes\cdots\otimes\begin{bmatrix}0&-qt^{n-1}\\1&(t^{n-1}+q)\end{bmatrix}\mathbf{R}(A). \tag{49}$$

Example 19. The ribbon Schur expansion of $\blacktriangledown(\mathbf{R}_{121}(A))$ is given by

$$\blacktriangledown(\mathbf{R}_{121}(A))=-q^2t^2\mathbf{R}_{22}(A)-(q^3t^2+q^2t^5)\mathbf{R}_{211}(A)-(q^5t^2+q^2t^3)\mathbf{R}_{112}(A)-(q^6t^2+q^5t^5+q^3t^3+q^2t^6)\mathbf{R}_{1111}(A).$$

In the case of commutative symmetric functions, A. Lascoux gives two conjectures that the commutative symmetric functions $\nabla\left(Q'_\lambda\left(X;\frac{1}{t}\right)\right)$ and $\nabla\left(\omega\left(Q'_\lambda(X;t)\right)\right)$ are Schur positive, up to a global sign, for any partition λ.

Example 20. The conjectures of Lascoux for the Hall–Littlewood function $Q'_{211}(X;t)$ are

$$\nabla\left(Q'_{211}\left(X;\frac{1}{t}\right)\right)=-qt^6s_{1111}(X)-(qt^5+qt^4)s_{211}(X)-qt^3s_{31}(X)-qt^4s_{22}(X),$$

and

$$\begin{aligned}\nabla\left(\omega(Q'_{211}(X;\ t))\right)=&\ (t^9+qt^6)s_{1111}+(t^8+t^7+t^6+qt^5+qt^4)s_{211}(X)+(t^7+t^5+qt^4)s_{22}(X)\\&+(t^6+t^5+t^4+qt^3)s_{31}(X)+t^3s_4(X).\end{aligned}$$

We can generalize the conjectures of Lascoux considering the expansion of the previous functions on the k-Schur basis in the parameter $1/t$.

Conjecture 21. *Let k be a non-negative integer and λ be a partition of size n such that $\lambda_1 \leq k$. For any integer k' such that $k' \geq k$, the commutative symmetric functions*

$$t^{n(n-1)/2}\nabla\left(Q'_\lambda\left(X;\frac{1}{t}\right)\right) \quad \text{and} \quad t^{n(\lambda)+n(n-1)/2}\nabla\left(\omega\left(Q'_\lambda(X;t)\right)\right)$$

are positive, up to a global sign, in the basis of the k'-Schur functions in the parameter $1/t$.

Example 22. For $\lambda = (311)$ and $k = 3$ and $k = 4$, we have the following expansions

$$t^{10}\nabla\left(Q'_{311}\left(X;\frac{1}{t}\right)\right) = (q^3t^3+q^2)s^{(3)}_{11111}\left(\frac{1}{t}\right) + (q^3t^5+q^2t^3+q^2t^2)s^{(3)}_{2111}\left(\frac{1}{t}\right) + (q^3t^4+q^2t^3)s^{(3)}_{221}\left(\frac{1}{t}\right) + q^2t^5s^{(3)}_{311}\left(\frac{1}{t}\right)$$

and

$$\begin{aligned} t^{13}\nabla(\omega(Q'_{311}(X;t))) = {} & (q^3t^7+q^2t^6+qt^4+qt^3+1)s^{(3)}_{11111}\left(\frac{1}{t}\right) \\ & + (q^3t^9+q^2t^8+q^2t^7+qt^7+2qt^6+qt^5+t^4+t^3+t^2)s^{(3)}_{2111}\left(\frac{1}{t}\right) \\ & + (q^2t^9+q^2t^8+qt^8+2qt^7+qt^6+t^6+t^5+t^4+t^3)s^{(3)}_{221}\left(\frac{1}{t}\right) \\ & + (q^2t^9+qt^9+qt^8+t^7+t^6+t^5)s^{(3)}_{311}\left(\frac{1}{t}\right) + t^8s^{(3)}_{32}\left(\frac{1}{t}\right). \end{aligned}$$

We prove an analog of this conjecture in the non-commutative case. We also prove that the functions $\blacktriangledown\mathbf{R}^{(\gamma)}_\alpha(A;t)$ are positive in the γ-Schur basis.

Theorem 23. *Let γ be a composition. For any composition α, the functions $\blacktriangledown\mathbf{R}^{(\gamma)}_\alpha\left(A;\frac{1}{t}\right)$ are positive in the γ-Schur basis in the parameter $1/t$. More precisely, we have*

$$\blacktriangledown\mathbf{R}^{(\gamma)}\left(A;\frac{1}{t}\right) = \bigotimes_{i\in D(\gamma)}\left[t^i\right]\bigotimes_{i\notin D(\gamma)}\begin{bmatrix}0 & -t^iq^{n-i}\\ 1 & t^i+q^{n-i}\end{bmatrix}\mathbf{R}^{(\gamma)}\left(A;\frac{1}{t}\right). \tag{50}$$

It is important to remark here that the order on the vector $\mathbf{R}(A)$ is the one given by ϕ_γ.

Proof. Inverting relation (44), we can express the γ-ribbon Schur functions in terms of modified Macdonald polynomials as follows

$$\mathbf{R}^{(\gamma)}\left(A;\frac{1}{t}\right) = \bigotimes_{i\in D(\gamma)}\left[\frac{1}{t^i}\right]\bigotimes_{i\notin D(\gamma)}\frac{1}{t^i-q^{n-i}}\begin{bmatrix}t^i & -q^{n-i}\\ -1 & 1\end{bmatrix}\widetilde{\mathbf{H}}|_\gamma(A;q,t), \tag{51}$$

where $\widetilde{\mathbf{H}}|_\gamma(A;q,t)$ represents the column vector of the $\widetilde{\mathbf{H}}_\alpha(A;q,t)$ for $\alpha \leq \gamma$.

Applying the linear operator $\blacktriangledown$, we obtain

$$\blacktriangledown\mathbf{R}^{(\gamma)}\left(A;\frac{1}{t}\right) = \bigotimes_{i\in D(\gamma)}\left[\frac{1}{t^i}\right]\bigotimes_{i\notin D(\gamma)}\frac{1}{t^i-q^{n-i}}\begin{bmatrix}t^i & -q^{n-i}\\ -1 & 1\end{bmatrix}\blacktriangledown\widetilde{\mathbf{H}}|_\gamma(A;q,t). \tag{52}$$

By the definition of the operator $\blacktriangledown$ on $\widetilde{\mathbf{H}}_\alpha(X;q,t)$, we have

$$\blacktriangledown\widetilde{\mathbf{H}}|_\gamma(A;q,t) = \bigotimes_{i\in D(\gamma)}\left[t^i\right]\bigotimes_{i\notin D(\gamma)}\begin{bmatrix}q^{n-i} & 0\\ 0 & t^i\end{bmatrix}\widetilde{\mathbf{H}}|_\gamma(A;q,t). \tag{53}$$

Consequently, we have

$$\blacktriangledown\mathbf{R}^{(\gamma)}\left(A;\frac{1}{t}\right) = \bigotimes_{i\in D(\gamma)}[1]\bigotimes_{i\notin D(\gamma)}\frac{1}{t^i-q^{n-i}}\begin{bmatrix}t^iq^{n-i} & -t^iq^{n-i}\\ -q^{n-i} & t^i\end{bmatrix}\widetilde{\mathbf{H}}|_\gamma(A;q,t). \tag{54}$$

By using Eq. (44), we obtain

$$\blacktriangledown\mathbf{R}^{(\gamma)}\left(A;\frac{1}{t}\right) = \bigotimes_{i\in D(\gamma)}\left[t^i\right]\bigotimes_{i\notin D(\gamma)}\begin{bmatrix}0 & -t^iq^{n-i}\\ 1 & t^i+q^{n-i}\end{bmatrix}\mathbf{R}^{(\gamma)}\left(A;\frac{1}{t}\right). \quad \square \tag{55}$$

Theorem 24. *The image of the non-commutative modified Hall–Littlewood functions by the operator ▼ is γ-Schur positive, up to a global sign. More precisely,*

$$\blacktriangledown\left(\widetilde{\mathbf{H}}|_{\gamma}(A;t)\right) = \bigotimes_{i\in D(\gamma)} \begin{bmatrix} t^{2i} \end{bmatrix} \bigotimes_{i\notin D(\gamma)} \begin{bmatrix} 0 & -t^i q^{n-i} \\ t^i & t^{2i} \end{bmatrix} \mathbf{R}^{(\gamma)}\left(A; \frac{1}{t}\right). \tag{56}$$

Proof. The specialization of Eq. (44) at $q = 0$ gives us

$$\widetilde{\mathbf{H}}|_{\gamma}(A;t) = \bigotimes_{i\in D(\gamma)} \begin{bmatrix} t^{i} \end{bmatrix} \bigotimes_{i\notin D(\gamma)} \begin{bmatrix} 1 & 0 \\ 1 & t^{i} \end{bmatrix} \mathbf{R}^{(\gamma)}\left(A; \frac{1}{t}\right). \tag{57}$$

Using the result of Theorem 23, we obtain Eq. (56). □

Example 25. For the non-commutative Hall–Littlewood function $\widetilde{\mathbf{H}}_{121}(A;t)$, we have

$$\blacktriangledown\left(\widetilde{\mathbf{H}}_{121}(A;t)\right) = -q^2t^6\mathbf{R}_{22}(A) - q^2t^9\mathbf{R}_{211}(A) - q^2t^7\mathbf{R}_{112}(A) - q^2t^{10}\mathbf{R}_{1111}. \tag{58}$$

Theorem 26. *The image of the γ-Schur functions in the parameter $1/t$ by the operator ▼ is ribbon Schur positive, up to a global sign. More precisely,*

$$\blacktriangledown\left(\mathbf{R}^{(\gamma)}\left(A; \frac{1}{t}\right)\right) = \bigotimes_{i\in D(\gamma)} \begin{bmatrix} 1 & t^{i} \end{bmatrix} \bigotimes_{i\notin D(\gamma)} \begin{bmatrix} 0 & -q^{n-i}t^i \\ 1 & (t^i + q^{n-i}) \end{bmatrix} \mathbf{R}(A). \tag{59}$$

Proof. By restricting Eq. (29) to the space $\mathbf{Sym}^{(\gamma)}$, we obtain

$$\widetilde{\mathbf{H}}|_{\gamma}(A;q,t) = \bigotimes_{i\in D(\gamma)} \begin{bmatrix} 1 & t^{i} \end{bmatrix} \bigotimes_{i\notin D(\gamma)} \begin{bmatrix} 1 & q^{n-i} \\ 1 & t^{i} \end{bmatrix} \mathbf{R}(A), \tag{60}$$

where the column vectors are ordered using ϕ_{γ}.

The theorem is finally obtained by the composition of Eqs. (52), (53) and (60). □

Remark. There are many ways of defining non-commutative analogs of commutative symmetric functions. The fact that ▼ of these analogs are ribbon-Schur positive, up to a global sign, is an interesting property which is shared with the commutative version as conjectured in [1]. On the commutative side, these results permit us to define some generalizations of the (q,t)-Catalan numbers.

5. Multivariate version of the γ-Schur functions

In all the previous definitions, it is possible to replace the powers of the parameter t by products of the sequence of parameters $t_1,\ldots,t_{n-1}$ and the parameter q using the sequence $q_1,\ldots,q_{n-1}$. Powers of the parameters t and q are always of the form $c(\alpha,\beta) = \sum_{i\in D(\alpha)\cap D(\beta)} i$, for some compositions α and β. The multivariate versions permit us to keep track of the descents which appear in $c(\alpha,\beta)$. We reserve the presentation of these multivariate versions as a side note, as these refined results detract us from the presentation of the previous sections.

Definition 27. For any composition α of n, multivariate non-commutative Hall–Littlewood functions are defined by

$$\mathbf{H}_{\alpha}(A;t_1,\ldots,t_{n-1}) = \sum_{\beta\succeq\alpha}\left(\prod_{i\in D(\alpha)\cap D(\beta^c)} t_i\right)\mathbf{R}_{\beta}(A). \tag{61}$$

These functions are related to the non-commutative Hall–Littlewood functions by the specialization $t_i \to t^i$

$$\mathbf{H}_{\alpha}(A;t) = \mathbf{H}_{\alpha}(A;t,t^2,\ldots,t^{n-1}). \tag{62}$$

Example 28. The expansion of the multivariate non-commutative Hall–Littlewood function $\mathbf{H}_{121}(A;t_1,t_2,t_3)$ in the ribbon Schur basis is

$$\mathbf{H}_{121}(A;t_1,t_2,t_3) = \mathbf{R}_{121}(A) + t_1\,\mathbf{R}_{31}(A) + t_3\,\mathbf{R}_{13}(A) + t_1t_3\,\mathbf{R}_4(A).$$

Definition 29. As for non-commutative Hall–Littlewood functions, we define a multivariate version of non-commutative Macdonald polynomials by

$$\mathbf{H}_{\alpha}(A;q_1,\ldots,q_{n-1},t_1,\ldots,t_{n-1}) = \sum_{\beta\models|\alpha|}\left(\prod_{i\in D(\alpha)\cap D(\beta^c)} t_i \prod_{i\in D(\alpha')\cap D(\overleftarrow{\beta})} q_i\right)\mathbf{R}_{\beta}(A). \tag{63}$$

The non-commutative multivariate modified Macdonald polynomials are defined by

$$\widetilde{\mathbf{H}}_\alpha(A; q_1, \dots, q_{n-1}, t_1, \dots, t_{n-1}) = \left(\prod_{i\in D(\alpha)} t_i\right) \mathbf{H}_\alpha\,(A; q_1, \dots, q_{n-1}, 1/t_1, \dots, 1/t_{n-1}) \tag{64}$$

$$= \sum_{\beta\models|\alpha|} \left(\prod_{i\in D(\alpha)\cap D(\beta)} t_i\right)\left(\prod_{i\in D(\alpha')\cap D(\overleftarrow{\beta})} q_i\right) \mathbf{R}_\beta(A). \tag{65}$$

The non-commutative Macdonald polynomials $\mathbf{H}_\alpha(X; q, t)$ (resp. $\mathbf{H}_\alpha(X, q_1, \dots, q_{n-1}, t_1, \dots, t_{n-1})$) and their modified versions $\widetilde{\mathbf{H}}_\alpha(X; q, t)$ (resp. $\widetilde{\mathbf{H}}_\alpha(X; q_1, \dots, q_{n-1}, t_1, \dots, t_{n-1})$) coincide under the specialization $t_i \to t^i$ and $q_i \to q^i$.

Example 30. The expansion of the multivariate non-commutative modified Macdonald polynomial $\widetilde{\mathbf{H}}_{31}(A; q_1, q_2, q_3, t_1, t_2, t_3)$ in the ribbon Schur basis is

$$\widetilde{\mathbf{H}}_{31}(A; q_1, q_2, q_3, t_1, t_2, t_3) = \mathbf{R}_4(A) + q_3\,\mathbf{R}_{13}(A) + q_2\,\mathbf{R}_{22}(A) + q_2q_3\,\mathbf{R}_{112}(A) + t_3\,\mathbf{R}_{31}(A) + q_3t_3\,\mathbf{R}_{121}(A) + q_2t_3\,\mathbf{R}_{211}(A) + q_2q_3t_3\,\mathbf{R}_{1111}(A).$$

For the multivariate non-commutative Macdonald polynomials and their modified version, matricial expressions are given by

$$\mathbf{H}(A; q, t) = \begin{bmatrix} 1 & q_1 \\ t_{n-1} & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & q_2 \\ t_{n-2} & 1 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & q_{n-1} \\ t_1 & 1 \end{bmatrix} \mathbf{R}(A), \tag{66}$$

$$\widetilde{\mathbf{H}}(A; q, t) = \begin{bmatrix} 1 & q_{n-1} \\ 1 & t_1 \end{bmatrix} \otimes \begin{bmatrix} 1 & q_{n-2} \\ 1 & t_2 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & q_1 \\ 1 & t_{n-1} \end{bmatrix} \mathbf{R}(A). \tag{67}$$

Definition 31. Let α and γ be two compositions of n such that $\alpha \le \gamma$. A multivariate version of non-commutative γ-Schur functions is defined by

$$\mathbf{R}_\alpha^{(\gamma)}(A; t_1, \dots, t_{n-1}) = \sum_{\substack{\beta\ge\alpha \\ D(\alpha)\setminus D(\beta)\subseteq D(\gamma)}} \left(\prod_{i\in D(\alpha)\cap D(\beta^c)} t_i\right) \mathbf{R}_\beta(A). \tag{68}$$

All the results stated in the previous sections can be generalized to the multivariate versions for the most part simply by changing $t^i \to t_i$. For practical notational purposes, it was convenient to state the results using only the two parameters q and t.

Proposition 32. *The action of the analog of the k-conjugation on the multivariate γ-Schur functions is given by*

$$\overleftarrow{\omega}\left(\mathbf{R}_\alpha^{(\gamma)}(A; t_1, \dots, t_{n-1})\right) = \mathbf{R}_{\overleftarrow{\alpha}}^{(\overleftarrow{\gamma})}(A; t_{n-1}, \dots, t_1). \tag{69}$$

Remark. The non-commutative analogs of Hall–Littlewood functions and Macdonald polynomials defined in [13] admit also a multivariate version defined in [14].

Acknowledgments

All the computations related to this work have been done using the combinatorics package MuPAD-Combinat for the algebra computer system MuPAD (see [15] for an introduction to this package and especially [4] for computations and implementations of symmetric functions).

Appendix

A.1. Tables of γ-Schur functions on ribbon Schur functions for weight 4

These tables are calculated from Definition 7. The columns of the table index the corresponding elements of the γ-ribbon Schur basis and the rows index the subscripts of the ribbon Schur basis.

	(31)	(121)	(211)	(1111)
(4)	t^3	·	·	·
(13)	·	t^3	·	·
(22)	·	·	t^3	·
(112)	·	·	·	t^3
(31)	1	·	·	·
(121)	·	1	·	·
(211)	·	·	1	·
(1111)	·	·	·	1

(31)-Schur functions

	(22)	(112)	(211)	(1111)
(4)	t^2	·	·	·
(13)	·	t^2	·	·
(22)	1	·	·	·
(112)	·	1	·	·
(31)	·	·	t^2	·
(121)	·	·	·	t^2
(211)	·	·	1	·
(1111)	·	·	·	1

(22)-Schur functions

	(13)	(112)	(121)	(1111)
(4)	t	·	·	·
(13)	1	·	·	·
(22)	·	t	·	·
(112)	·	1	·	·
(31)	·	·	t	·
(121)	·	·	1	·
(211)	·	·	·	t
(1111)	·	·	·	1

(13)-Schur functions

	(112)	(1111)
(4)	t^3	·
(13)	t^2	·
(22)	t	·
(112)	1	·
(31)	·	t^3
(121)	·	t^2
(211)	·	t
(1111)	·	1

(112)-Schur functions

	(121)	(1111)
(4)	t^4	·
(13)	t^3	·
(22)	·	t^4
(112)	·	t^3
(31)	t	·
(121)	1	·
(211)	·	t
(1111)	·	1

(121)-Schur functions

	(211)	(1111)
(4)	t^5	·
(13)	·	t^5
(22)	t^3	·
(112)	·	t^3
(31)	t^2	·
(121)	·	t^2
(211)	1	·
(1111)	·	1

(211)-Schur functions

A.2. Table of Macdonald polynomials in the γ-Schur basis in weight 4

The tables below show examples of Theorem 14. The columns indicate the index α of the function $\mathbf{H}_\alpha(A; q, t)$.

	(31)	(121)	(211)	(1111)
(31)	1	t	t^2	t^3
(121)	q^3	1	q^3t^2	t^2
(211)	q^2	q^2t	1	t
(1111)	q^5	q^2	q^3	1

$\mathbf{H}|_{(31)}$ in the (31)-Schur basis

	(22)	(112)	(211)	(1111)
(22)	1	t	t^3	t^4
(112)	q^3	1	q^3t^3	t^3
(211)	q	qt	1	t
(1111)	q^4	q	q^3	1

$\mathbf{H}|_{(22)}$ in the (22)-Schur basis

	(13)	(112)	(121)	(1111)
(13)	1	t^2	t^3	t^5
(112)	q^2	1	q^2t^3	t^3
(121)	q	qt^2	1	t^2
(1111)	q^3	q	q^2	1

$\mathbf{H}|_{(13)}$ in the (13)-Schur basis

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