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Ramification of Automorphisms of $k((t))$

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Let k be a field of characteristic p and let $\gamma \in \mathrm{Aut}_k(k((t)))$. For $m \geqslant 0$ define $i_m = v_t(\gamma^{p^m} t - t) - 1$. We show that if $p \nmid i_0$ is and $i_1 < (p^2 - p + 1)\, i_0$ then there exists

Let k be a field of characteristic p and set $K = k((t))$. For $\gamma \in \mathrm{Aut}_k(K)$ define $i(\gamma) = v_K((\gamma t - t)/t)$ and for $m \geqslant 0$ define $i_m = i(\gamma^{p^m})$.

In [2], Keating determines upper bounds for the i_m in certain cases where γ has infinite order. He uses only elementary methods and he suggests that his results "could certainly be proved as applications of Wintenberger's theory of the fields of norms."

The goal of this paper is to follow this suggestion to improve the results of [2].

Let K and γ be as above. The group $\mathrm{Aut}_k(K)$ is compact for its ramification topology. Let Γ be the closed subgroup of $\mathrm{Aut}_k(K)$ generated by γ. Let K^{sep} be a separable closure of K and let $L \subset K^{\mathrm{sep}}$ be an abelian extension of K such that γ can be extended to an automorphism $\tilde{\gamma}$ of L. Then the group Γ acts on $\mathrm{Gal}(L/K)$ by $\gamma * h = \tilde{\gamma} h \tilde{\gamma}^{-1}$ and the group $\mathrm{Aut}_\Gamma(L)$ constituted by the automorphisms of L, the $K-$restriction of which belongs to Γ, is an extension of Γ by $\mathrm{Gal}(L/K)$. Let $K(\gamma)$ denote the maximal abelian extension L of this kind such that Γ acts trivially on $\mathrm{Gal}(L/K)$. Then $\mathrm{Aut}_\Gamma(K(\gamma))$ is abelian and the sequence

$$1 \to \mathrm{Gal}(K(\gamma)/K) \to \mathrm{Aut}_\Gamma(K(\gamma)) \to \Gamma \to 1$$

splits exactly.

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In order to state our first theorem, we let $\gamma \in \mathrm{Aut}_k(K)$, set $i_n = i(\gamma^{p^n})$ as before, and let $b = (i_1 - i_0)/p$. For any group G we define the p-rank of G to be the $\mathbb{F}_p$-dimension of G/G^p.

THEOREM 1. *Assume k is the finite field* $\mathbb{F}_{p^f}$, $p \nmid i_0$ *and* $i_1 < (p^2 - p + 1)\, i_0$. *Then the p-rank of* $Gal(K(\gamma)/K)$ *is bf.*

Proof. In [6], Sen proved that the sequence (i_n/p^n) is strictly increasing and that, for every integer $m \geqslant 0$, $i_{m+1} \equiv i_m \bmod p^{m+1}$. It can be easily deduced that $\lim_{n \to +\infty}(i_n/p^n) = (p/p-1)\, e$ where e is an integer $\geqslant 1$ or $+\infty$. Let us denote by (u_n) the upper numbering of the ramification filtration of Γ:

$$u_n = i_0 + \frac{i_1 - i_0}{p} + \frac{i_2 - i_1}{p^2} + \cdots + \frac{i_n - i_{n-1}}{p^n}.$$

Then, if e is finite, for any large enough integer n we have $u_{n+1} = u_n + e$.

The conditions $p \nmid i_0$ and $i_1 < (p^2 - p + 1)\, i_0$ imply $u_0 = i_0 \geqslant 1$ and $u_1 - u_0 < (p-1)u_0$, therefore $u_1 < pu_0$ and, in particular, γ cannot be of finite order (see [5, Section 5]).

Let us consider the $\mathbb{Z}_p$-extension F/E corresponding to the automorphism γ of $k((t))$ by the equivalence of categories given by the field of norms functor of Fontaine and Wintenberger [7, 8].

Then E is a local field with residue field k The Galois group of F/E is identically ramified to Γ, that is to say that there exists an isomorphism which applies $\mathrm{Gal}(F/E)^u$ on Γ^u for every $u \in [0, +\infty[$. Therefore, because of the well-known results on the ramification of $\mathbb{Z}_p$-extensions, it is clear that E is a local field of characteristic 0 if and only if e is finite and, in this case, e is the absolute ramification index of E [7]. Here, since $u_1 < pu_0$, we have $\mathrm{char}(E) = 0$ (see [5, Section 5]).

Now there exists [7] an equivalence of categories W between the separable extensions of F and the separable extensions of K with separable finite embeddings, trivial on K or F respectively, as arrows and the following property holds: for every separable extension N/F which is Galois over E, we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Gal}(N/F) & \longrightarrow & \mathrm{Gal}(N/E) & \longrightarrow & \mathrm{Gal}(E/F) & \longrightarrow & 1 \\ & & \wr\vert & & \wr\vert & & \wr\vert & & \\ 1 & \longrightarrow & \mathrm{Gal}(W(N)/K) & \longrightarrow & \mathrm{Aut}_\Gamma(W(N))) & \longrightarrow & \Gamma & \longrightarrow & 1, \end{array}$$

where the vertical isomorphisms preserve the ramification filtration (even the median one with a natural definition for the ramification of $\mathrm{Aut}_\Gamma(W(N))$.).

Let E^a (resp. F^a) be the maximal abelian pro-p-extension of E(resp. F); then E^a is Galois over F and we set $A = \mathrm{Gal}(E^a/F)$. Clearly, $\mathrm{Gal}(F^a/F)$ is the local Iwasawa module of the extension F/E and A is the maximal trivial quotient module of $\mathrm{Gal}(F^a/F)$. Therefore by the equivalence of category W, (E^a) is the maximal pro-p-extension of K contained in $K(\gamma)$. Since the p-rank of $\mathrm{Gal}(E^a/E)$ is $[E: \mathbb{Q}_p] + 1 = ef + 1$, the p-rank of $\mathrm{Gal}(K(\gamma)/K)$ is ef.

Now the conditions $p \nmid u_0$ and $u_1 < pu_0$ imply $u_0 \geqslant e/(p-1)$ and $u_1 = u_0 + e$ (see [5] or [1, Prop. 4.3]). Therefore $b = u_1 - u_0 = e$. ∎

THEOREM 2. *Assume that* $p \nmid i_0$ *and* $i_1 < (p^2 - p + 1)\, i_0$. *Then for all* $m > 0$, $i_m = i_0 + bp + bp^2 + \cdots + bp^m$.

Proof. In the case where k is a finite field, the end of the previous proof shows that for all $n \geqslant 0$, $u_n > e/(p-1)$, therefore $u_{n+1} = u_n + e$ (see [5] or [1, Prop. 4.3]) with $e = b$, and the proof is achieved in that case.

In order to conclude in the general case, it suffices to apply Theorem 4 of [3] when $p \neq 2$ or Theorem 2 of [4] when $p = 2$. ∎

The example given in [2] is coherent with our new results.

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