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Bordered Riemann surfaces in $\mathbb{C}^2$

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To Edgar Lee Stout on the occasion of his 70th birthday

Abstract

We prove that the interior of any compact complex curve with smooth boundary in $\mathbb{C}^2$ admits a proper holomorphic embedding into $\mathbb{C}^2$. In particular, if D is a bordered Riemann surface whose closure admits a holomorphic embedding into $\mathbb{C}^2$, then D admits a proper holomorphic embedding into $\mathbb{C}^2$.

Résumé

On montre que l'intérieur d'une courbe complexe compacte avec bord lisse dans $\mathbb{C}^2$ admet un plongement holomorphe propre dans $\mathbb{C}^2$. En particulier, si D est une surface de Riemann avec bord dont la fermeture admet un plongement holomorphe dans $\mathbb{C}^2$, alors D admet un plongement holomorphe propre dans $\mathbb{C}^2$.

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1. Introduction

It is an old problem whether every open Riemann surface is biholomorphically equivalent to a topologically closed smooth complex curve in $\mathbb{C}^2$. Equivalently, *does every open Riemann surface embed properly holomorphically in* $\mathbb{C}^2$? (See Bell and Narasimhan [7, Conjecture 3.7, p. 20].) Such D always embeds in $\mathbb{C}^3$ and immerses in $\mathbb{C}^2$ [11,35,37].

A *bordered Riemann surface* is a compact one-dimensional complex manifold, $\overline{D}$, not necessarily connected, with smooth boundary bD consisting of finitely many closed Jordan curves. The embedding problem naturally decouples in the following two problems:

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(a) find a (non-proper) holomorphic embedding $f:\overline{D}\hookrightarrow\mathbb{C}^2$;
(b) push the boundary of the compact complex curve $\overline{\Sigma}=f(\overline{D})\subset\mathbb{C}^2$ to infinity without introducing any double points.

In this paper we give a complete solution to the second problem, also for curves with interior singularities. The following is our main result.

Theorem 1.1. *If $\overline{\Sigma}$ is a (possibly reducible) compact complex curve in $\mathbb{C}^2$ with boundary $b\Sigma$ of class $\mathcal{C}^r$ for some $r>1$, then the inclusion map $\iota:\Sigma=\overline{\Sigma}\backslash b\Sigma\hookrightarrow\mathbb{C}^2$ can be approximated, uniformly on compacts in Σ, by proper holomorphic embeddings $\Sigma\hookrightarrow\mathbb{C}^2$. In particular, a smoothly bounded relatively compact domain Σ in an affine complex curve $A\subset\mathbb{C}^2$ admits a proper holomorphic embedding in $\mathbb{C}^2$.*

The precise assumption on $\overline{\Sigma}$ is that locally near each boundary point $p\in b\Sigma$ it is a one-dimensional complex manifold with boundary of class $\mathcal{C}^r$, while the interior Σ is a pure one-dimensional analytic subvariety with at most finitely many singularities.

Theorem 1.1 is proved in Section 5. It includes the following result which gives an affirmative answer to the problem posed in [13, p. 686] and which contains all known results on embedding bordered Riemann surfaces properly holomorphically in $\mathbb{C}^2$. For Riemann surfaces with punctures see also Theorem 5.2 and Corollary 5.3 below.

Corollary 1.2. *Assume that $\overline{D}$ is a bordered Riemann surface with $\mathcal{C}^r$ boundary for some $r>1$ and that $f:\overline{D}\hookrightarrow\mathbb{C}^2$ is a $\mathcal{C}^1$ embedding which is holomorphic in D. Then f can be approximated, uniformly on compacts in D, by proper holomorphic embeddings $D\hookrightarrow\mathbb{C}^2$.*

Proof. A bordered Riemann surface $\overline{D}$ with $\mathcal{C}^r$ boundary is biholomorphic to a relatively compact smoothly bounded domain D' in an open Riemann surface R. Furthermore, if r is a noninteger then any biholomorphic map $D\to D'$ extends to a $\mathcal{C}^r$ diffeomorphism $\overline{D}\to\overline{D}'$. (See the discussion and the references in Section 6.) Hence we may assume that D is a relatively compact domain with smooth boundary in a Riemann surface R.

By Mergelyan's theorem we can approximate f in the $\mathcal{C}^1$ topology on $\overline{D}$ by a holomorphic map $\widetilde{f}:U\to\mathbb{C}^2$ from an open set $U\subset R$ containing $\overline{D}$. If the approximation is sufficiently close and U is chosen sufficiently small, then $\widetilde{f}$ is a holomorphic embedding of U onto a locally closed embedded complex curve without singularities $A=\widetilde{f}(U)$ in $\mathbb{C}^2$. It remains to apply Theorem 1.1 to the complex curve with smooth boundary $\overline{\Sigma}=\widetilde{f}(\overline{D})$. □

Corollary 1.2, together with the main result of [31], implies the following result on embeddings with interpolation on a discrete set.

Corollary 1.3. *Let D be a bordered Riemann surface satisfying the hypothesis of Corollary 1.2. Given discrete sequences of points $\{a_j\}\subset D$ and $\{b_j\}\subset\mathbb{C}^2$ without repetitions, there is a proper holomorphic embedding $\varphi:D\hookrightarrow\mathbb{C}^2$ such that $\varphi(a_j)=b_j$ for $j=1,2,\ldots$.*

In the remainder of this introduction we summarize the main earlier results on embedding Riemann surfaces in $\mathbb{C}^2$, and we give a few examples.

An open Riemann surface is the same thing as a one-dimensional Stein manifold. By the classical results (see [11,35,37]) every open Riemann surface embeds properly holomorphically in $\mathbb{C}^3$, and it immerses properly holomorphically in $\mathbb{C}^2$. According to Eliashberg and Gromov [16] and Schürmann [39], a Stein manifold of dimension $n>1$ admits a proper holomorphic embedding in $\mathbb{C}^N$ with $N=[\frac{3n}{2}]+1$. For $n=1$ this would predict that each open Riemann surface embeds properly into $\mathbb{C}^2$, but the proof in the mentioned papers breaks down in this lowest dimensional case. The main problem is that self-intersections (double points) of an immersed complex curve in $\mathbb{C}^2$ are stable under deformations.

The oldest results for embedding Riemann surfaces in $\mathbb{C}^2$ are due to Kasahara and Nishino [42] (for the disc $\mathbb{D}=\{z\in\mathbb{C}\colon |z|<1\}$), Laufer [32] (for annuli $A=\{r_1<|z|<r_2\}$), and Alexander [5] (for $\mathbb{D}$ and $\mathbb{D}\backslash\{0\}$); these were essentially the only known results at the time of the survey by Bell and Narasimhan [7]. In 1995, J. Globevnik and

B. Stensønes proved that every finitely connected planar domain $D \subset \mathbb{C}$ without isolated boundary points embeds properly holomorphically into $\mathbb{C}^2$ (see [24] and also [13,14]).

Considerably more general results were obtained by the second author in recent papers [45–47]. In [46], Corollary 1.2 was proved under the additional assumption that each boundary curve C_j of the image $\overline{\Sigma} = f(\overline{D})$ contains an *exposed point* $p_j = (p_j^1, p_j^2)$, meaning that the vertical line $\{p_j^1\} \times \mathbb{C}$ intersects the curve $\overline{\Sigma}$ only at p_j and the intersection is transverse. (See Definition 4.1 and Theorem 5.1 below.) By applying a shear $g(z, w) = (z, w + h(z))$, where h is a suitably chosen rational function with simple poles at the points p_j^1, the exposed points p_j are blown off to infinity and we obtain an unbounded embedded complex curve $\overline{X} = g(\overline{\Sigma}\backslash\{p_1, \dots, p_m\}) \subset \mathbb{C}^2$ whose boundary bX consists of the arcs $\lambda_j = g(C_j\backslash\{p_j\})$ stretching to infinity. By a sequence of holomorphic automorphisms Φ_n of $\mathbb{C}^2$ we then push bX to infinity, insuring that the sequence converges to a Fatou–Bieberbach map $\Phi = \lim_{n\to\infty} \Phi_n : \Omega \to \mathbb{C}^2$ such that $X \subset \Omega$ and $bX \subset b\Omega$. The restriction $\Phi|_X : X \hookrightarrow \mathbb{C}^2$ is then a proper holomorphic embedding of X (that is biholomorphic to D) into $\mathbb{C}^2$. The relevant results on automorphisms of $\mathbb{C}^2$ come from the papers [6,12,21]. A list of Riemann surfaces that can be embedded in $\mathbb{C}^2$ by Wold's method can be found in [31, Theorem 1].

We prove Theorem 1.1 in Section 5 by first modifying $\overline{\Sigma}$ to a biholomorphically equivalent complex curve which contains an exposed point in each boundary component (see Theorem 4.2); this is the main new technical result of this paper. The proof is then completed by Wold's method as in [46] (see Theorem 5.1).

A main difference between our construction in this paper and those of Globevnik and Stensønes [24] (for planar domains) and Wold [47] (for domains in tori) is that *the conformal structure on D does not change during the construction*, and hence we do not need the uniformization theory in order to complete the proof.

In Section 6 we sketch another possible proof of Corollary 1.2 by using Teichmüller spaces of bordered Riemann surfaces.

Example 1.4. Let R be a smooth closed algebraic curve in the projective plane $\mathbb{P}^2$. If $U_1, \dots, U_k$ are pairwise disjoint smoothly bounded discs in R whose union contains the intersection of R with a projective line $\mathbb{P}^1 \subset \mathbb{P}^2$, then the bordered Riemann surface $D = R\backslash\bigcup_{i=1}^k \overline{U}_i \subset \mathbb{P}^2\backslash\mathbb{P}^1 = \mathbb{C}^2$ embeds properly holomorphically into $\mathbb{C}^2$ according to Corollary 1.2. In particular, since every one-dimensional complex torus embeds as a smooth cubic curve in $\mathbb{P}^2$, with a given point going to the line at infinity, we see that any finitely connected subset without isolated boundary points in a torus embeds properly into $\mathbb{C}^2$. (This is the main theorem in [47].)

Example 1.5. A compact Riemann surface R is called *hyperelliptic* if it admits a meromorphic function of degree two, i.e., a two-sheeted branched holomorphic covering $R \to \mathbb{P}^1$ Such R is the normalization of a complex curve in $\mathbb{P}^2$ given by $w^2 = \prod_{j=1}^k (z - z_j)$ for some choice of points $z_1, \dots, z_k \in \mathbb{C}$ (see [17]). A bordered Riemann surface D is hyperelliptic if its double is hyperelliptic. (The double of D is obtained by gluing two copies of $\overline{D}$, the second one with the conjugate conformal structure, along their boundaries; see [41, p. 217].) Such $\overline{D}$ admits a holomorphic embedding into the closed bidisc $\overline{\mathbb{D}}^2 \subset \mathbb{C}^2$ by a pair of inner functions mapping bD to the torus $(b\,\mathbb{D})^2$ (see Rudin [38] and Gouma [22]). Hence Corollary 1.2 implies that *every hyperelliptic bordered Riemann surface D, and also every smoothly bounded domain in such D, admits a proper holomorphic embedding in* $\mathbb{C}^2$. The first statement is known [13, Corollary 1.3], but the second one is new.

For the general theory of Riemann surfaces see [4,17,26,41], and for the theory of Stein manifolds see [27].

2. Construction of a conformal diffeomorphism

The main result of this section is Theorem 2.3 which is one of our main tools in the proof of Theorem 1.1.

We begin with a lemma on conformal mappings. Denote by $\mathbb{D}$ the open unit disc in $\mathbb{C}$, and by $r\mathbb{D}$ the disc of radius $r > 0$.

Lemma 2.1. *Assume that R is a connected open Riemann surface, $G \Subset R$ is an open simply connected domain with smooth boundary, $V' \Subset V'' \subset R$ are small neighborhoods of a boundary point $a \in bG$, b is a point in $R\backslash\overline{G}$, γ is a smooth Jordan arc with endpoints a and b such that $\gamma \cap \overline{G} = \{a\}$ and the tangent lines to γ and bG at the point a are transverse, and V is a neighborhood of γ. Then there exists a sequence of smooth diffeomorphisms $\psi_n : \overline{G} \to \psi_n(\overline{G}) \subset R$ that are conformal on G and satisfy the following properties for $n = 1, 2, \dots$:*

(i) $\psi_n \to \mathrm{id}$ *locally uniformly on* G *as* $n \to \infty$,
(ii) $\psi_n(a) = b$,
(iii) $\psi_n(\overline{V'} \cap \overline{G}) \subset V'' \cup V$.

Proof. Since $\overline{G} \cup \gamma$ admits a simply connected neighborhood in R, and since we are going to construct maps with images near $\overline{G} \cup \gamma$, we might as well assume that we are working in the complex plane, that a is the origin, and that the strictly positive real axis lies in the complement of $\overline{G}$ near the origin.

For each $n \in \mathbb{N}$ let l_n denote the line segment between 0 and $\frac{1}{n}$ in $\mathbb{R} \subset \mathbb{C}$. Let $\widetilde{V}$ be a neighborhood of the origin with $V' \Subset \widetilde{V} \Subset V''$. By approximation there are neighborhoods U_n of $\overline{G} \cup l_n$ and holomorphic injections $f_n : U_n \to \mathbb{C}$ such that the following hold for all $n \in \mathbb{N}$:

(1) $f_n \to \mathrm{id}$ uniformly on $\overline{G}$ as $n \to \infty$,
(2) $f_n(l_n)$ approximates γ, with $f_n(\frac{1}{n}) = b$ and $f_n(l_n) \subset V$,
(3) $f_n(\overline{G} \cap \widetilde{V}) \subset V''$.

Of course property (3) is a consequence of (1) for large enough n. For the details of this approximation argument see e.g. [43] or [30, Theorem 3.2] (for $\mathcal{C}^0$ approximation), and [20, Theorem 3.2] for the general case with smooth approximation on l_n.

For small positive numbers ϵ we let Ω_ϵ denote domains obtained by adding an ϵ-strip around l_n to G, containing the point $\frac{1}{n}$ in the boundary $b\,\Omega_\epsilon$. We smooth corners to obtain smoothly bounded domains. We let R_ϵ denote the part of Ω_ϵ that is not in G.

Choose a sequence $\epsilon_n \searrow 0$ such that $\overline{\Omega}_{\epsilon_n} \subset U_n$ for each $n \in \mathbb{N}$. Write $\Omega_n = \Omega_{\epsilon_n}$ and $R_n = R_{\epsilon_n}$. By choosing the ϵ_n's small enough we get that

(4) $f_n(R_n) \subset V$ for each $n \in \mathbb{N}$.

Next we choose a point $p \in G$ and a sequence of conformal maps $g_n : G \to \Omega_n$ such that $g_n(p) = p$ and $g_n'(p) > 0$ for $n = 1, 2, \ldots$. Since our domains are smoothly bounded, the map g_n extends to a smooth diffeomorphism of $\overline{G}$ onto $\overline{\Omega}_n$. Furthermore, since the domains $\overline{\Omega}_n$ converge to $\overline{G}$ as $n \to \infty$, we conclude by Rado's theorem (see e.g. [36, Corollary 2.4, p. 22] or [25, Theorem 2, p. 59]) that

(5) $g_n \to \mathrm{id}$ uniformly on $\overline{G}$ as $n \to \infty$.

Hence for n large enough we have that $g_n(\overline{V'} \cap \overline{G}) \subset (\widetilde{V} \cap \overline{G}) \cup R_n$. Combining this with (3) and (4) we see that $f_n \circ g_n(\overline{V'} \cap \overline{G}) \subset V'' \cup V$ if n is large enough. Hence, by defining $\psi_n := f_n \circ g_n$ we get property (iii) for all large n, and we clearly also get property (i).

To see that property (ii) holds, let $a_n \in b\,G$ denote the point that g_n sends to $\frac{1}{n} \in b\Omega_n$. By (5) the sequence a_n has to converge to the origin, and so there is a sequence of conformal automorphisms φ_n of G fixing the point p, sending the origin to a_n, with $\varphi_n \to \mathrm{id}$ uniformly on $\overline{G}$. Replacing the maps g_n by $g_n \circ \varphi_n$ in the above argument also gives (ii). □

In the remainder of this section, R denotes a Riemann surface without boundary and D is a relatively compact, smoothly bounded domain with nonempty boundary in R, not necessarily connected. The following lemma provides the main inductive step in the proof of Theorem 4.2.

Lemma 2.2. *Given pairwise distinct points* $a_1, a_2, \ldots, a_k \in \overline{D}$ *with* $a_1 \in bD$, *a neighborhood* $U \subset R$ *of* a_1, *a point* $b \in R\backslash\overline{D}$ *in the same connected component of* $R\backslash D$ *as* a_1, *and a positive integer* $N \in \mathbb{N}$, *there is a smooth diffeomorphism* $\phi : \overline{D} \to \overline{D}' \subset R$ *satisfying the following*:

(1) $\phi : D \to D'$ *is biholomorphic,*
(2) $\phi(a_1) = b$,
(3) ϕ *is tangent to the identity map to order* N *at each of the points* $a_2, \ldots, a_k$,
(4) ϕ *is as close as desired to the identity map on* $\overline{D}\backslash U$ *in the smooth topology on the space of maps.*

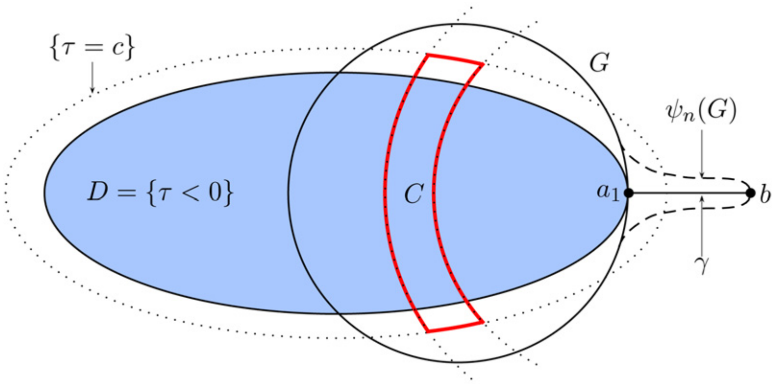

Fig. 1. The domains D and G.

Proof. We may assume that $N > 2$. Choose a smooth embedded Jordan arc $\gamma \subset R$ with the endpoints a_1 and b such that $\gamma \cap \overline{D} = \{a_1\}$, and the tangent line to γ at a_1 intersects the tangent line to bD at a_1 transversely. Then γ has an open, connected and simply connected neighborhood $W \subset R$ that is conformally equivalent to a bounded domain (a disc) in $\mathbb{C}$. Let z denote the corresponding holomorphic coordinate on W, chosen such that $z(a_1) = 0$. By shrinking the neighborhood U of the point a_1 we may assume that $\overline{U} \subset W$, that $\overline{U}$ does not contain any of the points $a_2, \ldots, a_k$, and that $z(U) = r\mathbb{D} \subset \mathbb{C}$ for some $r > 0$. Choose a number $r' \in (0, r)$ and let $U' \subset U$ be chosen such that $z(U') = r'\mathbb{D}$.

Choose a connected and simply connected domain $G \subset W$ with smooth boundary, with a defining function ρ such that $G = \{\rho < 0\}$ and $d\rho \neq 0$ on bG, satisfying the following properties (see Fig. 1):

(i) $\overline{D} \cap U \subset G \cup \{a_1\}$,
(ii) $-\rho(z) \geqslant \text{const}\cdot \operatorname{dist}(z, a_1)^2$ for points $z \in bD$ close to a_1, and
(iii) $\gamma \cap \overline{G} = \{a_1\}$.

Property (iii) can be achieved since the arc γ is transverse to bD at a_1.

Choose a smooth defining function τ for the domain D such that $D = \{\tau < 0\}$ and $d\tau \neq 0$ on $bD = \{\tau = 0\}$. Choose a small number $c > 0$ and let

$$A = \{\tau \leqslant c\} \backslash U', \qquad B = \{\tau \leqslant c\} \cap \overline{U}, \qquad C = \{\tau \leqslant c\} \cap (\overline{U} \backslash U').$$

Let be $c > 0$ small enough we insure that C is a compact set contained in G (see Fig. 1), and we have:

$$A \cup B = \{\tau \leqslant c\}, \qquad A \cap B = C.$$

On Fig. 1, the set C is bounded by the two circular arcs (left and right) and by the two arcs in the larger dotted ellipse representing the level set $\{\tau = c\}$. The set A is the part of the filled dotted ellipse lying on the left-hand side of the right boundary arc of C, and B is the part of the filled dotted ellipse on the right-hand side of the left boundary arc of C.

Choose small open neighborhoods $V' \Subset V''$ of the point a such that $\overline{V}''$ is contained in the interior of the set $B \backslash A$, and choose a small neighborhood V of γ such that $\overline{V} \cap \overline{(A \backslash B)} \cap \overline{D} = \emptyset$. Let $\psi_n : \overline{G} \to \psi_n(\overline{G})$ be a sequence of conformal maps furnished by Lemma 2.1, satisfying the properties of that lemma with respect to the sets V, V', V''. Recall that the compact set C is contained in G. Choose an open set $C' \Subset G$ containing C. On C' we write:

$$\mathrm{id} = \psi_n \circ \gamma_n, \quad \gamma_n = \psi_n^{-1}.$$

As $n \to +\infty$, ψ_n converges to the identity uniformly on C', and hence also in the smooth topology (by the Cauchy estimates). The same is then true for its inverse γ_n on a slightly smaller neighborhood of C.

We are now in position to apply [19, Theorem 4.1] to the map γ_n. For every sufficiently large $n \in \mathbb{N}$, the cited theorem furnishes a decomposition,

$$\gamma_n \circ \alpha_n = \beta_n \quad \text{near } C,$$

where α_n is a small holomorphic perturbation of the identity map on a fixed neighborhood of A (independent of n) that is tangent to the identity to order N at each of the points $a_2, \ldots, a_k$, and β_n is a small holomorphic perturbation of the identity map on a neighborhood of B that is tangent to the identity to order N at the point a_1. The closeness of α_n (resp. of β_n) to the identity in any $\mathcal{C}^r$ norm on A (resp. on B) can be estimated by the closeness of ψ_n to the identity on C'. (This Cartan-type decomposition lemma for biholomorphic maps close to the identity is one of the most essential results used in our construction. Its proof in [19] applies to Cartan pairs in an arbitrary Stein manifold.)

By combining the above two displays we obtain:

$$\alpha_n = \psi_n \circ \beta_n \quad \text{near } C.$$

If the approximations are sufficiently close (which holds for n large enough) then the two sides, restricted to $A \cap \overline{D}$ (resp. to $B \cap \overline{D}$), define a diffeomorphism $\phi_n : \overline{D} \to \phi_n(\overline{D}) \subset R$ that is holomorphic in D and such that

- $\phi_n(a_1) = b$,
- ϕ_n is tangent to the identity map to order N at each of the points $a_2, \ldots, a_k$,
- ϕ_n converges to the identity map uniformly on $\overline{D} \backslash U$ as $n \to +\infty$.

Indeed, both sides α_n and $\psi_n \circ \beta_n$ satisfy the stated properties on their respective domain. For α_n this is clear from the construction. For β_n we need a more precise argument to see that it maps $B \cap \overline{D}$ into $G \cup \{a\}$ for sufficiently large $n \in \mathbb{N}$. By the construction, its Taylor expansion in a local holomorphic coordinate z near a_1, with $z(a_1) = 0$, equals:

$$\beta_n(z) = z + M_n z^N + O\big(z^{N+1}\big).$$

The size of the constant M_n, and of the remainder term, can be estimated (using the Cauchy estimates) by $\mathrm{dist}(\beta_n, \mathrm{id})$ on B, and hence by $\mathrm{dist}(\psi_n, \mathrm{id})$ on the set C'. Since G osculates D from the outside to the second order at the point a_1 (see property (ii) above), it follows that for a sufficiently small neighborhood U_1 of the point a_1 and for all large enough $n \in \mathbb{N}$ we have:

$$\beta_n(\overline{D} \cap U_1) \subset \big(G \cup \{a_1\}\big) \cap V'. \tag{2.1}$$

On the complement $(B \cap \overline{D}) \backslash U_1$, β_n is close to the identity for large n, and hence it maps this set into a fixed compact set in G. Thus the composition $\psi_n \circ \beta_n$ is well defined on $B \cap \overline{D}$ and it satisfies the stated properties.

It is also easily seen that ϕ_n is injective if n is large enough. Indeed, each of the two expressions defining ϕ_n on $A \cap \overline{D}$ (resp. on $B \cap \overline{D}$) is injective by the construction, and hence it suffices to verify that no point from $(A \backslash B) \cap \overline{D}$ can get identified with a point from $(B \backslash A) \cap \overline{D}$ under ϕ_n. By the construction, the points from the first set remain nearby since α_n is close to the identity. Consider now points $x \in (B \backslash A) \cap \overline{D}$. If $x \in U_1$ then $\beta_n(x) \in (G \cup \{a_1\}) \cap V'$ by (2.1), and hence $\psi_n \circ \beta_n(x) \in V'' \cup V$ by property (iii) in Lemma 2.1. Since the set $V'' \cup V$ is at a positive distance from $(A \backslash B) \cap \overline{D}$, we see that $\psi_n \circ \beta_n(x)$ cannot coincide with $\alpha_n(x')$ for any point $x' \in (A \backslash B) \cap \overline{D}$ provided that n is large enough. The remaining set $((B \backslash A) \cap \overline{D}) \backslash U_1$ is compactly contained in $B \cap \overline{D} \cap G$ where $\psi_n \circ \beta_n$ is close to the identity for large n, and hence no point from this set can get identified with a point from $(A \backslash B) \cap \overline{D}$. □

Using Lemma 2.2 inductively we now prove the following result:

Theorem 2.3. *Assume that D is a relatively compact smoothly bounded domain in a Riemann surface R. Choose finitely many pairwise distinct points $a_1, \ldots, a_k \in bD$, $b_1, \ldots, b_k \in R \backslash \overline{D}$, and $c_1, \ldots, c_l \in \overline{D} \backslash \{a_1, \ldots, a_k\}$ such that for each $j = 1, \ldots, k$ the points a_j and b_j belong to the same connected component of $R \backslash D$. For every integer $N \in \mathbb{N}$ there exists a diffeomorphism $\phi : \overline{D} \to \overline{D}'$ onto a smoothly bounded domain $D' \subset R$ such that $\phi : D \to D'$ is biholomorphic, $\phi(a_j) = b_j$ for $j = 1, \ldots, k$, and ϕ is tangent to the identity map to order N at each point c_j. Furthermore, given a neighborhood U_j of a_j for every j, ϕ can be chosen as close as desired to the identity map in the smooth topology on $\overline{D} \backslash \bigcup_{j=1}^k U_j$.*

Proof. By decreasing the neighborhoods $U_j \ni a_j$ we may assume that their closures are pairwise disjoint and do not contain any of the points c_j. Choose smaller neighborhoods $U'_j \ni a_j$ with $\overline{U}'_j \subset U_j$ for $j = 1, \ldots, k$.

A map ϕ with the desired properties will be found as a composition,

$$\phi = \phi_k \circ \phi_{k-1} \circ \cdots \circ \phi_2 \circ \phi_1 : \overline{D} \to \overline{D}'.$$

In the first step, Lemma 2.2 furnishes a diffeomorphism $\phi_1 : \overline{D} \to \phi_1(\overline{D}) = \overline{D}_1$ onto a new domain $\overline{D}_1 \subset R$ such that

(1) ϕ_1 is biholomorphic in the interior,
(2) $\phi_1(a_1) = b_1$,
(3) ϕ_1 is tangent to the identity to order $N' = \max\{2, N\}$ at each of the points $a_2, \dots, a_k$ and $c_1, \dots, c_l$,
(4) ϕ_1 is uniformly close to the identity on $\overline{D} \backslash U_1'$.

Hence the points $b_1 = \phi_1(a_1), a_2, \dots, a_k$ lie on bD_1, and $c_j \in \overline{D}_1$ for $j = 1, \dots, l$.

In the second step we apply Lemma 2.2, with $\overline{D}$ replaced by $\overline{D}_1 = \phi_1(\overline{D})$, to find a diffeomorphism $\phi_2 : \overline{D}_1 \to \phi_2(\overline{D}_1) = \overline{D}_2$, holomorphic in the interior and close to the identity map on $\overline{D}_1 \backslash U_2'$, such that $\phi_2(a_2) = b_2$, ϕ_2 is tangent to the identity to order N' at the points $b_1, a_3, \dots, a_k$ and $c_1, \dots, c_l$, and ϕ_2 is close to the identity map on $\overline{D}_1 \backslash U_2'$.

Continuing inductively, we obtain after k steps a map ϕ satisfying the conclusion of Theorem 2.3 with $D' = D_k$. At the jth step of the construction, the action takes place near the point $a_j \in bD_{j-1}$ that is mapped by ϕ_j to the point $b_j \in bD_j = \phi_j(bD_{j-1})$. In addition, ϕ_j is tangent to the identity at the points $b_1, \dots, b_{j-1}, a_{j+1}, \dots, a_k$ and $c_1, \dots, c_l$, and ϕ_j is close to the identity map on $\overline{D}_{j-1} \backslash U_j'$.

The final domain $D' = D_k = \phi(D)$ contains the points $b_1, \dots, b_k$ in the boundary, while the points $c_1, \dots, c_l$ remained fixed during the construction. The domain D' is very close to D away from a small neighborhood of each point a_j, and at a_j it includes a spike reaching out to b_j. □

3. Normalization and stability of complex curves in $\mathbb{C}^2$

In this section we obtain some technical results that will be used in the proof of Theorem 1.1 in the case of curves with interior singularities.

The first lemma gives a normalization of a complex curve with smooth boundary by a bordered Riemann surface.

Lemma 3.1. *Let $\overline{\Sigma}$ be a compact complex curve with boundary of class $\mathcal{C}^r$ $(r \geqslant 1)$ in a complex manifold X. There exists a bordered Riemann surface $\overline{D}$ with $\mathcal{C}^r$ boundary and a $\mathcal{C}^r$ map $f : \overline{D} \to X$, with $f(\overline{D}) = \overline{\Sigma}$ and $f(bD) = b\Sigma$, such that f is a diffeomorphism near bD and $f : D \to \Sigma$ is a holomorphic normalization of Σ. In particular, f is biholomorphic over the regular locus of Σ.*

Proof. To get such D and f we simply normalize each singular point of Σ (see e.g. [15, p. 70] for curves without boundaries); we briefly describe this construction. The conditions imply that Σ has at most finitely many interior singularities $p_1, \dots, p_n \in \Sigma$ and no singularities on $b\Sigma$. Choose a small open set $B_j \subset X$ containing p_j (in local coordinates at p_j, B_j is a small ball) and let $\Sigma \cap B_j = \bigcup_{k=1}^{m_j} V_{j,k}$ be a decomposition into irreducible branches. By choosing B_j sufficiently small we insure that each $V_{j,k} \backslash \{0\}$ is regular, $V_{j,k} \cap V_{j,k'} = \{p_j\}$ when $k \neq k'$, and the normalization of each $V_{j,k}$ is a disc in $\mathbb{C}$. More precisely, there is an injective holomorphic map $\psi_{j,k} : \mathbb{D} \to V_{j,k}$, with $\psi_{j,k}(0) = p_j$, such that $\psi_{j,k} : \mathbb{D} \backslash \{0\} \to V_{j,k} \backslash \{p_j\}$ is biholomorphic. By surgery with $\psi_{j,k}$ we replace $V_{j,k} \subset \Sigma$ by the disc $\mathbb{D}$, hence $\Sigma \cap B_j$ is replaced by the disjoint union of m_j discs. To get D and f it suffices to perform this construction at every singular point p_j of Σ. □

The following lemma result is a special case of the classical results on universal denominators (see e.g. Whitney [44]). For completeness we provide a simple proof for curves in $\mathbb{C}^2$ by using a solution to $\bar\partial$-equation. (We thank J.-P. Rosay for suggesting such a proof.)

Lemma 3.2. *Let V be pure one-dimensional analytic subvariety near the origin in $\mathbb{C}^2$ with $V_{\mathrm{sing}} = \{0\}$. There is an integer $N \in \mathbb{N}$ such that every holomorphic function g on $V^* = V \backslash \{0\}$ satisfying $|g(z)| \leqslant C|z|^N$ for some $C > 0$ extends across 0 to a holomorphic function on V.*

Proof. Write $z = (z_1, z_2) \in \mathbb{C}^2$ and let $\pi_j(z_1, z_2) = z_j$ for $j = 1, 2$. After shrinking V and applying a linear change of coordinates on $\mathbb{C}^2$ we may assume that $\pi_1|_V : V \to U$ is a branched analytic covering over a disc $U = r\mathbb{D} \subset \mathbb{C}$ such that $|z_2| \leqslant |z_1|$ on V (see e.g. [15, §6.1]). By shrinking U around 0 we have for each $z_1 \in U^* = U\backslash\{0\}$,

$$\pi_2\big(V \cap \pi_1^{-1}(z_1)\big) = \big\{b_1(z_1), \dots, b_m(z_1)\big\} \subset \mathbb{C},$$

where the functions $b_j(z_1)$ are locally holomorphic and satisfy an estimate,

$$\big|b_j(z_1) - b_k(z_1)\big| \geqslant c|z_1|^\nu,$$

for some $\nu \geqslant 1$, $c > 0$ and for all $j \neq k \in \{1, \dots, m\}$. For irreducible V this estimate follows from the Puiseux series representation, see [15, p. 68]. In general we use that any two complex curves have a finite order of tangency at an isolated intersection point.

Let $W_j(z_1) \subset \mathbb{C}$ be the disc of radius $\frac{c}{4}|z_1|^\nu$ centered at $b_j(z_1)$, and let $W'_j(z_1)$ denote the disc of twice that radius; hence the larger discs are still pairwise disjoint. Set:

$$W = \big\{(z_1, z_2)\colon\ z_1 \in U^*,\ z_2 \in W_j(z_1) \text{ for some } j = 1, \dots, m\big\}.$$

Similarly we define the set $W' \supset W$ by taking the union of the discs $W'_j(z_1)$ in the fibers. Observe that the distance from a point $(z_1, z_2) \in W$ to the complement of W' is comparable to $|z_1|^\nu$ as $z_1 \to 0$, and hence there is a smooth function χ on $U^* \times \mathbb{C}$ with values in $[0, 1]$ such that $\chi = 1$ on W, $\operatorname{supp}\chi \subset W'$, and $|\bar\partial\chi(z)| \leqslant c'|z_1|^{-\nu}$ for some $c' > 0$.

Choose a holomorphic function $P(z_1, z_2)$ on $U \times \mathbb{C}$ with $V = \{P = 0\}$. In fact, P can be chosen as a Weierstrass polynomial in z_2, with coefficients holomorphic in $z_1 \in U$ (see e.g. Chirka [15, p. 25]).

Suppose that $g : V^* = V\backslash\{0\} \to \mathbb{C}$ is a holomorphic function. We extend g to a holomorphic function on the tube W' by taking a constant vertical extension on the fiber around each point $b_j(z_1)$. More precisely, for $(z_1, z_2) \in W'_j(z_1)$ we take $g(z_1, z_2) = g(z_1, b_j(z_1))$. Then χg is a well defined smooth function on $U^* \times \mathbb{C}$ which is holomorphic in W and agrees with the original function g on V^*. Note also that $\bar\partial(\chi g) = g\bar\partial\chi$ is supported in $W'\backslash W$ and satisfies $|g\bar\partial\chi| \leqslant c'|g||z_1|^{-\nu}$. We seek a holomorphic extension G of g in the form $G = \chi g - uP$; this implies $G = g$ on V^*. The holomorphicity condition $0 = \bar\partial G = g\bar\partial\chi - P\bar\partial u$ is equivalent to:

$$\bar\partial u = \alpha := \frac{1}{P} g\bar\partial\chi = \alpha_1\, d\bar z_1 + \alpha_2\, d\bar z_2.$$

On the support of α (in $W'\backslash W$) we have $|P(z_1, z_2)| \geqslant |z_1|^\mu$ for some $\mu > 0$. If $N \geqslant \mu + \nu + 2$, the estimate $|g(z_1, z_2)| \leqslant |z_1|^N$ for $(z_1, z_2) \in V^*$ implies that $|\alpha(z_1, z_2)| \leqslant c'|z_1|^2$. For such α, the equation $\bar\partial u = \alpha$ has a solution on $U \times \mathbb{C}$ given by:

$$u(z_1, z_2) = \frac{1}{2\pi i} \iint\limits_{t\in\mathbb{C}} \frac{\alpha_2(z_1, t)}{t - z_2}\, dt \wedge d\bar t.$$

(The integrand is compactly supported for each fixed $z_1 \in U$, and it vanishes for $z_1 = 0$.) This yields a desired holomorphic extension G of g. □

Our next result shows that the biholomorphic type of a holomorphic image of a bordered Riemann surface in $\mathbb{C}^2$ does not change under a perturbation of the map that is tangent to a sufficiently high order over every singularity.

Lemma 3.3. *Let $\overline{D}$ be a bordered Riemann surface with $\mathcal{C}^1$ boundary and $f : \overline{D} \to \mathbb{C}^2$ be a $\mathcal{C}^1$ map that is an embedding near bD and is holomorphic in D, with $f(D) \cap f(bD) = \emptyset$. Let $\Sigma = f(D)$, let $p_1, \dots, p_k \in \Sigma$ be all its singular points, and let $\{q_1, \dots, q_l\} = f^{-1}(\{p_1, \dots, p_k\}) \subset D$. Then there exists an integer $N \in \mathbb{N}$ with the following property. For every $\mathcal{C}^1$ map $f' : \overline{D} \to \mathbb{C}^2$ which is sufficiently $\mathcal{C}^1$ close to f, holomorphic in D and tangent to f to order N at each of the points $q_1, \dots, q_l$, the image $\Sigma' = f'(D)$ is biholomorphically equivalent to Σ.*

Proof. The conditions imply that f and f' are injective holomorphic embeddings of $D' = D\backslash\{q_1, \dots, q_l\}$ into $\mathbb{C}^2$, and hence the map,

$$\Phi = f' \circ f^{-1} : \Sigma\backslash\{p_1, \dots, p_k\} \to \Sigma'\backslash\{p_1, \dots, p_k\},$$

is biholomorphic. It remains to show that Φ and Φ^{-1} extend holomorphically across the singular points p_j, provided that f and f' are tangent to a sufficiently high order at all points in $f^{-1}(p_j) \subset D$.

The problem being local, we fix a point $q = q_j \in D$ and let $p = f(q) = f'(q) \in \Sigma_{\mathrm{sing}} \subset \mathbb{C}^2$. In suitable local holomorphic coordinates we have $q = 0 \in \mathbb{C}$, $p = 0 \in \mathbb{C}^2$, and $f(\zeta) = (f_1(\zeta), f_2(\zeta))$ is an injective local holomorphic map with the only branch point at $\zeta = 0$. Let $V \subset \Sigma$ be the local image of f, so V is a local irreducible complex curve in $\mathbb{C}^2$ whose only singular point is the origin $0 \in \mathbb{C}^2$. For $z \in V^* = V\backslash\{0\}$ let $\zeta(z) = f^{-1}(z)$, a holomorphic function on V^*. We have $|\zeta(z)| \leqslant |z|^\alpha$ for some $\alpha > 0$. Then $\Phi(z) = f'(\zeta(z))$ for $z \in V^*$. From $f'(\zeta) = f(\zeta) + O(\zeta^N)$ we get for $z \in V^*$,

$$\Phi(z) = f\big(\zeta(z)\big) + O\big(\zeta(z)^N\big) = z + g(z), \tag{3.1}$$

where g is a holomorphic function on V^* satisfying:

$$\big|g(z)\big| = O\big(\big|\zeta(z)\big|^N\big) = O\big(|z|^{N\alpha}\big), \quad z \to 0.$$

The same argument applies to every local irreducible component of Σ at the singular point p. If $N > 0$ is sufficiently large then the function g in (3.1), which is defined and holomorphic on a deleted neighborhood of p in Σ, extends holomorphically across p by Lemma 3.2. It follows that Φ extends holomorphically to Σ for all large N. The same argument applies to Φ^{-1}, so $\Phi : \Sigma \to \Sigma'$ is biholomorphic. □

4. Exposing boundary points

In this section we prove a result on exposing boundary points of complex curves in $\mathbb{C}^2$. Theorem 4.2 below is a main new technical result of this paper. It also hold in $\mathbb{C}^n$, with essentially the same proof.

We shall need the following notion introduced in [46]. Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ denote a $\mathbb{C}$-linear map onto $\mathbb{C}$; we may assume that $\pi(z_1, z_2) = z_1$.

Definition 4.1. Let $\Sigma \subset \mathbb{C}^2$ be a locally closed complex curve, possibly with boundary. A point $p = (p_1, p_2) \in \Sigma$ is *exposed* (with respect to the projection π) if the complex line,

$$\Lambda_p = \pi^{-1}\big(\pi(p)\big) = \big\{(p_1, \zeta):\ \zeta \in \mathbb{C}\big\},$$

intersects Σ precisely at p and the intersection is transverse: $T_p\Lambda_p \cap T_p\Sigma = \{0\}$. If $\Sigma = f(R)$, where R is a Riemann surface (with or without boundary) and $f : R \to \mathbb{C}^2$ is a holomorphic map, then a point $a \in R$ is said to be f-exposed if the point $p = f(a) \in \Sigma$ is exposed.

Theorem 4.2. *Let $\overline{D}$ be a bordered Riemann surface with $\mathcal{C}^r$ boundary for some $r > 1$. Assume that $f : \overline{D} \to \mathbb{C}^2$ is a $\mathcal{C}^1$ map which is holomorphic in D and is an embedding near bD, with $f(D) \cap f(bD) = \emptyset$. Then f can be approximated, uniformly on compacts in D, by a map $F : \overline{D} \to \mathbb{C}^2$ with the same properties such that the complex curve $F(D) \subset \mathbb{C}^2$ is biholomorphic to the curve $f(D)$, and such that every boundary curve of $F(\overline{D})$ contains an exposed point. Furthermore, F can be chosen to agree with f to a given finite order at a prescribed finite set of points $c_1, \ldots, c_l \in D$; if these points are f-exposed then F can be chosen such that they are also F-exposed.*

Proof. We begin with a few reductions.

The hypotheses imply that $\overline{\Sigma} = f(\overline{D})$ is a compact complex curve in $\mathbb{C}^2$ with embedded $\mathcal{C}^1$ boundary $b\Sigma = f(bD)$ and with finitely many interior singularities. Let $\{d_1, \ldots, d_s\} = f^{-1}(\Sigma_{\mathrm{sing}}) \subset D$.

We realize $\overline{D}$ as a domain with smooth boundary in an open Riemann surface R; the corresponding biholomorphic map is of class $\mathcal{C}^1$ up to the boundary. By Mergelyan's theorem we can find a holomorphic map $g : U \to \mathbb{C}^2$ from an open neighborhood $U \subset R$ of $\overline{D}$ into $\mathbb{C}^2$ such that g approximates f arbitrarily well in the $\mathcal{C}^1(\overline{D})$ topology, and g agrees with f to a given order at each of the points $c_1, \ldots, c_l, d_1, \ldots, d_s$. By Lemma 3.3 we may assume that the complex curve $g(D)$ is biholomorphic to $f(D)$. Replacing g by f and R by a sufficiently small open neighborhood of $\overline{D}$ in R we may therefore assume that $f : R \to \mathbb{C}^2$ is a holomorphic map which is an embedding (injective immersion) on $R\backslash\{d_1, \ldots, d_s\}$.

We have $bD = \bigcup_{j=1}^m C_j$, each C_j being a closed curve. For every j we choose a point $a_j \in C_j$ and a smooth embedded arc $\gamma_j \subset R$ that is attached with one of its endpoints to $\overline{D}$ at a_j, and such that the intersection of γ_j and C_j

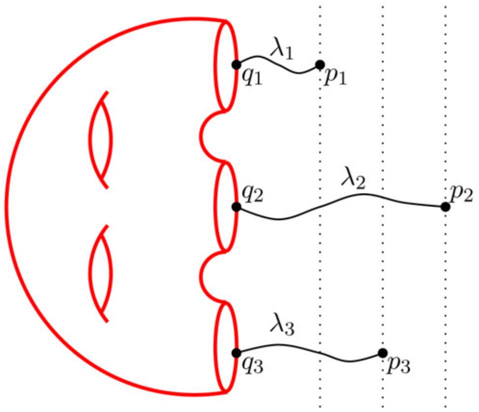

Fig. 2. A Riemann surface with exposed tails.

is transverse at a_j. The rest of the arc, $\gamma_j \backslash \{a_j\}$, is contained in $R\backslash\overline{D}$. Let b_j denote the other endpoint of γ_j. Choose an open set $U \subset R$ that contains $\overline{D}$ and such that $\overline{U}$ does not contain any of the points $b_1, \dots, b_m$. We also insure that the set $\gamma_j \cap U = \widetilde{\gamma}_j$ is an arc with an endpoint a_j.

In $\mathbb{C}^2$ we choose for every $j = 1, \dots, m$ a smooth embedded arc λ_j that agrees with the arc $f(\widetilde{\gamma}_j)$ near the endpoint $q_j = f(a_j)$, while the rest of it, $\lambda_j \backslash f(\widetilde{\gamma}_j)$, does not intersect $f(U)$. We also insure that the arcs $\lambda_1, \dots, \lambda_m$ are pairwise disjoint, they do not intersect any of the vertical complex lines through the points $f(c_1), \dots, f(c_l)$, and the other endpoint p_j of λ_j is an exposed point for the set $f(\overline{D}) \cup (\bigcup_{j=1}^m \lambda_j) \subset \mathbb{C}^2$ (see Fig. 2). In particular, the complexified tangent line to the arc λ_j at p_j is transverse to the vertical line through p_j. We may begin with an arbitrary set of points $p_1, \dots, p_m \in \mathbb{C}^2$ such that the vertical lines through them are pairwise disjoint and do not intersect $f(\overline{U})$, and then find arcs λ_j from $q_j = f(a_j)$ to p_j as above.

Let $K = \overline{D} \cup (\bigcup_{j=1}^m \gamma_j)$, a compact set in the Riemann surface R. Let $f' : U \cup (\bigcup_{j=1}^m \gamma_j) \to \mathbb{C}^2$ be a smooth map that agrees with f on U and that maps each arc $\gamma_j \subset R$ diffeomorphically onto the corresponding arc $\lambda_j \subset \mathbb{C}^2$. In particular, the endpoint b_j of γ_j is mapped by f' to the exposed endpoint p_j of λ_j.

By Mergelyan's theorem (see e.g. [20, Theorem 3.2]) we can approximate f', uniformly on a neighborhood of $\overline{D}$ in R and in the $\mathcal{C}^1$ topology on each of the arcs γ_j, by a holomorphic map $\widetilde{f} : V \to \mathbb{C}^2$ from an open neighborhood of K in R. At the same time we insure that $\widetilde{f}$ agrees with f' to a high order at each of the points $a_1, \dots, a_m$, $b_1, \dots, b_m$, $c_1, \dots, c_l$, $d_1, \dots, d_s$. If the approximation is close enough, the neighborhood $V \supset K$ is chosen small enough, and the interpolation at the indicated points is to a sufficiently high order, then $\widetilde{f} : V \to \mathbb{C}^2$ is a (non-proper) holomorphic embedding except at the points $d_1, \dots, d_s$, the complex curve $\widetilde{\Sigma} = \widetilde{f}(D) \subset \mathbb{C}^2$ is biholomorphic to the curve $\Sigma = f(D)$ according to Lemma 3.3, and the points $p_j = \widetilde{f}(b_j)$ and $\widetilde{f}(c_j) = f(c_j)$ are exposed in $\widetilde{f}(V)$.

Now Theorem 2.3 furnishes a diffeomorphism $\phi : \overline{D} \to \phi(\overline{D}) \subset V$ that is holomorphic in D, that sends the point $a_j \in bD$ to the point b_j for every $j = 1, \dots, m$, that is tangent to the identity to a desired (high) order at each of the points $c_1, \dots, c_l$, $d_1, \dots, d_s$, and that is close to the identity map outside a small neighborhood of $\{a_1, \dots, a_m\}$. The composition,

$$F = \widetilde{f} \circ \phi : \overline{D} \to \mathbb{C}^2,$$

maps $\overline{D}$ onto the domain $F(\overline{D})$ in the complex curve $\widetilde{f}(V) \subset \mathbb{C}^2$ such that each point $p_j = F(a_j)$ for $j = 1, \dots, m$, is an exposed boundary point of $F(\overline{D})$, and the points $F(c_j) = f(c_j)$ are also exposed in $F(\overline{D})$.

Let $\Sigma' = F(D)$. Note that ϕ induces a biholomorphic map:

$$\widetilde{\phi} = F \circ (\widetilde{f})^{-1} = \widetilde{f} \circ \phi \circ (\widetilde{f})^{-1} : \widetilde{\Sigma}_{\mathrm{reg}} \to \Sigma'_{\mathrm{reg}}.$$

If ϕ is chosen tangent to the identity map to a sufficiently high order at each of the points $d_1, \dots, d_s$, then $\widetilde{\phi}$ is tangent to the identity to a high order at each of the points in $\widetilde{\Sigma}_{\mathrm{sing}}$, and hence Lemma 3.3 shows that $\widetilde{\phi}$ extends to a biholomorphic map $\Phi : \widetilde{\Sigma} \to \Sigma'$. Thus $\Sigma' = F(D)$ is biholomorphic to $\widetilde{\Sigma}$, and hence to $\Sigma = f(D)$. $\square$

5. Proofs of main results

In this section we prove Theorem 1.1 and obtain some further corollaries.

By Theorem 4.2 in Section 4 we may assume that the complex curve $\overline{\Sigma}$ in Theorem 1.1 admits an exposed point in each of its boundary curves. To complete the proof of Theorem 1.1 it therefore suffices to show the following.

Theorem 5.1. *Let $\overline{\Sigma} \subset \mathbb{C}^2$ be as in Theorem* 1.1. *If every boundary component of Σ contains an exposed point* (*see Definition* 4.1) *then the conclusion of Theorem* 1.1 *holds.*

Proof. In the special case when Σ has no interior singularities, Theorem 5.1 is due to the second author (see [46, Theorem 1]). We shall now show that the proof given there also holds for curves with singularities.

Lemma 3.1 furnishes a smoothly bounded domain D in a Riemann surface R and a $\mathcal{C}^r$ map $f:\overline{D} \to \overline{\Sigma}$ such that $f(\overline{D}) = \overline{\Sigma}$, $f(bD) = b\Sigma$, f is diffeomorphic near bD, and $f: D \to \Sigma$ is a holomorphic normalization of Σ.

Let $bD = \bigcup_{j=1}^m C_j$, and assume that $a_j \in C_j$ is an f-exposed point for each $j = 1, \dots, m$ (with respect to the first projection $\pi_1(z,w) = z$). Let $\pi_2 : \mathbb{C}^2 \to \mathbb{C}$ be the second projection $\pi_2(z,w) = w$. Define a rational shear map g of $\mathbb{C}^2$ by:

$$g(z,w) = \left(z, w + \sum_{j=1}^m \frac{\alpha_j}{z - \pi(f(a_j))}\right). \tag{5.1}$$

The numbers $\alpha_j \in \mathbb{C}\backslash\{0\}$ can be chosen such that π_2 maps the (unbounded) curves,

$$\lambda_j = (g \circ f)\big(C_j \backslash \{a_j\}\big) \subset \mathbb{C}^2,$$

to unbounded curves $\gamma_j = \pi_2(\lambda_j) \subset \mathbb{C}$, and $\pi_2 : \lambda_j \to \gamma_j$ is a diffeomorphism near infinity. Furthermore, for every sufficiently large number $\rho > 0$, the set $\rho\overline{\mathbb{D}} \cup \bigcup_{j=1}^m \gamma_j \subset \mathbb{C}$ has no bounded complementary connected components. This is achieved by a careful choice of the arguments of α_j's, while their absolute values $|\alpha_j|$ can be taken as small as desired.

Consider the map $g \circ f : \overline{D}\backslash\{a_j\}_{j=1}^m \to \mathbb{C}^2$. Fix a compact set L in D. By choosing the numbers α_j small enough we insure that $g \circ f$ is close to f on L. The complex curve $X = (g \circ f)(D) \subset \mathbb{C}^2$, with boundary

$$bX = (g \circ f)\big(bD\backslash\{a_j\}_{j=1}^m\big) = \bigcup_{j=1}^m \lambda_j,$$

is then biholomorphic to $\Sigma = f(D)$, and it enjoys the following properties:

(1) X admits an exhaustion $K_1 \subset K_2 \subset \cdots \subset \bigcup_{j=1}^\infty K_j = X$ by compact sets K_j that are polynomially convex in $\mathbb{C}^2$, with $(g \circ f)(L) \subset K_1$.

To see this, it suffices to show that any smoothly bounded compact set $K \subset X$ that is holomorphically convex in X is also polynomially convex in $\mathbb{C}^2$. Since $\widehat{K} = \widehat{bK}$ and bK is a union of smooth curves, the set $A = \widehat{K}\backslash bK$ is an analytic subvariety of $\mathbb{C}^2\backslash bK$ containing $K\backslash bK$ (see [43]). If $A \neq K\backslash bK$, then A contains a local extension of K in X near a boundary component of K. Hence $\widehat{K}$ contains at least one connected component of $X\backslash K$, a contradiction since each of these components is unbounded in $\mathbb{C}^2$. Thus $\widehat{K} = K$ as claimed.

(2) A similar argument shows that for any compact polynomially convex set $K \subset \mathbb{C}^2\backslash bX$, $K \cup K_j$ is also polynomially convex for all large $j \in \mathbb{N}$.

(3) For every compact polynomially convex set K contained in $\mathbb{C}^2\backslash bX$, and for every pair of numbers $\epsilon > 0$ (small) and $R > 0$ (large) there exists a holomorphic automorphism ϕ of $\mathbb{C}^2$ such that

$$\sup_{x\in K}\big|\phi(x) - x\big| < \epsilon \quad \text{and} \quad \phi(bX) \subset \mathbb{C}^2\backslash R\mathbb{B}.$$

(Here $\mathbb{B}$ is the unit ball in $\mathbb{C}^2$.) This property of X is invariant under holomorphic automorphisms of $\mathbb{C}^2$ as is seen by a conjugation argument.

The construction of such ϕ can be found in [45] (see Lemma 1 and the proof of Theorem 4 in [45]); the main point to note here is that the construction depends on the geometric assumptions on the curves λ_j—it has nothing to do with whether or not X is smooth.

Using properties (1)–(3) we find a sequence of holomorphic automorphisms $\Phi_j = \phi_j \circ \phi_{j-1} \circ \cdots \circ \phi_1 \in \mathrm{Aut}\,\mathbb{C}^2$ ($j = 1, 2, \ldots$) carrying bX to infinity and converging on X to a proper holomorphic embedding $X \hookrightarrow \mathbb{C}^2$. The inductive step is the following. Fix $j \in \mathbb{N}$ and assume inductively that $\Phi_j(bX) \cap j\overline{\mathbb{B}} = \emptyset$. (This trivially holds for $j = 0$ with $\Phi_0 = \mathrm{id}$.) Choose $m_j \in \mathbb{N}$ large enough such that the compact set $L_j = j\overline{\mathbb{B}} \cup \Phi_j(K_{m_j})$ is polynomially convex (this is possible by property (2)). By property (3) there is for any $\epsilon_j > 0$ an automorphism $\phi_{j+1} \in \mathrm{Aut}\,\mathbb{C}^2$ such that

- $|\phi_{j+1}(x) - x| < \epsilon_j$ for all $x \in L_j$,
- $|\phi_{j+1}(x)| > j + 1$ for all $x \in \Phi_j(bX)$.

Setting $\Phi_{j+1} = \phi_{j+1} \circ \Phi_j$ completes the induction step.

Suitable choices of the sequences $\epsilon_j \searrow 0$ and $m_j \nearrow +\infty$ insure that the sequence $\Phi_j \in \mathrm{Aut}\,\mathbb{C}^2$ converges locally uniformly on the domain,

$$\Omega = \bigcup_{j=1}^{\infty} \Phi_j^{-1}(j\overline{\mathbb{B}}) \subset \mathbb{C}^2,$$

to a biholomorphic map $\Phi : \Omega \to \mathbb{C}^2$ onto $\mathbb{C}^2$ (a Fatou–Bieberbach map), and we have $X \subset \Omega$ and $bX \subset b\Omega$ (see [18, Proposition 5.1]). The restriction $\varphi = \Phi|_X : X \hookrightarrow \mathbb{C}^2$ is then a proper holomorphic embedding of X into $\mathbb{C}^2$. Since X is biholomorphic to $\Sigma = f(D)$, this proves Theorem 5.1. □

Proof of Corollary 1.3. Let $\{a_j\} \subset D$ and $\{b_j\} \subset \mathbb{C}^2$ be discrete sequences without repetition. If $f : \overline{D} \hookrightarrow \mathbb{C}^2$ is a holomorphic embedding such that each boundary component of D admits an f-exposed point, it was proved in [31, Theorem 3] that there is a proper holomorphic embedding $\varphi : D \to \mathbb{C}^2$ such that $\varphi(a_j) = b_j$ for $j = 1, 2, \ldots$. By Theorem 4.2 such an embedding $f : \overline{D} \hookrightarrow \mathbb{C}^2$ with exposed boundary points exists for every Riemann surface D satisfying the hypothesis of Corollary 1.2. □

We also have the following embedding result for certain bordered Riemann surfaces with punctures.

Theorem 5.2. *Assume that $f : \overline{D} \to \mathbb{C}^2$ is as in Corollary* 1.2*, $\pi : \mathbb{C}^2 \to \mathbb{C}$ is a $\mathbb{C}$-linear projection, $b_1, \ldots, b_k \in \mathbb{C}$,*

$$\{c_1, \ldots, c_l\} = (\pi \circ f)^{-1}\big(\{b_1, \ldots, b_k\}\big) \subset D.$$

Then $D \backslash \{c_1, \ldots, c_l\}$ embeds properly holomorphically in $\mathbb{C}^2$.

Proof. By a linear change of coordinates on $\mathbb{C}^2$ we may assume that π is the first coordinate projection. Theorem 4.2 furnishes a new embedding $F : \overline{D} \hookrightarrow \mathbb{C}^2$ with an exposed point $a_j \in bD$ in each boundary component, taking care to insure that $F(c_j) = f(c_j)$ for $j = 1, \ldots, l$. The construction also shows that we can avoid creating any new intersections of $F(\overline{D})$ with the finitely many complex lines $\pi^{-1}(b_j)$ for $j = 1, \ldots, k$, so that we have

$$(\pi \circ F)^{-1}\big(\{b_1, \ldots, b_k\}\big) = \{c_1, \ldots, c_l\} \subset D.$$

Let g be a shear (5.1) with simple poles at all points $(\pi \circ F)(a_j)$ ($j = 1, \ldots, m$) and $b_1, \ldots, b_k$. Then $g \circ F$ embeds the punctured domain $D' = D \backslash \{c_1, \ldots, c_l\}$ onto a complex curve $X \subset \mathbb{C}^2$. The rest of the proof (pushing bX to infinity) is exactly as in the proof of Theorem 5.1. □

Corollary 5.3. *Assume that the embedding $f : \overline{D} \hookrightarrow \mathbb{C}^2$ satisfies the hypotheses of Corollary* 1.2*. If $c_1, \ldots, c_l \in D$ are f-exposed points* (*with respect to some linear projection $\pi : \mathbb{C}^2 \to \mathbb{C}$*)*, then the domain $D' = D \backslash \{c_1, \ldots, c_l\}$ admits a proper holomorphic embedding in $\mathbb{C}^2$.*

In particular, every finitely connected planar domain with finitely many punctures embeds properly in $\mathbb{C}^2$, a result first proved by Wold [45] (for the punctured disc see also Alexander [5] and Globevnik [23]).

6. Teichmüller spaces of bordered Riemann surfaces

In this section we outline another possible proof of Corollary 1.2 by using the theory of Teichmüller spaces. Although not nearly as explicit as our main proof, it sheds additional light on the subject. The main idea was already used by Globevnik and Stensønes (see [24]) for planar domains (genus $g = 0$), and by the second author (see [47]) for domains in complex tori (genus $g = 1$). Here we focus on domains of genus $g > 1$.

Let R be a connected, closed, oriented smooth surface of genus $g > 1$. The set of all equivalence classes of complex structures on R is the quotient T_g/Γ_g, where T_g is the Teichmüller space of R (a complex manifold of complex dimension $3g-3$ that is biholomorphic to a bounded domain in $\mathbb{C}^{3g-3}$ and is homeomorphic to the ball), and Γ_g is a properly discontinuous group of holomorphic automorphisms of T_g. (For a precise description and the construction of the Teichmüller space T_g see [1–3,8–10] and the monographs [33,34].) Each element of T_g can be represented uniquely as the quotient $\mathbb{D}/G$ of the unit disc $\mathbb{D} \subset \mathbb{C}$ by a suitably normalized *Fuchsian group* G, that is, a group of fractional linear transformations preserving the circle $b\mathbb{D}$ and acting properly discontinuously and without fixed points on both discs forming the complement of $b\mathbb{D}$ in the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. By fixing a marked reference surface $R_0 = \mathbb{D}/G_0 \in T_g$, we may view T_g as the space of group isomorphisms $\theta : G_0 \to G$ of normalized Fuchsian groups, with the coefficients of the generators of $\theta(G_0)$ serving as the coordinates (see [9, Theorem 2]).

There exists a holomorphic submersion $\pi : Z \to T_g$ of a complex manifold Z onto the Teichmüller space T_g such that the fiber $\pi^{-1}(\theta)$ over any point $\theta \in T_g$ is the Riemann surface $R_\theta = \mathbb{D}/\theta(G_0)$; hence Z is a *universal family of closed Riemann surfaces of genus* g. One takes Z as the quotient of $X = T_g \times \mathbb{D}$ obtained by replacing each fiber $\{\theta\} \times \mathbb{D} \subset X$ by the Riemann surface $\mathbb{D}/\theta(G_0)$. Ahlfors showed that, in the complex structure on X, the maps $(\theta, z) \mapsto \theta$ and $(\theta, z) \mapsto (\theta, \theta(a)z)$ are holomorphic for a fixed $a \in G_0$ (see [3]), and this gives a complex structure to Z.

We now consider connected domains $D \subset R$ obtained by removing $m \geqslant 1$ discs (homeomorphic images of the closed disc $\overline{\mathbb{D}}$) from R. The boundary bD of any such domain is the union of m closed Jordan curves, each bounding a complementary disc that was removed from R. We shall write R_θ for the Riemann surface obtained by endowing R with the complex structure determined by a point $\theta \in T_g$.

He and Schramm proved (see [28,29]) that every domain $D \subset R_\theta$ as above is conformally equivalent to a domain D' in another Riemann surface $R' = R_{\theta'}$ such that the preimage of D' in the universal covering $\mathbb{D}$ of R' is a domain in $\mathbb{D}$ all of whose complementary components are geometric (round) discs; we shall call such D' a *circle domain*. Moreover, the operation mapping D to D' is continuous, in the sense that domains close to D are mapped to circle domains close to D' in Riemann surfaces close to R'. For connected planar domains with at most countably many boundary components, this solved a famous conjecture of Koebe from 1908 to the effect that every planar domain is conformally equivalent to a circle domain. Known as the *Kreisnormierungsproblem*, this conjecture was the subject of considerable effort over many decades.

Using the result of He and Schramm, one can give the following description of the Teichmüller space $T_{g,m}$ of bordered Riemann surfaces of genus $g \geqslant 2$ with $m \geqslant 1$ boundary components. Every element of $T_{g,m}$ is represented by a circle domain D in a closed Riemann surface R_θ of genus g, determined by a point $\theta \in T_g$. We represent D by a choice of representatives $(z, r) = (z_1, \ldots, z_m, r_1, \ldots, r_m) \in \mathbb{D}^m \times (0, \infty)^m$ of the centers $z_j \in \mathbb{D}$ and the radii $r_j > 0$ of the complementary components of the preimage of D in $\mathbb{D}$; such triples (θ, z, r) then parametrize the points in $T_{g,m}$. Although this representation of D is clearly not unique as we may choose different representatives of the removed discs, it is locally unique in the following sense: If $\epsilon > 0$ is small enough then the triples (θ', z', r') that are ϵ-close to (θ, z, r) determine pairwise distinct elements of $T_{g,m}$. (This is seen by observing that the Fuchsian group $G = \theta(G_0)$ acts properly discontinuously and without fixed points on $\mathbb{D}$, and for each removed disc $\Delta \subset \mathbb{D}$ we also remove all its images $g(\Delta)$ for $g \in \theta$.) In this way we define on $T_{g,m}$ the structure of a real $(6g-6+3m)$-dimensional manifold.

Let $E_{g,m}$ denote the set of all circle domains D in Riemann surfaces R_θ $(\theta \in T_g)$ such that $\overline{D}$ admits an injective immersion $f : \overline{D} \hookrightarrow \mathbb{C}^2$ that is holomorphic in D. In other words, $E_{g,m}$ is the set of elements of the Teichmüller space $T_{g,m}$ that satisfy the hypothesis of Corollary 1.2.

Proposition 6.1. *The set $E_{g,m}$ is nonempty and open in $T_{g,m}$.*

Proof. That $E_{g,m}$ is nonempty was proved in [13, Theorem 1.1], and it also follows from our results: Any compact Riemann surface R admits an immersion to $\mathbb{P}^2$, and by cutting out a suitably chosen open disc $U \subset R$ one obtains a holomorphic embedding of $\overline{D}_0 = R \backslash U$ into $\mathbb{C}^2$. Removing $m-1$ additional pairwise disjoint closed discs from D_0 we obtain a point in $E_{g,m}$.

To see that $E_{g,m}$ is open, choose a point $(\theta, z, r) \in E_{g,m}$ and let $D \subset R_\theta$ denote the corresponding circle domain. Let $f : \overline{D} \hookrightarrow \mathbb{C}^2$ be an embedding as in Corollary 1.2. We can approximate f in the $\mathcal{C}^1(\overline{D})$ topology by a holomorphic map $f : U \to \mathbb{C}^2$ from an open set $U \subset R_\theta$ containing $\overline{D}$.

Consider R_θ as the fiber $\pi^{-1}(\theta)$ in the fibration $\pi : Z \to T_g$ defined above. The set $\overline{D} \subset R_\theta$ admits an open Stein neighborhood in R_θ (just remove a point from each connected component of $R_\theta \backslash \overline{D}$), and hence it has a basis of open Stein neighborhoods $\Omega \subset Z$ by Siu's theorem [40, Theorem 1]. Choose Ω small enough such that $\Omega \cap R_\theta \subset U$. By Cartan's extension theorem, the map $f : U \cap \Omega \to \mathbb{C}^2$ extends to a holomorphic map $F : \Omega \to \mathbb{C}^2$. The restriction of F to any domain $\overline{D}' \subset R_{\theta'}$ sufficiently near D (in a fiber $R_{\theta'}$ of Z that is sufficiently close to the initial fiber R_θ) is then a holomorphic embedding of $\overline{D}'$ into $\mathbb{C}^2$, and hence such D' belongs to $E_{g,m}$. This completes the proof of Proposition 6.1. □

Problem 6.2. Is the set $E_{g,m}$ closed in $T_{g,m}$?

An affirmative answer would imply that every bordered Riemann surface embeds properly holomorphically into $\mathbb{C}^2$. For the time being this seems entirely out of reach.

Sketch of an alternative proof of Corollary 1.2. Fix a circle domain $D \subset \mathbb{R}_\theta$ satisfying the hypothesis of the corollary. The argument in the proof of Proposition 6.1 above give a smooth family of holomorphic embeddings of (the closures of) all nearby circle domains into $\mathbb{C}^2$. Proposition 3 in [47] gives another continuously varying family of holomorphically embedded surfaces in $\mathbb{C}^2$, close to the original one, whose members all have an exposed point in each boundary component, and hence they all embed properly holomorphically into $\mathbb{C}^2$ by Theorem 5.1. (This construction of exposed points is reminiscent of what we did in the proof of Proposition 4.2 above, but less precise as it entails a small cut of each domain, thereby changing its conformal structure. At this point one must use that the normalization provided by He and Schramm is a continuous operation.)

An argument as in [24] and [47], using the Brouwer fixed point theorem, now shows that there is a domain in the new family that is conformally equivalent to the original domain D, thereby concluding the proof. For domains in tori the details of this argument can be found in [47]. □

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