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Common fixed points of four maps in partially ordered metric spaces

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ABSTRACT

In this paper, common fixed points of four mappings satisfying a generalized weak contractive condition in the framework of partially ordered metric space are obtained. We also provide examples of new concepts introduced herein.

1. Introduction and preliminaries

Alber and Guerre-Delabrere [1] introduced the concept of weakly contractive mappings and proved that weakly contractive mapping defined on a Hilbert space is a Picard operator. Rhoades [2] proved that the corresponding result is also valid when Hilbert space is replaced by a complete metric space. Dutta et al. [3] generalized the weak contractive condition and proved a fixed point theorem for a selfmap, which in turn generalizes Theorem 1 in [2] and the corresponding result in [1]. The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity. The area of common fixed point theory, involving four single valued maps, began with the assumption that all of the maps commuted. Introducing weakly commuting maps, Sessa [4] generalized the concept of commuting maps. Then Jungck generalized this idea, first to compatible mappings [5] and then to weakly compatible mappings [6]. There are examples that show that each of these generalizations of commutativity is a proper extension of the previous definition. On the other hand, Beg and Abbas [7] obtained a common fixed point theorem extending weak contractive conditions for two maps. In this direction, Zhang and Song [8] introduced the concept of a generalized φ- weak contraction condition and obtained a common fixed point for two maps. In 2009, Đorić [9] proved a common some fixed point theorem for generalized (ψ, φ)- weakly contractive mappings. Abbas and Đorić [10] obtained a common fixed point theorem for four maps that satisfy a contractive condition which is more general than that given in [8].

Existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [11], and then by Nieto and Lopez [12]. Further results in this direction under weak contraction conditions were proved, e.g. [13,7,14–17,2].

Recently, Radenović and Kadelburg [17] presented a result for generalized weak contractive mappings in partially ordered metric spaces.

The aim of this paper is to initiate the study of common fixed points for four mappings under generalized weak contractions in complete partially ordered metric space. Our result extend, unify and generalize the comparable results in [7,9,3,8].

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Consistent with Altun [18] the following definitions and results will be needed in what follows.

Definition 1.1 (*[18]*). Let $(X, \preceq)$ be a partially ordered set. A pair (f, g) of selfmaps of X is said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$.

Now we give a definition of partially weakly increasing pair of mappings.

Definition 1.2. Let $(X, \preceq)$ be a partially ordered set and f and g be two selfmaps on X. An ordered pair (f, g) is said to be partially weakly increasing if $fx \preceq gfx$ for all $x \in X$.

Note that a pair (f, g) is weakly increasing if and only if ordered pair (f, g) and (g, f) are partially weakly increasing.

Following is an example of an ordered pair (f, g) of selfmaps f and g which is partially weakly increasing but not weakly increasing.

Example 1.3. Let $X = [0, 1]$ be endowed with usual ordering and $f, g : X \to X$ be defined by $fx = x^2$ and $gx = \sqrt{x}$. Clearly, (f, g) is partially weakly increasing. But $gx = \sqrt{x} \npreceq x = fgx$ for $x \in (0, 1)$ implies that (g, f) is not partially weakly increasing.

Definition 1.4. Let $(X, \preceq)$ be a partially ordered set. A mapping f is a called weak annihilator of g if $fgx \preceq x$ for all $x \in X$.

Example 1.5. Let $X = [0, 1]$ be endowed with usual ordering and $f, g : X \to X$ be defined by $fx = x^2$, $gx = x^3$. Obviously, $fgx = x^6 \leq x$ for all $x \in X$. Thus f is a weak annihilator of g.

Definition 1.6. Let $(X, \preceq)$ be a partially ordered set. A mapping f is called dominating if $x \preceq fx$ for each x in X.

Example 1.7. Let $X = [0, 1]$ be endowed with usual ordering and $f : X \to X$ be defined by $fx = x^{\frac{1}{3}}$. Since $x \leq x^{\frac{1}{3}} = fx$ for all $x \in X$. Therefore f is a dominating map.

Example 1.8. Let $X = [0, \infty)$ be endowed with usual ordering and $f : X \to X$ be defined by $fx = \sqrt[n]{x}$ for $x \in [0, 1)$ and $fx = x^n$ for $x \in [1, \infty)$, for any $n \in N$. Clearly, for every x in X we have $x \leq fx$.

Example 1.9. Let $X = [0, 4]$, endowed with usual ordering. Let $f, g : X \to X$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in [0, 1) \\ 1, & \text{if } x \in [1, 3] \\ 3, & \text{if } x \in (3, 4) \\ 4, & \text{if } x = 4, \end{cases} \qquad g(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \in (0, 1] \\ 3, & \text{if } x \in (1, 3] \\ 4, & \text{otherwise}. \end{cases}$$

The pair (f, g) is partially weakly increasing and the dominating map g is a weak annihilator of f.

Theorem 1.10 (*[8]*). *Let (X, d) be a complete metric space, and let $f, g : X \to X$ be two self-mappings such that for all $x, y \in X d(fx, gy) \leq M(x, y) - \varphi(M(x, y))$, where $\varphi : [0, \infty) \to [0, \infty)$ is a lower semicontinuous function with $\varphi(t) > 0$ for $t \in (0, +\infty)$ and $\varphi(0) = 0$,*

$$M(x, y) = \max \left\{ d(x, y), d(fx, x), d(gy, y), \frac{d(x, gy) + d(fx, y)}{2} \right\}.$$

Then there exists a unique point $u \in X$ such that $u = fu = gu$.

Definition 1.11 (*[9]*). The control functions ψ and φ are defined as

(a) $\psi : [0, \infty) \to [0, \infty)$ is a continuous nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$,
(b) $\varphi : [0, \infty) \to [0, \infty)$ is a lower semicontinuous function with $\varphi(t) = 0$ if and only if $t = 0$.

A subset W of a partially ordered set X is said to be well ordered if every two elements of W are comparable.

2. Common fixed point results

We start with the following result.

Theorem 2.1. *Let $(X, \preceq, d)$ be an ordered complete metric space. Let f, g, S and T be selfmaps on X, (T, f) and (S, g) be partially weakly increasing with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$, dominating maps f and g are weak annihilators of T and S, respectively. Suppose that there exist control functions ψ and φ such that for every two comparable elements $x, y \in X$,*

$$\psi(d(fx, gy)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \tag{2.1}$$

is satisfied where

$$M(x,y)=\max\left\{d(Sx,Ty),d(fx,Sx),d(gy,Ty),\frac{d(Sx,gy)+d(fx,Ty)}{2}\right\}.$$

If for a nondecreasing sequence $\{x_n\}$ *with* $x_n \preceq y_n$ *for all* n *and* $y_n \to u$ *implies that* $x_n \preceq u$ *and either*

(a) $\{f,S\}$ *are compatible,* f *or* S *is continuous and* $\{g,T\}$ *are weakly compatible or*
(b) $\{g,T\}$ *are compatible,* g *or* T *is continuous and* $\{f,S\}$ *are weakly compatible,*

then f,g,S *and* T *have a common fixed point. Moreover, the set of common fixed points of* f,g,S *and* T *is well ordered if and only if* f,g,S *and* T *have one and only one common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n-1}=fx_{2n-2}=Tx_{2n-1}$, and $y_{2n}=gx_{2n-1}=Sx_{2n}$. By given assumptions, $x_{2n-2}\preceq fx_{2n-2}=Tx_{2n-1}\preceq fTx_{2n-1}\preceq x_{2n-1}$, and $x_{2n-1}\preceq gx_{2n-1}=Sx_{2n}\preceq Sgx_{2n}\preceq x_{2n}$. Thus, for all $n\ge 1$ we have $x_n\preceq x_{n+1}$. We suppose that $d(y_{2n},y_{2n+1})>0$, for every n. If not then $y_{2n}=y_{2n+1}$, for some n. From (2.1), we obtain

$$\psi(d(y_{2n+1},y_{2n+2}))=\psi(d(fx_{2n},gx_{2n+1}))\le\psi(M(x_{2n},x_{2n+1}))-\varphi(M(x_{2n},x_{2n+1})),\tag{2.2}$$

where

$$\begin{aligned}M(x_{2n},x_{2n+1})&=\max\left\{d(Sx_{2n},Tx_{2n+1}),d(fx_{2n},Sx_{2n}),d(gx_{2n+1},Tx_{2n+1}),\frac{d(Sx_{2n},gx_{2n+1})+d(fx_{2n},Tx_{2n+1})}{2}\right\}\\&=\max\left\{d(y_{2n},y_{2n+1}),d(y_{2n+1},y_{2n}),d(y_{2n+2},y_{2n+1}),\frac{d(y_{2n},y_{2n+2})+d(y_{2n+1},y_{2n+1})}{2}\right\}\\&=\max\left\{0,0,d(y_{2n+2},y_{2n+1}),\frac{d(y_{2n+1},y_{2n+2})}{2}\right\}=d(y_{2n+1},y_{2n+2}).\end{aligned}$$

Hence, $\psi(d(y_{2n+1},y_{2n+2}))\le\psi(d(y_{2n+1},y_{2n+2}))-\varphi(d(y_{2n+1},y_{2n+2}))$, implies that $\varphi(d(y_{2n+1},y_{2n+2}))=0$. As, $\varphi(t)=0$ if and only if $t=0$ $y_{2n+1}=y_{2n+2}$. Following the similar arguments, we obtain $y_{2n+2}=y_{2n+3}$ and so on. Thus $\{y_n\}$ becomes a constant sequence and y_{2n} is the common fixed point of f,g,S and T.

Take, $d(y_{2n},y_{2n+1})>0$ for each n. Since x_{2n} and x_{2n+1} are comparable, from (2.1) we obtain

$$\begin{aligned}\psi(d(y_{2n+2},y_{2n+1}))&=\psi(d(y_{2n+1},y_{2n+2}))=\psi(d(fx_{2n},gx_{2n+1}))\\&\le\psi(M(x_{2n},x_{2n+1}))-\varphi(M(x_{2n},x_{2n+1}))\le\psi(M(x_{2n},x_{2n+1})).\end{aligned}$$

Therefore

$$d(y_{2n+1},y_{2n+2})\le M(x_{2n},x_{2n+1}),\tag{2.3}$$

where

$$\begin{aligned}M(x_{2n},x_{2n+1})&=\max\left\{d(Sx_{2n},Tx_{2n+1}),d(fx_{2n},Sx_{2n}),d(gx_{2n+1},Tx_{2n+1}),\frac{d(Sx_{2n},gx_{2n+1})+d(fx_{2n},Tx_{2n+1})}{2}\right\}\\&=\max\left\{d(y_{2n},y_{2n+1}),d(y_{2n+1},y_{2n}),d(y_{2n+2},y_{2n+1}),\frac{d(y_{2n},y_{2n+2})+d(y_{2n+1},y_{2n+1})}{2}\right\}\\&\le\max\left\{d(y_{2n+1},y_{2n}),d(y_{2n+2},y_{2n+1}),\frac{d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})}{2}\right\}\\&=\max\{d(y_{2n+1},y_{2n}),d(y_{2n+2},y_{2n+1})\}.\end{aligned}$$

If $\max\{d(y_{2n+1},y_{2n}),d(y_{2n+2},y_{2n+1})\}=d(y_{2n+2},y_{2n+1})$, then (2.3) gives that $M(x_{2n},x_{2n+1})=d(y_{2n+2},y_{2n+1})$, and

$$\begin{aligned}\psi(d(y_{2n+2},y_{2n+1}))&\le\psi(M(x_{2n},x_{2n+1}))-\varphi(M(x_{2n},x_{2n+1}))\\&=\psi(d(y_{2n+2},y_{2n+1}))-\varphi(d(y_{2n+2},y_{2n+1})),\end{aligned}$$

gives a contradiction. Hence $d(y_{2n+2},y_{2n+1})\le d(y_{2n+1},y_{2n})$. Moreover $M(x_{2n},x_{2n+1})\le d(y_{2n},y_{2n+1})$. But, since $M(x_{2n},x_{2n+1})\ge\max\{d(y_{2n},y_{2n+1}),d(y_{2n+2},y_{2n+1})\}=M(x_{2n},x_{2n+1})$.

Similarly, $d(y_{2n+3},y_{2n+2})\le d(y_{2n+2},y_{2n+1})$. Thus the sequence $\{d(y_{2n+1},y_{2n})\}$ is nonincreasing and so there exists $\lim_{n\to\infty}d(y_{2n+1},y_{2n})=L\ge 0$. Suppose that $L>0$. Then, $\psi(d(y_{2n+2},y_{2n+1}))\le\psi(M(x_{2n+1},x_{2n}))-\varphi(M(x_{2n+1},y_{2n}))$, and lower semicontinuity of φ gives that

$$\limsup_{n\to\infty}\psi(d(y_{2n+2},y_{2n+1}))\le\limsup_{n\to\infty}\psi(M(x_{2n+1},x_{2n}))-\liminf_{n\to\infty}\varphi(M(x_{2n+1},x_{2n})),$$

which implies that $\psi(L)\le\psi(L)-\varphi(L)$, a contradiction. Therefore $L=0$. So we conclude that

$$\lim_{n\to\infty}d(y_{2n+1},y_{2n})=0.\tag{2.4}$$

Now we show that $\{y_n\}$ is Cauchy sequence. For this it is sufficient to show that $\{y_{2n}\}$ is Cauchy in X. If not, there is $\varepsilon > 0$, and there exist even integers $2n_k$ and $2m_k$ with $2m_k > 2n_k > k$ such that

$$d(y_{2m_k}, y_{2n_k}) \geq \varepsilon, \tag{2.5}$$

and $d(y_{2m_k-2}, y_{2n_k}) < \varepsilon$. Since

$$\varepsilon \leq d(y_{2m_k}, y_{2n_k}) \leq d(y_{2n_k}, y_{2m_k-2}) + d(y_{2m_k-1}, y_{2m_k-2}) + d(y_{2m_k-1}, y_{2m_k})$$

now (2.4) and (2.5) implies that

$$\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \varepsilon. \tag{2.6}$$

Also (2.4) and inequality $d(y_{2m_k}, y_{2n_k}) \leq d(y_{2m_k}, y_{2m_k-1}) + d(y_{2m_k-1}, y_{2n_k})$ gives that $\varepsilon \leq \lim_{k\to\infty} d(y_{2m_k-1}, y_{2n_k})$, while (2.4) and inequality $d(y_{2m_k-1}, y_{2n_k}) \leq d(y_{2m_k-1}, y_{2m_k}) + d(y_{2m_k}, y_{2n_k})$ yields $\lim_{k\to\infty} d(y_{2m_k-1}, y_{2n_k}) \leq \varepsilon$, and hence

$$\lim_{k\to\infty} d(y_{2m_k-1}, y_{2n_k}) = \varepsilon. \tag{2.7}$$

As

$$\begin{aligned} M(x_{2n_k}, x_{2m_k-1}) &= \max\left\{d(Sx_{2n_k}, Tx_{2m_k-1}), d(fx_{2n_k}, Sx_{2n_k}), d(gx_{2m_k-1}, Tx_{2m_k-1}), \frac{d(Sx_{2n_k}, gx_{2m_k-1}) + d(fx_{2n_k}, Tx_{2n_k})}{2}\right\} \\ &= \max\left\{d(y_{2n_k}, y_{2m_k-1}), d(y_{2n_k+1}, y_{2n_k}), d(y_{2m_k}, y_{2m_k-1}), \frac{d(y_{2n_k}, y_{2m_k}) + d(y_{2n_k+1}, y_{2n_k})}{2}\right\}, \end{aligned}$$

thus $\lim_{k\to\infty} M(x_{2n_k}, x_{2m_k-1}) = \max\{\varepsilon, 0, 0, \frac{\varepsilon}{2}\} = \varepsilon$. From (2.1), we obtain

$$\psi(d(y_{2n_k+1}, y_{2m_k})) = \psi(d(fx_{2n_k}, gx_{2m_k-1})) \leq \psi(M(x_{2n_k}, x_{2m_k-1})) - \varphi(M(x_{2n_k}, x_{2m_k-1})).$$

Taking limit as $k \to \infty$ implies that $\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon)$, which is a contradiction as $\varepsilon > 0$.

It follows that $\{y_{2n}\}$ is a Cauchy sequence and since X is complete, there exists a point z in X, such that y_{2n} converges to z. Therefore,

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} fx_{2n} = z, \quad \text{and} \quad \lim_{n\to\infty} y_{2n+2} = \lim_{n\to\infty} Sx_{2n+2} = \lim_{n\to\infty} gx_{2n+1} = z.$$

Assume that S is continuous. Since $\{f, S\}$ are compatible, we have

$$\lim_{n\to\infty} fSx_{2n+2} = \lim_{n\to\infty} Sfx_{2n+2} = Sz.$$

Also, $x_{2n+1} \preceq gx_{2n+1} = Sx_{2n+2}$. Now

$$\psi(d(fSx_{2n+2}, gx_{2n+1})) \leq \psi(M(Sx_{2n+2}, x_{2n+1})) - \varphi(M(Sx_{2n+2}, x_{2n+1})), \tag{2.8}$$

where

$$\begin{aligned} M(Sx_{2n+2}, x_{2n+1}) = \max\Big\{&d(SSx_{2n+2}, Tx_{2n+1}), d(fSx_{2n+2}, SSx_{2n+2}), \\ &d(gx_{2n+1}, Tx_{2n+1}), \frac{d(SSx_{2n+2}, gx_{2n+1}) + d(fSx_{2n+2}, Tx_{2n+1})}{2}\Big\}. \end{aligned}$$

On taking limit as $n \to \infty$, we obtain $\psi(d(Sz, z)) \leq \psi(d(Sz, z)) - \varphi(d(Sz, z))$, and $Sz = z$.

Now, $x_{2n+1} \preceq gx_{2n+1}$ and $gx_{2n+1} \to z$ as $n \to \infty$, $x_{2n+1} \preceq z$ and (2.1) becomes $\psi(d(fz, gx_{2n+1})) \leq \psi(M(z, x_{2n+1})) - \varphi(M(z, x_{2n+1}))$, where

$$M(z, x_{2n+1}) = \max\left\{d(Sz, Tx_{2n+1}), d(fz, Sz), d(gx_{2n+1}, Tx_{2n+1}), \frac{d(Sz, gx_{2n+1}) + d(fz, Tx_{2n+1})}{2}\right\}.$$

On taking limit as $n \to \infty$, we have $\psi(d(fz, z)) \leq \psi(d(fz, z)) - \varphi(d(fz, z))$, and $fz = z$.

Since $f(X) \subseteq T(X)$, there exists a point $w \in X$ such that $fz = Tw$. Suppose that $gw \neq Tw$. Since $z \preceq fz = Tw \preceq fTw \preceq w$ implies $z \preceq w$. From (2.1), we obtain

$$\psi(d(Tw, gw)) = \psi(d(fz, gw)) \leq \psi(M(z, w)) - \varphi(M(z, w)), \tag{2.9}$$

where

$$\begin{aligned} M(z, w) &= \max\left\{d(Sz, Tw), d(fz, Sz), d(gw, Tw), \frac{d(Sz, gw) + d(fz, Tw)}{2}\right\} \\ &= \max\left\{d(z, z), d(z, z), d(gw, Tw), \frac{d(Tw, gw) + d(Tw, Tw)}{2}\right\} = d(Tw, gw). \end{aligned}$$

Now (2.9) becomes $\psi(d(Tw, gw)) \leq \psi(d(Tw, gw)) - \varphi(d(Tw, gw))$, a contradiction. Hence, $Tw = gw$. Since g and T are weakly compatible, $gz = gfz = gTw = Tgw = Tfz = Tz$. Thus z is a coincidence point of g and T.

Now, since $x_{2n} \preceq fx_{2n}$ and $fx_{2n} \to z$ as $n \to \infty$, implies that $x_{2n} \preceq z$, from (2.1) $\psi(d(fx_{2n}, gz)) \leq \psi(M(x_{2n}, z)) - \varphi(M(x_{2n}, z))$, where

$$M(x_{2n}, z) = \max\left\{d(Sx_{2n}, Tz), d(fx_{2n}, Sx_{2n}), d(gz, Tz), \frac{d(Sx_{2n}, gz) + d(fx_{2n}, Tz)}{2}\right\}$$
$$= \max\left\{d(z, gz), d(z, z), d(gz, gz), \frac{d(z, gz) + d(z, gz)}{2}\right\} = d(z, gz).$$

On taking limit as $n \to \infty$, we have $\psi(d(z, gz)) \leq \psi(d(z, gz)) - \varphi(d(z, gz))$, and $z = gz$. Therefore $fz = gz = Sz = Tz = z$. The proof is similar when f is continuous.

Similarly, the result follows when (b) holds.

Now suppose that the set of common fixed points of f, g, S and T is well ordered. We claim that common fixed point of f, g, S and T is unique. Assume on contrary that, $fu = gu = Su = Tu = u$ and $fv = gv = Sv = Tv = v$ but $u \neq v$. By supposition, we can replace x by u and y by v in (2.1) to obtain

$$\psi(d(u, v)) = \psi(d(fu, gv)) \leq \psi(M(u, v)) - \varphi(M(u, v)),$$

where

$$M(u, v) = \max\left\{d(Su, Tv), d(fu, Su), d(gv, Tv), \frac{d(Su, gv) + d(fu, Tv)}{2}\right\}$$
$$= \max\left\{d(u, v), 0, 0, \frac{d(u, v) + d(u, v)}{2}\right\} = d(u, v),$$

and $\psi(d(u, v)) \leq \psi(d(u, v)) - \varphi(d(u, v))$, a contradiction. Hence $u = v$. Conversely, if f, g, S and T have only one common fixed point then the set of common fixed point of f, g, S and T being singleton is well ordered. □

Example 2.2. Consider $X = [0, 1] \cup \{2, 3, 4, \ldots\}$ with usual ordering and

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y \in [0, 1], \text{ and } x \neq y \\ x + y & \text{if at least one of } x \text{ or } y \notin [0, 1] \text{ and } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

Then $(X, \preceq, d)$ is a complete partially ordered metric space [19]. Let $\psi, \varphi : [0, \infty) \to [0, \infty)$ be defined by $\psi(x) = \begin{cases} 2x \text{ if } 0 \leq x \leq \frac{1}{2} \\ 1 \text{ if } x \in (\frac{1}{2}, 1] \\ x \text{ otherwise} \end{cases}$ and $\varphi(x) = \begin{cases} \frac{1}{4} - x^2 \text{ if } 0 \leq x < \frac{1}{2} \\ 0 \text{ otherwise} \end{cases}$ and selfmaps f, g, S and T on X be given by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{2}, & \text{if } x \in \left(0, \frac{1}{2}\right] \\ 1, & \text{if } x \in \left(\frac{1}{2}, 1\right] \\ x, & \text{if } x \in \{2, 3, 4, \ldots\}, \end{cases} \qquad g(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{2}, & \text{if } x \in \left(0, \frac{1}{2}\right] \\ x, & \text{if } x \in \left(\frac{1}{2}, 1\right] \cup \{2, 3, 4, \ldots\}, \end{cases}$$

$$T(x) = \begin{cases} 0, & \text{if } x \leq \frac{1}{2} \\ \frac{1}{2}, & \text{if } x \in \left(\frac{1}{2}, 1\right] \\ x - 1, & \text{if } x \in \{2, 3, 4, \ldots\}, \end{cases} \qquad S(x) = \begin{cases} 0, & \text{if } x \leq \frac{1}{2} \\ 2x - 1, & \text{if } x \in \left(\frac{1}{2}, 1\right] \\ x, & \text{if } x \in \{2, 3, 4, \ldots\}. \end{cases}$$

Note that, f, g, S and T satisfy all the conditions given in Theorem 2.1. Moreover, 0 is a unique common fixed point of f, g, S and T.

Corollary 2.3. *Let $(X, \preceq, d)$ be an ordered complete metric space. Let f, S and T be selfmaps on X, (T, f) and (S, f) be partially weakly increasing with $f(X) \subseteq T(X)$, $f(X) \subseteq S(X)$, and dominating map f is weak annihilator of T and S. Suppose that there exists control functions ψ and φ such that for every two comparable elements $x, y \in X$,*

$$\psi(d(fx, fy)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \tag{2.10}$$

where

$$M(x, y) = \max\left\{d(Sx, Ty), d(fx, Sx), d(fy, Ty), \frac{d(Sx, fy) + d(fx, Ty)}{2}\right\}$$

is satisfied. If for a nondecreasing sequence $\{x_n\}$ *with* $x_n \preceq y_n$ *for all n and* $y_n \to u$ *implies* $x_n \preceq u$ *and either*

(a) $\{f, S\}$ *are compatible,* f *or* S *is continuous and* $\{f, T\}$ *are weakly compatible or*
(b) $\{f, T\}$ *are compatible,* f *or* T *is continuous and* $\{f, S\}$ *are weakly compatible,*

then f, S *and* T *have a common fixed point. Moreover, the set of common fixed points of* f, S *and* T *is well ordered if and only if* f, S *and* T *have one and only one common fixed point.*

Corollary 2.3 *is a special case of* Theorem 2.1, *obtained by setting* $f = g$.

Corollary 2.4. *Let* $(X, \preceq, d)$ *be an ordered complete metric space. Let* f, g *and* T *be selfmaps on* X, (T, f) *and* (T, g) *be partially weakly increasing with* $f(X) \subseteq T(X)$, $g(X) \subseteq T(X)$, *and dominating maps* f *and* g *are weak annihilators of* T. *Suppose that there exist control functions* ψ *and* φ *such that for every two comparable elements* $x, y \in X$,

$$\psi(d(fx, gy)) \le \psi(M_1(x, y)) - \varphi(M_1(x, y)), \tag{2.11}$$

$$\text{where } M_1(x, y) = \max\left\{d(Tx, Ty), d(fx, Tx), d(gy, Ty), \frac{d(Tx, gy) + d(fx, Ty)}{2}\right\}$$

is satisfied. If for a nondecreasing sequence $\{x_n\}$ *with* $x_n \preceq y_n$ *for all n and* $y_n \to u$ *implies* $x_n \preceq u$ *and either*

(a) $\{f, T\}$ *are compatible,* f *or* T *is continuous and* $\{g, T\}$ *are weakly compatible or*
(b) $\{g, T\}$ *are compatible,* g *or* T *is continuous and* $\{f, T\}$ *are weakly compatible,*

then f, g *and* T *have a common fixed point. Moreover, the set of common fixed points of* f, g *and* T *is well ordered if and only if* f, g *and* T *have one and only one common fixed point.*

Corollary 2.5. *Let* $(X, \preceq, d)$ *be an ordered complete metric space. Let* f *and* T *be selfmaps on* X, (T, f) *be partially weakly increasing with* $f(X) \subseteq T(X)$, *dominating map* f *is weak annihilator of* T. *Suppose that there exist control functions* ψ *and* φ *such that for every two comparable elements* $x, y \in X$,

$$\psi(d(fx, gy)) \le \psi(M(x, y)) - \varphi(M(x, y)), \tag{2.12}$$

$$\text{where } M(x, y) = \max\left\{d(Tx, Ty), d(fx, Tx), d(fy, Ty), \frac{d(Tx, fy) + d(fx, Ty)}{2}\right\}$$

is satisfied. If for a nondecreasing sequence $\{x_n\}$ *with* $x_n \preceq y_n$ *for all n and* $y_n \to u$ *implies* $x_n \preceq u$. *If* $\{f, T\}$ *are compatible,* f *or* T *is continuous and* $\{f, T\}$ *are weakly compatible, then* f *and* T *have a common fixed point. Moreover, the set of common fixed points of* f *and* T *is well ordered if and only if* f *and* T *have one and only one common fixed point.*

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