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# A matching pursuit technique for computing the simplest normal forms of vector fields

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**Abstract**

This paper presents a matching pursuit technique for computing the simplest normal forms of vector fields. First a simple, explicit recursive formula is derived for general differential equations, which reduces computation to the minimum. Then a matching pursuit technique is introduced and applied to the Takens–Bogdanov dynamical singularity. It is shown that unlike other methods for computing normal forms, the technique using matching pursuit does not need any algebraic constraints which are required for the existence of the simplest normal form. The efficient method and matching pursuit technique, which have been implemented using Maple, can be "automatically" executed on various computer systems. A number of examples are presented to demonstrate the advantages of the technique. 

## 1. Introduction

Normal form theory has been widely used in the study of nonlinear vector fields in order to simplify the analysis of the original system (Chow et al., 1994; Cushman and Sanders, 1988; Golubisky and Schaeffer, 1985; Guckenheimer and Holmes, 1993; Nayfeh, 1993). It provides a convenient tool to transform a given system to an equivalent system, whose dynamical behavior is easier to analyze. (Note that the normal form used in this paper particularly refers to the Birkhoff normal form.) Consider the following general system:

$$\dot{\boldsymbol{x}} = J\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}) \equiv J\boldsymbol{x} + \sum_{k=2}^{N} \boldsymbol{f}_k(\boldsymbol{x}) \equiv \boldsymbol{v}_1 + \sum_{k=2}^{N} \boldsymbol{a}_k \boldsymbol{x}^k, \tag{1}$$

where $\boldsymbol{x} \in \boldsymbol{R}^n$ and $\boldsymbol{f} : \boldsymbol{R}^n \to \boldsymbol{R}^n$, $N$ is an arbitrary positive integer and $\boldsymbol{v}_1 \equiv J\boldsymbol{x}$ represents the linear term, where $J$ is the Jacobian matrix of the system evaluated at the

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the origin **0**—an equilibrium of the system. The $J$ is assumed, without loss of generality, in Jordan canonical form. Function $\boldsymbol{f}$ is analytic and can thus be expanded in Taylor series. $\boldsymbol{f}_k$ denotes the $k$th degree homogeneous vector polynomials of $\boldsymbol{x}$. $\boldsymbol{x}^k$ denotes $x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$ satisfying $k_1 + k_2 + \cdots k_n = k$ for all possible non-negative $k_j$'s. The coefficients $\boldsymbol{a}_k$ can be (rational or irrational) numbers, or symbolic notations, or a combination of both numbers and notations. More specifically, $J \in Q^{n,n}$, $\boldsymbol{f}_k \in (Q[\boldsymbol{a}_k][\boldsymbol{x}]_k)^n$ and $\boldsymbol{f} \in (Q[\boldsymbol{a}][\boldsymbol{x}])^n$, where $\boldsymbol{a} = (\boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_N)$.

The basic procedure in the computation of normal forms employs a near-identity nonlinear transformation to obtain a simpler form which is qualitatively equivalent to the original system. However, the conventional normal form has been found not the simplest form and further reductions using a similar near-identity nonlinear transformation are possible, leading to the simplest normal form (e.g. see Algaba et al., 1997; Baider and Churchill, 1988; Baider and Sanders, 1992; Baider, 1989; Chua and Kokubu, 1988a,b; Kokubu et al., 1996; Ushiki, 1984; Wang, 1993; Wang et al., 2000; Yu, 1999; Yu and Yuan, 2000, 2001; Yuan and Yu, 2001). The fundamental difference between the computations of the conventional normal form and the simplest normal form can be roughly explained as follows. First note that computing the coefficients of the normal form and associated nonlinear transformation needs to solve a set of linear algebraic equations at each order. Since in general the number of the variables—the coefficients of the nonlinear transformation—is larger than the number of the algebraic equations, some coefficients of the nonlinear transformation are not determined. In conventional normal form theory, the coefficients of the $k$th order nonlinear transformation are only used to possibly remove the $k$th order nonlinear terms of the system and the undetermined $k$th order coefficients are set to zero at order $k$ (and therefore, the nonlinear transformation is simplified). However, in the computation of the simplest normal form, the undetermined coefficients can be used to further simplify the normal form. They are not set to zero but carried over to higher order equations so that they may be used to eliminate nonlinear terms in higher order normal forms. In other words, the $k$th order coefficients are not only used to simplify the $k$th order terms of the system, but are also used to eliminate higher order nonlinear terms. This is the key idea of the simplest normal form theory. At each order, the simplest normal form computation keeps the minimum number of terms retained in the final form, which cannot be further reduced by any other near-identity nonlinear transformations. In addition, in this paper a recursive algorithm is formulated for efficient computation. The formula is applicable for arbitrary dynamical singularity, and is employed to solve the Takens–Bogdanov singularity in this paper.

It has been noticed that the computation of the simplest normal form is much more complicated than that of the conventional normal form, and thus computer algebra systems such as Maple, Mathematica, Reduce, etc. must be used (e.g. see Algaba et al., 1997; Yu, 1999; Yu and Yuan, 2000, 2001; Yuan and Yu, 2001). Even with the aid of computer algebra systems, computational efficiency is still the main concern in the computation of the simplest normal form. Recently, we have paid attention to developing efficient methodologies and efficient algorithms for computing the simplest normal form (e.g. see Yu, 2002; Yu and Yuan, 2003). Since Ushiki (1984) introduced the method of infinitesimal deformation in 1984 to study the simplest normal form of vector fields, many researchers have applied Lie algebra to consider the computation of the simplest normal form.

However, only very few singularities have been investigated so far. Hopf and generalized Hopf bifurcations were completely solved (e.g. see Baider and Churchill, 1988; Yu, 1999), and explicit formulas as well as "automatic" Maple programs were developed (Yu, 1999). The 1:2 resonant case (double Hopf) was also considered in detail (Sanders and van der Meer, 1990; Yuan and Yu, 2002). The main attention, however, has been concentrated on the Takens–Bogdanov dynamical singularity (an algebraic double but geometric simple zero eigenvalue) (Baider and Sanders, 1992; Chen and Della Dora, 2000; Chua and Kokubu, 1988a,b; Kokubu et al., 1996; Ushiki, 1984; Wang et al., 2000; Yuan and Yu, 2001). For this case, the Jacobian matrix given in Eq. (1) may be assumed to include a double zero eigenvalue, given in the form:

$$J = \operatorname{diag}\left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \alpha_1 \alpha_2 \cdots \alpha_p \begin{bmatrix} \alpha_{p+1} & \omega_1 \\ -\omega_1 & \alpha_{p+1} \end{bmatrix} \begin{bmatrix} \alpha_{p+2} & \omega_2 \\ -\omega_2 & \alpha_{p+2} \end{bmatrix} \cdots \begin{bmatrix} \alpha_{p+q} & \omega_q \\ -\omega_q & \alpha_{p+q} \end{bmatrix}\right], \tag{2}$$

where $\alpha_j < 0$, $j = 1, 2, \ldots, p+q$; $\omega_k > 0$, $k = 1, 2, \ldots, q$, and $2 + p + 2q = n$, $p, q, \alpha_j$ and $\omega_k$ are given fixed numbers. Note that for most physical systems, the unstable manifold is assumed null. Then by normal form theory, the conventional normal form of system (1) is of the form:

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= \sum_{j=2}^{n} a_{2j0} y_1^j + a_{2(j-1)1} y_1^{j-1} y_2, \end{aligned} \tag{3}$$

where $a_{2jk}$'s are explicitly expressed in terms of the derivatives of the original function $\boldsymbol{f}$ evaluated at $\boldsymbol{x} = \boldsymbol{0}$.

Baider and Sanders (1992) gave a detailed study for the Takens–Bogdanov dynamical singularity and classified the normal forms into three cases according to the relation between $\mu$ and $\nu$: (I) $\mu < 2\nu$, (II) $\mu > 2\nu$ and (III) $\mu = 2\nu$, where the $\mu$ and $\nu$ are defined by the $a$ coefficients of system (3): $a_{220} = a_{230} = \cdots = a_{2\mu 0} = 0$, but $a_{(2\mu+1)0} \neq 0$, and $a_{211} = a_{221} = \cdots = a_{2(\nu-1)1} = 0$, but $a_{2\nu 1} \neq 0$. They provided a fair detailed analysis on the first two cases and obtained the "forms" of the simplest normal form for most of the sub-cases (Baider and Sanders, 1992). Later, Kokubu et al. (1996) and Wang et al. (2000) considered case (III) and also obtained the "form" of the simplest normal form. Recently, Wang et al. (2001) investigated a special sub-case of case (I). However, some special sub-cases are still unsolved. Moreover, even for a classified case, certain non-algebraic number conditions must be satisfied in order for the algebraic equations to be solvable (e.g. see Wang et al., 2000; Yu and Yuan, 2000; Yuan and Yu, 2001). Unfortunately, such non-algebraic number conditions cannot be known before determining the "form" of the simplest normal form. Therefore, regardless of the methods used, there always exist unsolvable special cases if certain non-algebraic number conditions are not assumed appropriately. Otherwise, one must specify the non-algebraic number conditions case by case in the process of computing the simplest normal form. (It will be seen more clearly in Section 5.) When the non-algebraic number conditions are violated, the commonly developed computer programs such as those given in Li et al. (2001) and Yuan and Yu (2001) fail to obtain the simplest normal form, since a "zero division" problem occurs when the programs are executed up to such an order.

A novel approach called matching pursuit technique has been developed to solve this difficulty. Here, the “matching” means that for any given vector fields, the algorithm can match a “form” of the simplest normal form to a special non-algebraic number condition, and the “pursuit” means that the algorithm (program) has been designed to automatically search the right “matching” between the simplest normal form and the non-algebraic number conditions. Symbolic programs are coded using Maple, which can be used to “automatically” compute the simplest normal form of any given vector fields associated with the Takens–Bogdanov singularity.

Before we describe the matching pursuit technique, we present an efficient approach for computing the simplest normal form in the next section. Section 3 deals with the computation of the simplest normal form for the Takens–Bogdanov dynamical singularity. The matching pursuit technique is discussed in detail in Section 4, and the algorithm is also outlined in this section. Various examples are shown in Section 5 to demonstrate the advantage of the matching pursuit technique, and conclusions are given in Section 6.

## 2. An efficient approach for computing the simplest normal form

Consider the general system (1). The basic idea of normal form theory is to find a near-identity nonlinear transformation, given by

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y}) \equiv \boldsymbol{y} + \sum_{k=2}^{N} \boldsymbol{h}_k(\boldsymbol{y}) \equiv \boldsymbol{y} + \sum_{k=2}^{N} \boldsymbol{h}_k \boldsymbol{y}^k \tag{4}$$

such that the resulting system

$$\dot{\boldsymbol{y}} = J\boldsymbol{y} + \boldsymbol{g}(\boldsymbol{y}) \equiv J\boldsymbol{y} + \sum_{k=2}^{N} \boldsymbol{g}_k(\boldsymbol{y}) \equiv J\boldsymbol{y} + \sum_{k=2}^{N} \boldsymbol{g}_k \boldsymbol{y}^k \tag{5}$$

becomes as simple as possible. Here $\boldsymbol{h}_k(\boldsymbol{y}) \in (Q[\boldsymbol{h}_k][\boldsymbol{y}]_k)^n$ and $\boldsymbol{g}_k(\boldsymbol{y}) \in (Q[\boldsymbol{g}_k][\boldsymbol{y}]_k)^n$ denote the general $k$th degree homogeneous vector polynomials of $\boldsymbol{y}$ with the coefficients $\boldsymbol{h}_k$ and $\boldsymbol{g}_k$ to be determined.

To apply normal form theory, we define the linear vector space $\mathcal{H}_k$ which consists of the $k$th degree homogeneous vector polynomials $\boldsymbol{f_k}(\boldsymbol{x})$. Further define the homological operator $L_k$, induced by the linear vector $\boldsymbol{v_1}$, as

$$\begin{aligned} &L_k : \mathcal{H}_k \mapsto \mathcal{H}_k \\ &U_k \in \mathcal{H}_k \mapsto L_n(U_k) = [U_k, \boldsymbol{v_1}] \in \mathcal{H}_k, \end{aligned} \tag{6}$$

where the operator $[U_k, \boldsymbol{v_1}]$ is called the Lie bracket, defined by

$$[U_k, \boldsymbol{v_1}] = DU_k \cdot \boldsymbol{v_1} - D\boldsymbol{v_1} \cdot U_k, \tag{7}$$

where $D$ is a Frechét differential operator, and $D\boldsymbol{v_1} = J$.

Next, we define the space $\mathcal{R}_k$ as the range of $L_k$, and $\mathcal{K}_k$ as the complementary space of $\mathcal{R}_k$. Thus,

$$\mathcal{H}_k = \mathcal{R}_k \oplus \mathcal{K}_k, \tag{8}$$

and we can then choose the vector space bases for $\mathcal{R}_k$ and $\mathcal{K}_k$. Consequently, a homogeneous vector polynomial $\boldsymbol{f_k}(\boldsymbol{x}) \in \mathcal{H}_k$ can be split into two parts: one is spanned by the vector space basis of $\mathcal{R}_k$ and the other by that of $\mathcal{K}_k$.

By applying Takens normal form theory (Takens, 1974), one can find the $k$th order normal form $\boldsymbol{g}_k(\boldsymbol{y})$, while the part belonging to $\mathcal{R}_k$ can be removed by appropriately choosing the coefficients of the nonlinear transformation, $\boldsymbol{h}_k(\boldsymbol{y})$. The "form" of the normal form $\boldsymbol{g}_k(\boldsymbol{y})$ depends upon the vector space basis of the complementary space $\mathcal{K}_k$, which is determined by the linear vector $\boldsymbol{v}_1$. We may apply the matrix method (Guckenheimer and Holmes, 1993) to find the vector space basis of $\mathcal{R}_k$ and then determine the basis of the complementary space $\mathcal{K}_k$. Once the vector space basis of $\mathcal{K}_k$ is chosen, the form of $\boldsymbol{g}_k(\boldsymbol{y})$ can be determined. The idea of further reduction of the conventional normal form is to find an appropriate $\boldsymbol{h}_k(\boldsymbol{y})$ such that some coefficients of $\boldsymbol{g}_k(\boldsymbol{y})$ can be eliminated, leading to the simplest normal form.

Once the "form" of the normal form is determined, in order to find the explicit expression of the conventional normal form or the simplest normal form, in general one needs to use Eqs. (1) and (4) to find a set of algebraic equations at each order. Suppose the normal form and associated nonlinear transformation have been obtained up to $(k-1)$ order, we want to find the $k$th order normal form. To do this, usually one may assume a general form for the $k$th order nonlinear transformation and substitute it back to the original system (1). Then with the aid of the obtained normal form one can derive the $k$th order algebraic equations by balancing the coefficients of the homogeneous polynomial terms. From this way, the solution procedure generates the expressions which contain not only lower order terms, but also higher order terms. This dramatically increases the time and space complexity of the computation. Therefore, a crucial step in the computation of the simplest normal form is to derive the $k$th order algebraic equations as simply as possible, i.e. only the $k$th order nonlinear terms should be calculated.

The following theorem gives an efficient recursive formula for computing the $k$th order algebraic equations, which can be used to determine the $k$th order normal form and associated nonlinear transformation for any kind of singularity.

**Theorem 1.** *The recursive formula for computing the* $k$th *order algebraic equations is given by*

$$\begin{aligned}\boldsymbol{g}_k = \boldsymbol{f}_k + [\boldsymbol{h}_k, \boldsymbol{v}_1] + \sum_{i=2}^{k-1} \{[\boldsymbol{h}_{k-i+1}, \boldsymbol{f}_i] + D\boldsymbol{h}_i(\boldsymbol{f}_{k-i+1} - \boldsymbol{g}_{k-i+1})\} \\ + \sum_{m=2}^{[\frac{k}{2}]} \sum_{i=m}^{k-m} D^m \boldsymbol{f}_i \sum_{\substack{q_1 l_1 + q_2 l_2 + \cdots + q_p l_p = k-(i-m) \\ 2 \le l_p < l_{p-1} < \cdots l_1 \le (k-(i-m))/m}} \frac{\boldsymbol{h}_{l_1}^{q_1} \boldsymbol{h}_{l_2}^{q_2} \cdots \boldsymbol{h}_{l_p}^{q_p}}{q_1! q_2! \cdots q_p!},\end{aligned} \tag{9}$$

*where* $k = 2, 3, \ldots,$ *and* $\boldsymbol{f}_k$, $\boldsymbol{h}_k$ *and* $\boldsymbol{g}_k$ *are the* $k$th *degree homogeneous vector polynomials of* $\boldsymbol{y}$ *(where* $\boldsymbol{y}$ *has been dropped for simplicity).*

**Notes.** The notation $D^m \boldsymbol{f}_i$ denotes the $m$th order terms of the Taylor expansion of $\boldsymbol{f}_i(\boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y}))$ about $\boldsymbol{y}$. More precisely,

$$D^m \boldsymbol{f}_i(\boldsymbol{y} + \boldsymbol{h}) = D(D(\ldots D((D\boldsymbol{f}_i)\boldsymbol{h}_{l_1})\boldsymbol{h}_{l_2}) \cdots \boldsymbol{h}_{l_{m-1}})\boldsymbol{h}_{l_m}, \tag{10}$$

where each differential operator $D$ affects only function $\boldsymbol{f}_i$, not $\boldsymbol{h}_{l_j}$ (i.e. $\boldsymbol{h}_{l_j}$ is treated as a constant vector in the process of the differentiation), and thus $m \leq i$. At each level of the differentiation, the Frechét derivative operator, $D$, results in a matrix, which is multiplied with a vector to generate another vector, and then to another level of Frechét derivative, and so on.

The proof of Theorem 1 can follow a similar proof given by Yu and Yuan (2003) and thus only the main steps are outlined below: first differentiate Eq. (4) and then substitute Eqs. (1) and (5) into the resulting equation, and then apply Eq. (4) again and finally employ Taylor expansion about $\boldsymbol{y}$ to obtain

$$\sum_{i=2}^{\infty} \boldsymbol{g}_i(\boldsymbol{y}) = \sum_{i=2}^{\infty} \boldsymbol{f}_i(\boldsymbol{y}) + \sum_{i=2}^{\infty}[\boldsymbol{h}_i(\boldsymbol{y}), \boldsymbol{v}_1(\boldsymbol{y})] + \sum_{i=2}^{\infty}\sum_{j=2}^{\infty} D\boldsymbol{h}_j(\boldsymbol{y})\{\boldsymbol{f}_i(\boldsymbol{y}) - \boldsymbol{g}_i(\boldsymbol{y})\}$$
$$+ \sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\{D\boldsymbol{f}_i(\boldsymbol{y})\boldsymbol{h}_j(\boldsymbol{y}) - D\boldsymbol{h}_j(\boldsymbol{y})\boldsymbol{f}_i(\boldsymbol{y})\} + \boldsymbol{T}_f, \tag{11}$$

where

$$\boldsymbol{T}_f = \sum_{i=2}^{\infty}\sum_{j=k}^{\infty} \frac{1}{k!} D^k \boldsymbol{f}_j(\boldsymbol{y})\boldsymbol{h}^j(\boldsymbol{y}). \tag{12}$$

It is easy to find the formulas for the 2nd, 3rd and 4th order equations as follows:

$$\begin{aligned}
\boldsymbol{g}_2 &= \boldsymbol{f}_2 + [\boldsymbol{h}_2, \boldsymbol{v}_1],\\
\boldsymbol{g}_3 &= \boldsymbol{f}_3 + [\boldsymbol{h}_3, \boldsymbol{v}_1] + [\boldsymbol{h}_2, \boldsymbol{f}_2] + D\boldsymbol{h}_2(\boldsymbol{f}_2 - \boldsymbol{g}_2),\\
\boldsymbol{g}_4 &= \boldsymbol{f}_4 + [\boldsymbol{h}_4, \boldsymbol{v}_1] + [\boldsymbol{h}_3, \boldsymbol{f}_2] + [\boldsymbol{h}_2, \boldsymbol{f}_3]\\
&\quad + D\boldsymbol{h}_2(\boldsymbol{f}_3 - \boldsymbol{g}_3) + D\boldsymbol{h}_3(\boldsymbol{f}_2 - \boldsymbol{g}_2) + \tfrac{1}{2}D^2\boldsymbol{f}_2\boldsymbol{h}_2^2.
\end{aligned} \tag{13}$$

For $k \geq 5$, one needs to carefully consider $\boldsymbol{T}_f$ and separate the $k$th order terms, which finally leads to Eq. (9). Note that $\boldsymbol{g}_k \in Q(\boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_k, \boldsymbol{h}_2, \boldsymbol{h}_3, \ldots, \boldsymbol{h}_k)$.

## 3. The simplest normal form for the Takens–Bogdanov dynamical singularity

In this section, we consider the Takens–Bogdanov dynamical singularity and derive the general formula for computing the simplest normal form. For simplicity, we may choose the system described on a 2-dimensional center manifold, given by the equations:

$$\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_1, x_2),\\
\dot{x}_2 &= f_2(x_1, x_2),
\end{aligned} \tag{14}$$

where $f_1, f_2 \in C^{\infty}$, which vanish, together with their first derivatives, at the origin. Note that if the system is not given in the 2-dimensional center manifold, but in the form of Eq. (1), one may first apply center manifold theory or normal form theory to obtain either the 2-dimensional center manifold (14) or the conventional normal form (3). A more sophisticated approach is to directly compute the simplest normal form from the original system (1). We will not discuss such an approach here, but the idea of the method can be found in Yu (2003).

The vector field of system (14) can be written as

$$\boldsymbol{v} = (x_2 + f_1(x_1, x_2))\partial_{x_1} + f_2(x_1, x_2)\partial_{x_2}, \tag{15}$$

and the homological operator is defined in Eq. (6), where the linear part $\boldsymbol{v}_1$ now becomes $\boldsymbol{v}_1 = (x_2, 0)^T$.

To obtain the explicit formulas, we may find the vector space basis:

$$\{x_1^{k-1}x_2\partial_{x_1}, \ldots, x_2^k\partial_{x_1}, -x_1^k\partial_{x_1} + kx_1^{k-1}x_2\partial_{x_2}, x_1^{k-2}x_2^2\partial_{x_2}, \ldots, x_2^k\partial_{x_2}\} \tag{16}$$

for $R_k$, and that:

$$\{x_1x_2^{k-1}\partial_{x_1} + x_2^k\partial_{x_2}, x_2^k\partial_{x_2}\} \tag{17}$$

for $K_k$. However, we may use a more convenient vector space basis for the complementary space to $R_k$, denoted by $C_k$ which is spanned by

$$\{x_1^k\partial_{x_2}, x_1^{k-1}x_2\partial_{x_2}\}. \tag{18}$$

Thus the $k$th order conventional normal form, $\boldsymbol{g}_k(\boldsymbol{y})$, can be assumed in the form of

$$\boldsymbol{g}_k(\boldsymbol{y}) = \begin{pmatrix} 0 \\ g_{2k0}y_1^k + g_{2(k-1)1}y_1^{k-1}y_2 \end{pmatrix}, \tag{19}$$

where $g_{2k0}$ and $g_{2(k-1)1}$ are two coefficients to be determined. For the conventional normal form, these two coefficients are generally non-zero and retained in the normal form. In the further reduction of the conventional normal form leading to the simplest normal form, we try to use the coefficients of nonlinear transformation to eliminate as many as possible of the $g$ coefficients.

Now we shall use the formulas given in the previous section and the idea stated above to compute the simplest normal form for the Takens–Bogdanov dynamical singularity. First, let the general forms of $\boldsymbol{f}_k$ and $\boldsymbol{h}_k$ be given respectively by

$$\boldsymbol{f}_k(\boldsymbol{y}) = \begin{pmatrix} a_{1k0}y_1^k + a_{1(k-1)1}y_1^{k-1}y_2 + \cdots + a_{11(k-1)}y_1y_2^{k-1} + a_{10k}y_2^k \\ a_{2k0}y_1^k + a_{2(k-1)1}y_1^{k-1}y_2 + \cdots + a_{21(k-1)}y_1y_2^{k-1} + a_{20k}y_2^k \end{pmatrix}, \tag{20}$$

and

$$\boldsymbol{h}_k(\boldsymbol{y}) = \begin{pmatrix} h_{1k0}y_1^k + h_{1(k-1)1}y_1^{k-1}y_2 + \cdots + h_{11(k-1)}y_1y_2^{k-1} + h_{10k}y_2^k \\ h_{2k0}y_1^k + h_{2(k-1)1}y_1^{k-1}y_2 + \cdots + h_{21(k-1)}y_1y_2^{k-1} + h_{20k}y_2^k \end{pmatrix}. \tag{21}$$

Then for $k = 2$, applying the formula $\boldsymbol{g}_2 = \boldsymbol{f}_2 + [\boldsymbol{h}_2, \boldsymbol{v}_1]$ yields

$$\begin{aligned} g_{220} &= a_{220}, & g_{211} &= a_{211} + 2a_{120}, \\ h_{120} &= \tfrac{1}{2}(a_{111} + a_{202}), & h_{111} &= h_{202} + a_{102}, \\ h_{220} &= -a_{120}, & h_{211} &= a_{202}, \end{aligned} \tag{22}$$

which indicates that none of the two 2nd order $g$ coefficients can be eliminated. In other words, the 2nd order normal form cannot be simplified. It is also noted that the coefficients

$h_{102}$ (which does not appear in the equations) and $h_{202}$ are undetermined and may thus be used in high order equations to remove normal form coefficients $g_{2k0}$ and $g_{2(k-1)1}$.

Next consider $k = 3$. Similarly we can apply the formula $\boldsymbol{g}_3 = \boldsymbol{f}_3 + [\boldsymbol{h}_3, \boldsymbol{v}_1] + [\boldsymbol{h}_2, \boldsymbol{f}_2] + D\boldsymbol{h}_2(\boldsymbol{f}_2 - \boldsymbol{g}_2)$ to obtain eight algebraic equations. It is noted that six of the eight equations, which do not involve the two coefficients $g_{230}$ and $g_{221}$, can be used to determine six of the eight 3rd order $h$ coefficients:

$$\begin{aligned}
h_{230} &= -A_{130} + a_{220}h_{202}, \\
h_{221} &= \tfrac{1}{2}(A_{212} - 4a_{120}h_{202} + 2a_{220}h_{102}), \\
h_{212} &= a_{203} + 2a_{202}h_{202} + a_{211}h_{102}, \\
h_{130} &= \tfrac{1}{3}(A_{121} - a_{211}h_{202} - 2a_{220}h_{102}) + h_{221}, \\
h_{121} &= \tfrac{1}{2}\left[A_{112} + 2a_{111}h_{202} - 2(a_{120} + a_{211})h_{102}\right] + h_{212}, \\
h_{112} &= a_{103} + 2a_{102}h_{202} + a_{111}h_{102} + h_{203},
\end{aligned} \tag{23}$$

where $A_{ijk}$ are known coefficients related to the original system.

The remaining two equations, which may be called *key* equations and can be used to determine the normal form coefficients $g_{230}$ and $g_{221}$, are given as follows:

$$\begin{aligned}
&g_{230} - a_{230} - a_{111}a_{220} + a_{120}a_{211} = 0, \\
&g_{221} - a_{221} - 3a_{220}h_{202} + 3a_{130} - 5a_{102}a_{220} \\
&\qquad + 7a_{120}a_{202} - \tfrac{1}{2}(a_{111} + a_{202}) = 0.
\end{aligned} \tag{24}$$

The first equation of (24) indicates that $g_{230}$ must be retained in the normal form, given by

$$g_{230} = a_{230} + a_{111}a_{220} - a_{120}a_{211}. \tag{25}$$

On the other hand, the second equation of (24) suggests that one may set

$$g_{221} = 0, \tag{26}$$

under the condition $a_{220} \neq 0$, and then the 2nd order coefficient $h_{202}$ can be used to solve the equation, uniquely determined as

$$h_{202} = -\frac{1}{3a_{220}}\left[a_{221} - 3a_{130} + 5a_{102}a_{220} - 7a_{120}a_{202} + \frac{1}{2}(a_{111} + a_{202})\right]. \tag{27}$$

It is observed from the above procedure that the coefficient $h_{202}$ which is not determined in the 2nd order equation has been used to eliminate the 3rd order conventional normal form coefficient $g_{221}$. This clearly shows the basic idea of the simplest normal form computation: *lower order nonlinear transformation coefficients are used to eliminate higher order normal form coefficients*.

However, it is noted in the 3rd order equations that the 2nd order coefficient $h_{102}$ is not determined, and in addition, two 3rd order coefficients $h_{103}$ and $h_{203}$ are undetermined. It can be shown that $h_{102}$ will be used in the 4th order equation to remove the normal form coefficient $g_{231}$ under the condition $a_{211} + 2a_{120} \neq 0$. Further, the coefficient $h_{203}$ will be used to eliminate the 5th order normal form coefficient $g_{241}$, and so on.

For an arbitrary $k$th order equation, we want to use the $h$ coefficients which are not determined in lower order equations to eliminate the $k$th order normal form coefficients $g_{2k0}$ and $g_{2(k-1)1}$. Similarly applying Eq. (9) results in $2k+2$ linear algebraic equations, among which two equations do not involve the $k$th order $h$ coefficients but contain the two $g$ coefficients $g_{2k0}$ and $g_{2(k-)1}$ as well as some lower order $h$ coefficients. It can be shown that the lower order $h$ coefficients can be used to eliminate either one or both of the two $g$ coefficients. Under the assumption: $a_{220}(a_{211}+2a_{120}) \neq 0$, the general rule for choosing the nonlinear transformation coefficients $h_{10k}$ and $h_{20k}$ to eliminate the normal form coefficients $g_{2k0}$ and $g_{2(k-1)1}$ are given as follows (for proof see Yu and Yuan, 2003):

$$\begin{array}{lll} \text{For} & k=3, & h_{202} \Longrightarrow g_{221}=0, \\ \text{For} & k=3m+1, & h_{102m} \Longrightarrow g_{2(k-1)1}=0, \\ \text{For} & k=3m+2, & h_{20(2m+1)} \Longrightarrow g_{2(k-1)1}=0, \\ \text{For} & k=3m+3, & h_{20(2m+2)} \Longrightarrow g_{2(k-1)1}=0, \\ & & h_{10(2m+1)} \Longrightarrow g_{2k0}=0, \end{array} \tag{28}$$

where $m \geq 1$. The meaning of notation "$\Longrightarrow$" means "imply", for example, $h_{202} \Longrightarrow g_{221}=0$ indicates that $g_{221}$ can be set zero by appropriately choosing the coefficient $h_{202}$.

Once the two *key* equations are solved, the remaining $2k$ equations can be solved using the $2k$ $h$ coefficients as follows:

$$\begin{aligned} -h_{2k0} &= A_{1k0} + \alpha_{2k0} h_{20(k-1)} + \beta_{2k0} h_{10(k-1)}, \\ (k-j)h_{2(k-j)j} &= A_{2(k-j-1)(j+1)} + \alpha_{2(k-j)j} h_{20(k-1)} \\ &\quad + \beta_{2(k-j)j} h_{10(k-1)}, \\ (k-j+1)h_{1(k-j+1)(j-1)} - h_{2(k-j)j} &= A_{1(k-j)j} + \alpha_{1(k-j+1)(j-1)} h_{20(k-1)} \\ &\quad + \beta_{1(k-j+1)(j-1)} h_{10(k-1)}, \\ h_{11(k-1)} - h_{20k} &= A_{10k} + \alpha_{11(k-1)} h_{20(k-1)} + \beta_{11(k-1)} h_{10(k-1)}, \end{aligned} \tag{29}$$

where $j = 1, 2, \ldots, k-1$, and $A_{ijk}$ are known coefficients. Note that the first and the last equations of (29) are decoupled from the other $(2k-2)$ equations. The first equation can be used to solve $h_{2k0}$, while the last equation may be used to determine $h_{20k}$.

Summarizing the above results yields the following theorem.

**Theorem 2.** *The generic simplest normal form of system* (14) *for Takens–Bogdanov dynamical singularity up to an arbitrary order is given by*

$$\begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= a_{220} u_1^2 + (a_{211}+2a_{120}) u_1 u_2 + g_{230} u_1^3 \\ &\quad + \sum_{j=1}^{m} (g_{2(3j+1)0} + g_{2(3j+2)0} u_1) u_1^{3j+1}, \end{aligned} \tag{30}$$

*if* $a_{220}(a_{211}+2a_{120}) \neq 0$, *where the coefficients* $g_{2k0}$*'s are expressed explicitly in terms of the coefficients* $a_{ijk}$*'s of the original system* (14).

**Notes.** The simplest normal form given in the above theorem is for a general system described on a 2-dimensional center manifold, given by Eq. (14). However, in many cases the original system is given in the conventional normal form (3) in which only $a_{2k0}$ and $a_{2(k-1)1}$ are non-zero. This is a particular case of the general system (14). In this particular case, the condition required for the generic simplest normal form reduced to $a_{220}a_{211} \neq 0$, as expected (e.g. see Yuan and Yu, 2001).

It should be pointed out that the basic rule given in Eq. (28) is the same regardless of whether the general system (14) or the particular system (3) is used. This can be easily shown by using conventional normal form theory to transform system (14) into system (3) with a nonlinear transformation. In fact, we can find the following nonlinear transformation:

$$\begin{aligned} x_1 &= y_1 + \tfrac{1}{2}(a_{111} + a_{202})y_1^2 + a_{102}y_1y_2 \\ &\quad + \tfrac{1}{6}[a_{212} + 2a_{121} + a_{111}^2 + a_{202}(3a_{111} + 2a_{202}) - a_{102}(a_{211} + 4a_{120})]y_1^3 \\ &\quad + \tfrac{1}{2}[a_{112} + a_{203} + a_{102}(a_{111} + 2a_{202})]y_1^2y_2 + a_{103}y_1y_2^2 + \cdots \\ x_2 &= y_2 - a_{120}y_1^2 + a_{202}y_1y_2 - (a_{130} + a_{120}a_{202} - a_{102}a_{220})y_1^3 \\ &\quad + \tfrac{1}{2}(a_{212} + 2a_{202}^2 + a_{102}a_{211})y_1^2y_2 + a_{203}y_1y_2^2 + \cdots \end{aligned} \tag{31}$$

to transform system (14) into the following conventional normal form:

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= \tilde{a}_{220}y_1^2 + \tilde{a}_{211}y_1y_2 + \tilde{a}_{230}y_1^3 + \tilde{a}_{221}y_1^2y_2 + \tilde{a}_{240}y_1^4 + \tilde{a}_{231}y_1^3y_2 + \cdots \end{aligned} \tag{32}$$

which is in the form of (3), where $\tilde{a}_{ijk}$'s are explicitly given in terms of $a_{ijk}$' s. Thus the generic condition, $a_{220}(a_{211} + 2a_{120}) \neq 0$, required for system (14) becomes $\tilde{a}_{220}\tilde{a}_{211} \neq 0$ for the new system (32), as expected. If system (14) is given in the form of the conventional normal form (3), then $\tilde{a}_{2k0} = a_{2k0}$ and $\tilde{a}_{2(k-1)1} = a_{2(k-1)1}$. Therefore, the degenerate cases discussed on the basis of the conventional normal form (3) may be unlikely to occur for the general system (14) since the coefficients $\tilde{a}_{230}$, $\tilde{a}_{221}$, $\tilde{a}_{240}$, etc. are generally not zero if the function $f_1$ given in Eq. (14) is non-zero.

The above discussion is for the generic case. The same argument can be applied to non-generic cases, and thus the conclusion is true for any case. That is, considering systems (3) and (14) equivalent and gives the same rule for eliminating the $k$th order normal form coefficients $g_{2k0}$ and $g_{2(k-1)1}$ by using the $h$ nonlinear transformation coefficients.

## 4. The matching pursuit technique for computing the simplest normal form

In the previous section we have discussed the computation of the simplest normal form for the Takens–Bogdanov dynamical singularity and obtained the explicit formulas for computing the coefficients of the simplest normal form and the associated nonlinear transformation. However, the results are obtained under the assumption that $a_{220}(a_{211} + 2a_{120}) \neq 0$ when the system is described by the general equation (14), or $a_{220}a_{211} \neq 0$ if the system is given in the conventional normal form (3). As shown in the previous section, the rule for choosing the nonlinear transformation coefficients to eliminate the two normal form coefficients $g_{2k0}$ and $g_{2(k-1)1}$ is the same regardless of the type of the original system.

Therefore, without loss of generality, we will use Eq. (3) throughout this section for the convenience of discussion.

Although since 1984 many researchers studied the simplest normal form of the Takens–Bogdanov dynamical singularity, the problem is not completely solved. Not only because few results are obtained for computing the simplest normal form, but also because the analytical "form" for some special cases are not found. Even suppose one can classify all sub-cases and find all of the analytical "forms", there still exists the non-algebraic number problem (Wang et al., 2000). Roughly speaking, some non-algebraic number conditions must be satisfied at certain order equations to make the equations solvable. Unfortunately, such non-algebraic number conditions are not predictable. In other words, unless the simplest normal form is explicitly computed, it is impossible to find or determine the non-algebraic number conditions. Therefore, no matter what methods are used, there always exist unsolvable special cases if certain non-algebraic number conditions are not assumed appropriately.

The computation approaches recently developed (e.g. see Algaba et al., 2001; Li et al., 2001; Yuan and Yu, 2001) are based on explicit analytical formulas. Thus only the cases for which the explicit formulas have been obtained are computable. Even for the limited cases, the non-algebraic number problem is not solved because the obtained formulas do not take account of this. Therefore, from the computational point of view, a natural question would arise: can we design a computational approach or an algorithm to solve the problem completely? More precisely, can we develop a program with the aid of computer algebra, which can be used to compute the simplest normal form of the Takens–Bogdanov dynamical singularity for a given general system without requiring any non-algebraic number conditions or assumptions? Fortunately, the answer is *yes*. The advantage for developing such algorithms is obvious: for a given system, one does not need to worry about what case it might be and one can always find the simplest normal form up to any desired order. The matching pursuit technique has been developed and "automatic" Maple programs have been coded. It has been shown that this approach is indeed very powerful, and many systems have been tested to give correct results. Unlike many other programs which depend upon explicit formulas, this algorithm does not need to specify cases in the input file and is very convenient for users. Therefore, this matching pursuit technique has completely solved the problem of computing the simplest normal form for the Takens–Bogdanov dynamical singularity.

### 4.1. *The matching pursuit technique*

Now we turn to discuss the matching pursuit technique. The basic idea of the technique is based on the following observation: both the non-algebraic number problem and the necessity for Baider and Sanders to classify the three cases are due to the same cause. Recall that the computation of the $k$th order simplest normal form of the Takens–Bogdanov dynamical singularity (described in the previous section) is to use the lower order $\boldsymbol{h}$ coefficients to eliminate the two $k$th order $\boldsymbol{g}$ coefficients, ($g_{2k0}$ and $g_{2(k-1)1}$). Further, note that there are only two *key* equations at each order which contain the two $\boldsymbol{g}$ coefficients. So the further reduction leading to the simplest normal form can be achieved by using the $\boldsymbol{h}$ coefficients involved in the two *key* equations to remove as many of the $k$th order $\boldsymbol{g}$ coefficients as possible. In the generic case, under the basic assumption $a_{220}(a_{211} + 2a_{120}) \neq 0$ (with no extra non-algebraic number conditions), the rule of choosing the $h$ coefficients is given in Eq. (28). It is shown that starting from the 3rd order at least $g_{2(k-1)1}$ can be removed, and for order $k = 3m + 3$, both the two $k$th order $g$ coefficients can be eliminated. The basic assumption becomes clear in the following discussion. When we determine one $h$ coefficient from a *key* equation, we actually solve a linear algebraic equation for the $h$ coefficient. It is thus obvious that the linear equation is solvable as long as the coefficient of the $h$ variable is non-zero, which generates the non-algebraic number conditions. For example, consider the second equation of (24), which contains $g_{221}$ and $-3a_{220}h_{202}$ terms. Hence, if $a_{220} \neq 0$, we can set $g_{221} = 0$ and then uniquely determine $h_{202}$. That is why we need to assume $a_{220} \neq 0$ for the generic case. The second condition $a_{211} + 2a_{120} \neq 0$ comes from one of the 4th order *key* equations. For simplicity, instead of Eq. (14), we use Eq. (3) in the following analysis. Then the second condition becomes $a_{211} \neq 0$ and the *key* equation is of the form:

$$g_{231} - a_{231} + \frac{4}{3}a_{220}a_{211}h_{102} + \frac{a_{221}}{9a_{220}}(9a_{230} + a_{211}^2) = 0, \tag{33}$$

which clearly shows that as long as the coefficient of $h_{102}$ is non-zero, i.e. $a_{220}a_{211} \neq 0$, we can set $g_{231} = 0$ and uniquely determine $h_{102}$, as the rule given in Eq. (28) shows.

Further it can be shown that for the generic case the only condition required is $a_{220}a_{211} \neq 0$ (remember that we are now using Eq. (3)) no other non-algebraic number conditions are required. In other words, under the assumption $a_{220}a_{211} \neq 0$, all the $h$ coefficients can be uniquely determined to remove the $g$ coefficients by following the rule given in Eq. (28). However, this is not always true, i.e. when the basic condition, $a_{220}a_{211} \neq 0$, does not hold, some extra non-algebraic number conditions must be satisfied. For example, consider $a_{220} = 0$, but $a_{230} \neq 0$ and $a_{211} \neq 0$. Here, $\mu = 2$ and $\nu = 1$, so it belongs to case (III) $\mu = 2\nu$. Then the rule given in Eq. (28) cannot be followed. The 2nd and 3rd order equations show that $g_{220} = 0, g_{211} = a_{211}, g_{230} = a_{230}, g_{211} = a_{211}$. Compared with Eq. (33), this *key* equation at the 4th order becomes

$$g_{231} - a_{231} + \tfrac{1}{3}(9a_{230} + a_{211}^2)h_{202} = 0 \tag{34}$$

which indicates that if $9a_{230} + a_{211}^2 \neq 0$, then one can set $g_{231} = 0$ to uniquely determine $h_{202}$ (note that here it is $h_{202}$, *not* $h_{102}$ like the generic case). Further, one of the 5th order *key* equations is found to be

$$g_{241} - a_{241} + \frac{5}{4}a_{211}a_{230}h_{102} + \frac{a_{231}(18a_{240} + 5a_{211}a_{221})}{2(9a_{230} + a_{211}^2)} = 0 \tag{35}$$

which implies that in order to set $g_{241} = 0$ by choosing $h_{102}$, one needs $a_{211}a_{230} \neq 0$, in addition to $9a_{230} + a_{211}^2 \neq 0$. Therefore, this case (when $a_{220} = 0$) not only requires the basic assumption $a_{211}a_{230} \neq 0$, but it also needs the non-algebraic number condition $9a_{230} + a_{211}^2 \neq 0$ at the 4th order. In fact, it can be shown using the program developed by Yuan and Yu (2001) that more non-algebraic number conditions need to be satisfied at higher orders (see Example 4 in the next section).

In general, for the $k$th order equation we may find a set of algebraic equations, written in the matrix form:

$$\left[\begin{array}{cccccc|cccccc}
0 & & & & 0 & -1 & & & & & \\
k & & & & 0 & & -1 & & & & \\
 & k\text{-}1 & & & 0 & & & -1 & & & \\
 & & \ddots & & \vdots & & & & \ddots & & \\
 & & & 1 & 0 & & & & & & -1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & k & & & & & \\
 & & & & 0 & & k\text{-}1 & & & & \\
 & & & & 0 & & & k\text{-}2 & & & \\
 & & & & \vdots & & & & \ddots & & \\
 & & & & 0 & & & & & 1 & 0
\end{array}\right]
\begin{pmatrix} h_{1k0} \\ h_{1(k-1)1} \\ h_{1(k-2)2} \\ \vdots \\ h_{10k} \\ \hline h_{2k0} \\ h_{2(k-1)1} \\ h_{2(k-2)2} \\ h_{2(k-3)3} \\ \vdots \\ h_{20k} \end{pmatrix} = \boldsymbol{w}, \tag{36}$$

where the $2(k+1)$-dimensional vector $\boldsymbol{w}$ contains the undetermined $h$ coefficients, one or two of them are solved at the current order, while others will be determined in higher order equations. It is seen from Eq. (36) that the coefficient $h_{2k0}$ can be solved first from the first equation. Note that the coefficient $h_{10k}$ does not appear in the equations, while $h_{20k}$ is only involved in the $(k+1)$th equation and can thus be chosen arbitrarily. The two *key* equations are the $(k+2)$th and $(k+3)$th equations which contain the two coefficients $g_{2k0}$ and $b_{2(k-1)1}$. The remaining $(2k-2)$ equations can be used to determine the remaining $(2k-2)$ $h$ coefficients: $h_{1k0}, h_{1(k-1)1}, \ldots, h_{12(k-2)}$ and $h_{2(k-1)1}, h_{2(k-2)2}, \ldots, h_{21(k-1)}$.

Summarizing the above discussions gives the following theorem.

**Theorem 3.** *The rule for choosing the nonlinear transformation coefficients, $h$, to eliminate the normal form coefficients, $g$, is determined by the two* key *equations. The solvable non-algebraic number conditions are determined by the coefficients of the $h$ variables which are involved in the two* key *equations.*

It should be noted that the conditions determined by the coefficients of the $h$ variables include not only the non-algebraic number conditions, but also the simple conditions (in terms of $a_{2k0}$ and $a_{2(k-1)1}$) for classifying the three cases due to Baider and Sanders (1992). So strictly speaking, there is no difference between the simple classifying conditions and the non-algebraic number conditions, and thus it is not necessary to consider the non-algebraic number conditions separately. Since, as discussed before, the non-algebraic number conditions are not predictable, the classification to the three cases (Baider and Sanders, 1992) is not enough and there should exist infinite sub-cases. However, it becomes quite simple when considering the problem from the computational point of view. For a given system, suppose the vector field of the system is explicitly given, then at each order one only needs to investigate the $h$ coefficients involved in the two *key* equations. It is straightforward to use the $h$ coefficients to possibly remove the two $g$ coefficients $g_{2k0}$ and $g_{2(k-1)1}$.

Now the only remaining problem is: when a degenerate case occurs (i.e. when some non-algebraic number condition is not satisfied), some $h$ coefficient is not present and will appear in high order equations, how can we determine when this $h$ coefficient becomes useless? In general, if one of the $h$ coefficients is not used at the current order, it may be used later in higher order equations. However, this $h$ coefficient may become nonlinear as the order of the equations increases. Here we assume to obey the same rule in computing normal forms: At each order, we only solve *linear* algebraic equations with respect to the $h$ variables. Therefore, one can establish a rule for discarding an $h$ coefficient: once an $h$ coefficient appears in higher order equations and becomes at least quadratic, set it to zero.

By summarizing the above discussion, we can establish the rules for using the matching pursuit technique to find the $k$th order simplest normal form for the Takens–Bogdanov dynamical singularity as follows.

(1) First solve $h_{2k0}$ from the first equation given in Eq. (36) since the result may contain the lower order $h$ coefficients which may be used at the current order.
(2) Solve the $(k+2)$th and $(k+3)$th equations of (36) using $h$ coefficients linearly to possibly remove $g_{2k0}$ and $g_{2(k-1)1}$.
(3) If a lower order $h$ coefficient is not present in lower order equations but appears in higher order equations due to a degenerate condition (i.e. a non-algebraic number condition is not satisfied), then carry it over until either (i) it can be used to linearly solve a higher order equation, or (ii) it can be set to zero if it becomes nonlinear.

Note that the above rules are applicable for a given explicitly described system. For a system not described numerically but in symbolic notations, it is usually assumed that all the unknown non-algebraic number conditions are satisfied. That is, one may assume that any algebraic expressions on denominators are non-zero so that the "zero division" problem is avoided.

### 4.2. Outline of the matching pursuit technique algorithm

It is straightforward to follow the discussion and the established rules given above to design an algorithm using computer algebra systems. In fact, Maple has been used to develop programs for computing the simplest normal form of a given vector field associated with the Takens–Bogdanov dynamical singularity. They can be conveniently executed on various computer systems and only require a minimum preparation for an input from a user.

Input: The input gives an index, *CASE*, for classifying irrational numbers, the order, Ord, for the computation of the simplest normal form, and the original differential equations given in homogeneous polynomials. The reason for defining *CASE* to identify irrational numbers is that more careful treatment should be taken when arithmetic operations involve irrational numbers. In particular, rationalization must be performed whenever an expression involves irrational numbers on its denominator. Other steps are outlined below.

(A) For a sub-order $k (2 \le k \le \text{Ord})$, compute the algebraic equations using the efficient method.

  (a) Build the procedures for computing the Lie bracket, vector multiplication and equation solver.
  (b) Separate the original different equations to obtain homogeneous vector polynomials. Set general forms for the $k$th order nonlinear transformation and normal form.

(c) Use the recursive formula (9) to find the $k$th order equation which only contains the $k$th order terms. The variable COF is used to transfer non-definite multiple loops to single loops so the searching scheme can be handled by the regular program routines.
(d) Get the coefficients of the monomials from the $k$th order equation, which consists of the $k$th order algebraic equations.

(B) Call the subroutine for computing the simplest normal form of the Takens–Bogdanov dynamical singularity. For a sub-order $k(2 \le k \le \text{Ord})$, recursively calculate the coefficients of the simplest normal form and the corresponding nonlinear transformation.

(a) Build several procedures for computing the index and solving the two *key* equations. *Index*1 and *Index*2 are used to record the relation between $h_{10p}$ and $h_{20q}(p, q \le \text{Ord})$ as well as the number of $h_{10p}$'s and $h_{20q}$'s which have been used.
(b) Set the two key coefficients $h_{20k} = s_{2k-3}$ and $h_{10k} = s_{2k-2}$ for a consistent identifying process. *Control_no* is a counter to record the number of $s$ coefficients which have been used.
(c) Solve the equation for the initial order ($k = 2$), and find the 2nd order normal form coefficients, $g_{21}$ and $g_{22}$. (Note: The notations $g_{2k0}$ and $g_{2(k-1)1}$ used in the text are replaced by $g_{k1}$ and $g_{k2}$ respectively, in the Maple program for convenience.)
(d) For a sub-order $3 \le k \le \text{Ord}$, get the coefficients of $s_m$'s from the expressions $cof_{2k0}$ and $cof_{2(k-1)1}$.
(e) Classify the cases based on the information obtained in (d), solve the $s$ coefficients and determine whether or not to carry the unsolved $s$ coefficients to higher order equations.
(f) Determine the rule to eliminate $g_{k1}$ and $g_{k2}$.
(g) Call the procedure to solve the $k$th order non-key nonlinear transformation coefficients, $h_{ijk}$.

Output: The simplest normal form is expressed in polynomials which contain minimum terms with coefficients given in rational functions of the original coefficients of $a_{ik}$'s.

The Maple source code and a sample input can be downloaded from the website: *http://pyu1.apmaths.uwo.ca/~pyu/pub/preprints*. (The file names are *matching_maple* and *matching_input*.)

## 5. Examples

In this section we shall present several examples for the computation of the simplest normal form using the matching pursuit technique and the Maple programs developed in this paper. The first example shows the computation starting from original $n$-dimensional differential equation, while others are based on a general conventional normal form. In particular, it is shown that unlike other theory or methods which require certain non-algebraic number conditions, our matching pursuit technique and the Maple program do not have any limitations. In principle, the Maple program can be used to compute the simplest normal form of the Takens–Bogdanov dynamical singularity up to any order. However, in practice, due to limitations of computer memory, it always stops at a certain order. The results given in this paper are up to the 12th order. It should be pointed out that our program computes not only the simplest normal form but also the associated nonlinear transformation. Also it is noted that the Maple program can treat both numerical (rational or irrational) numbers and symbolic notations. The following examples use numerical numbers (but still handle them symbolically) for the convenience of presenting higher order results.

In the following computations, if the original system is described by Eq. (1) we shall first use normal form theory to find the conventional normal form given in form (3), and then apply the results presented in the previous sections to obtain the simplest normal form. If the original system is already given in the conventional normal form (3), then the formulas and programs developed in this paper are directly employed to find the simplest normal form. Five examples are present in this section.

### 5.1. Example 1

Consider the following 6-dimensional differential equation, given by

$$\begin{aligned}
\dot{x}_1 &= x_2 + x_1^2 + 5x_2x_3x_4 - x_3^2 + \tfrac{1}{3}x_2^3,\\
\dot{x}_2 &= 2x_2x_3 + \tfrac{3}{7}x_3x_5 + \tfrac{1}{2}, x_4^2 - 11x_1x_5,\\
\dot{x}_3 &= -\tfrac{2}{7}x_3 + \tfrac{2}{3}x_2x_4 + \tfrac{1}{2}x_5^2,\\
\dot{x}_4 &= -\tfrac{1}{3}x_4 + 11x_1x_6 + 7x_4^2,\\
\dot{x}_5 &= -5x_5 + x_6 + x_2x_3 + \tfrac{1}{3}x_6x_4x_5,\\
\dot{x}_6 &= -x_5 - 5x_6 + \tfrac{3}{5}x_2^2 + \tfrac{1}{11}x_1x_3.
\end{aligned} \tag{37}$$

The Jacobian of the system evaluated at the equilibrium $x_i = 0$ is in Jordan canonical form, having a double zero eigenvalue, $\lambda_1 = \lambda_2 = 0$, two real eigenvalues, $\lambda_3 = -\frac{2}{7}$ and $\lambda_4 = -\frac{1}{3}$, and a complex conjugate eigenvalue, $\lambda_{5,6} = -5 \pm i$. The conventional normal form of system (37) can be found by using the Maple program developed by Bi and Yu (1999) as follows (up to 12th order):

$$\begin{aligned}
\dot{y}_1 &= y_2,\\
\dot{y}_2 &= 2y_1y_2 + \frac{11}{26}y_1^3y_2 - \frac{33}{130}y_1^5 + \frac{115961}{338000}y_1^5y_2 + \frac{363}{16900}y_1^7 - \frac{7381}{54925}y_1^6y_2\\
&\quad + \frac{1089}{21970}y_1^8 - \frac{2787053907}{45697600}y_1^7y_2 + \frac{39599857}{2197000}y_1^9 + \frac{165961642011}{4158481600}y_1^8y_2\\
&\quad - \frac{1320167799}{114244000}y_1^{10} + \frac{291338703339460741}{417036297600000}y_1^9y_2 - \frac{2197367304}{1160290625}y_1^{11}\\
&\quad - \frac{6117027761700617401}{527087542800000}y_1^{10}y_2 - \frac{190417469981733}{5406026080000}y_1^{12}\\
&\quad + \frac{6339649201319919511132899}{5168469848256000000}y_1^{11}y_2.
\end{aligned} \tag{38}$$

The coefficients given in the above equation can be written in the form of $a_{2j0}$ and $a_{2(j-1)1}$ according to formula (3). By noting that $a_{220} = a_{230} = a_{240} = 0$, $a_{250} \neq 0$, $a_{211} \neq 0$, we know that this is a non-generic case. According to the notation of Baider and Sanders (1992), this belongs to case (II) $\mu > 2\nu$. For this example, $\mu = 4$, $\nu = 1$. Executing our Maple program based on the matching pursuit technique yields the simplest normal form:

$$\begin{aligned}
\dot{u}_1 &= u_2,\\
\dot{u}_2 &= 2u_1u_2 - \frac{33}{130}u_1^5 + \frac{1089}{8450}u_1^7 + \frac{1089}{21970}u_1^8 + \frac{19730051}{1098500}u_1^9\\
&\quad - \frac{9304535517}{799708000}u_1^{10} - \frac{276693133299}{29703440000}u_1^{11} - \frac{14527959542023}{1351506520000}u_1^{12}.
\end{aligned} \tag{39}$$

## 5.2. Example 2

In the previous example, although the original system is a general $n$-dimensional system ($n > 2$), one first needs to use a method to find the conventional normal form on the 2-dimensional center manifold, and then apply the approach developed in this paper to find the simplest normal form from the conventional normal form. Note that with the approach developed in this paper, one does not require the equations to be described on the center manifold to be given in the conventional normal form. For an example, consider the following system with randomly chosen coefficients up to 12th degree homogeneous polynomial:

$$\begin{aligned}
\dot{x}_1 &= x_2 + x_1^2 + \tfrac{1}{2}x_1x_2 + 2x_2^2 + 2x_1^3 + \tfrac{1}{7}x_1^2x_2 + \tfrac{5}{3}x_1x_2^2 + \tfrac{1}{2}x_2^3 + 5x_1^4 + \tfrac{1}{3}x_1^3x_2\\
&\quad - 15x_1^2x_2^2 + \tfrac{7}{3}x_1x_2^3 + 2x_2^4 - 2x_1^5 + 5x_1^4x_2 + \tfrac{1}{4}x_1^3x_2^2 + x_1^2x_2^3 + \tfrac{7}{4}x_1x_2^4 + 20x_2^5\\
&\quad + \cdots\\
\dot{x}_2 &= 3x_1^2 + \tfrac{1}{4}x_1x_2 + 5x_2^2 + \tfrac{2}{5}x_1^3 + 3x_1^2x_2 + 10x_1x_2^2 + \tfrac{4}{7}x_2^3 + \tfrac{4}{3}x_1^4 - \tfrac{2}{3}x_1^3x_2\\
&\quad + 10x_1^2x_2^2 + 3x_1x_2^3 + x_2^4 + 7x_1^5 - \tfrac{3}{5}x_1^4x_2 + 7x_1^2x_2^3 + \tfrac{3}{4}x_1x_2^4 + \tfrac{1}{8}x_2^5\\
&\quad + \cdots
\end{aligned} \tag{40}$$

The complete description of the above equation can be found from the input given in *http://pyu1.apmaths.uwo.ca/~pyu/pub/preprints*. (The file name is *matching_input*.) Executing the Maple program takes only about a few seconds on a PC to obtain the following simplest normal form:

$$\begin{aligned}
\dot{u}_1 &= u_2,\\
\dot{u}_2 &= 3u_1^2 + \frac{9}{4}u_1u_2 + \frac{33}{20}u_1^3 + \frac{7330723}{134400}u_1^3u_2\\
&\quad + \frac{27908277}{256000}u_1^4u_2 + \frac{4028573967382003}{3612672000000}u_1^6u_2\\
&\quad - \frac{611689589037424603663387}{682795008000000000}u_1^7u_2
\end{aligned} \tag{41}$$

$$
- \frac{2136699101955403817686368261611713}{569811028049657856000000000} u_1^9 u_2
- \frac{1264850044225914971746326926326209573613}{50143370468369891328000000000000} u_1^{10} u_2 .
$$

In the next two examples, the computation of the simplest normal form is based on the following general conventional normal form, say, up to 12th order:

$$
\begin{aligned}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= a_{220} y_1^2 + a_{211} y_1 y_2 + a_{230} y_1^3 + a_{221} y_1^2 y_2 + \cdots + a_{2120} y_1^{12} + a_{2111} y_1^{11} y_2 .
\end{aligned}
\tag{42}
$$

## 5.3. *Example 3*

First consider $\mu = 1, \nu = 2$, which, according to the classification, satisfies $\mu < 2\nu$. This implies that $a_{211} = 0$, $a_{220} \neq 0$, $a_{221} \neq 0$. Li et al. (2001) have computed the simplest normal form for this case and shown that the following non-algebraic number condition:

$$
183 a_{230} (a_{230} a_{221} - a_{220} a_{231}) + 110 a_{220} (a_{220} a_{241} - a_{240} a_{221}) \neq 0 \tag{43}
$$

must be satisfied. In fact, we can show that this condition is not required until the 9th order.

Now suppose that condition (43) is satisfied, then one may use either the Maple program developed by Yuan and Yu (2001) or the program developed based on the matching pursuit technique to find the following explicit expressions for the coefficients of the simplest normal form (only the non-zero coefficients are listed):

$$
\begin{aligned}
g_{220} &= a_{220}, \\
g_{221} &= a_{221}, \\
g_{231} &= a_{231} - \frac{a_{230} a_{221}}{a_{220}}, \\
g_{241} &= a_{241} - \frac{a_{240} a_{221}}{a_{220}}, \\
g_{251} &= a_{260} - \frac{1330 a_{230} a_{250} + 560 a_{240}^2 + 85 a_{230} a_{221}^2 - 50 a_{220} a_{221} a_{231} - \frac{2268 a_{230}^2 a_{240}}{a_{220}}}{500 a_{220}}, \\
g_{261} &= a_{261} - \frac{28 a_{241} a_{240} + 35 a_{230} a_{251} + 12 a_{250} a_{231} + 20 a_{221} a_{260} + 4 a_{221}^2 a_{231}}{20 a_{220}} \\
&\quad - \frac{231 a_{240} a_{230} (a_{230} a_{221} - a_{231} a_{220}) - 5 a_{220} a_{221} (4 a_{221}^2 a_{230} + 28 a_{240}^2 + 47 a_{250} a_{230})}{100 a_{220}^3}, \\
&\ \ \vdots
\end{aligned}
\tag{44}
$$

However, if condition (43) is not held, for example, let

$$
a_{231} = \frac{a_{230} a_{221}}{a_{220}} + \frac{110 (a_{220} a_{241} - a_{240} a_{221})}{183 a_{230}},
$$

then the Maple program given in Yuan and Yu (2001) will experience a "zero division" problem when it is executed up to the 9th order. The Maple program using the matching pursuit technique can overcome this difficulty and produce the unique simplest normal form. To demonstrate this and avoid massive expressions, we use the following numerical conventional normal form:

$$\begin{aligned}\dot{y}_1 &= y_2, \\ \dot{y}_2 &= a_{220}y_1^2 + a_{211}y_1y_2 + a_{230}y_1^3 + a_{221}y_1^2y_2 + g(y_1, y_2),\end{aligned} \tag{45}$$

where

$$\begin{aligned}g(y_1, y_2) = {} & y_1^4 + y_1^3y_2 + \tfrac{2}{3}y_1^5 + y_1^4y_2 + \tfrac{1}{2}y_1^6 + \tfrac{1}{2}y_1^5y_2 + 5y_1^7 + 2y_1^6y_2 + 7y_1^8 \\ & + 3y_1^7y_2 + \tfrac{3}{7}y_1^9 + 11y_1^8y_2 + \tfrac{2}{9}y_1^{10} + \tfrac{5}{9}y_1^9y_2 + \tfrac{1}{7}y_1^{11} + \tfrac{5}{11}y_1^{10}y_2 \\ & + 3y_1^{12} + \tfrac{2}{3}y_1^{11}y_2.\end{aligned} \tag{46}$$

We choose $a_{211} = 0, a_{220} = a_{230} = \frac{1}{2} \neq 0, a_{221} = -\frac{73}{37}$, and $a_{240} = a_{231} = a_{241} = 1$, which violates condition (43). Executing the Maple program results in the following simplest normal form:

$$\begin{aligned}\dot{u}_1 = {} & u_2, \\ \dot{u}_2 = {} & \frac{1}{2}u_1^2 + \frac{1}{2}u_1^3 + \frac{110}{37}u_1^3u_2 + \frac{183}{37}u_1^4u_2 + \frac{336001}{2053500}u_1^6 - \frac{52435501}{6078360}u_1^6u_2 \\ & + \frac{3772692223}{151959000}u_1^7u_2 - \frac{35707023869779}{1443103970000}\boxed{u_1^9} \\ & + \frac{6838151186754887664549850 6669}{2211283083977214661299 6000}u_1^9u_2 \\ & + \frac{7525814427323419465150553 4919139}{7567502109610912396447520000}u_1^{10}u_2.\end{aligned} \tag{47}$$

It should be pointed out that the violation of condition (43) would, in general, yield one more term $u_1^9$ (marked by a box in Eq. (43)) than the simplest normal form obtained when condition (43) is satisfied. Suppose condition (43) is held. For example, let $a_{231} = 2, a_{241} = 5$, instead of $a_{231} = a_{241} = 1$, then one can find the second equation of the simplest normal form given as follows:

$$\begin{aligned}\dot{u}_2 = {} & \frac{1}{2}u_1^2 + \frac{1}{2}u_1^3 + \frac{147}{37}u_1^3u_2 + \frac{331}{37}u_1^4u_2 - \frac{69149}{2053500}u_1^6 - \frac{533790509}{30391800}u_1^6u_2 \\ & + \frac{1665621781}{50653000}u_1^7u_2 + \frac{1589267410929106809 91}{69236127146865000}u_1^9u_2 \\ & + \frac{7444055008477339875641}{823348539043800000}u_1^{10}u_2.\end{aligned} \tag{48}$$

It is clearly seen from Eqs. (47) and (48) that Eq. (47) has one more term, $u_1^9$, than Eq. (48), due to the violation of the condition at the 9th order at which an $h$ coefficient does not appear and thus cannot be used at this order. In general, if some non-algebraic number condition like the one given in Eq. (43) is not satisfied at the $k$th order, then one more term than the regular simplest normal form is retained at the $k$th order normal form.

Also, it should be noted that by a method such as used by Algaba et al. (1997), Chen and Della Dora (2000), Li et al. (2001), Yu (2002) and Yuan and Yu (2001), higher order simplest normal forms may require more non-algebraic number conditions like the one given by Eq. (43). There is no way to find all such non-algebraic number conditions for the simplest normal form of a system up to an arbitrary order. However, with the matching pursuit technique and the Maple program, one does not need to worry about these non-algebraic number conditions, and the simplest normal form can be obtained even when these unknown non-algebraic number conditions are not satisfied.

### 5.4. Example 4

We now turn to consider a case: $\mu = 2, \nu = 1$ which belongs to case (III) $\mu = 2\nu$, i.e. $a_{220} = 0, a_{211} \neq 0, a_{230} \neq 0$. It can be shown that the following algebraic conditions should be held, which are found using the Maple program given in Yuan and Yu (2001):

$$\begin{array}{ll} 9a_{230} + a_{211}^2 \neq 0 & \text{at 4th order,} \\ 62a_{230} + 3a_{211}^2 \neq 0 & \text{at 6th order,} \\ 315a_{230}^2 - 229a_{230}a_{211}^2 - 6a_{211}^4 \neq 0 & \text{at 8th order.} \end{array} \tag{49}$$

The condition for the 4th order has been given by Algaba et al. (2001). We can use the matching pursuit technique to find the simplest normal forms for the above three cases when the conditions are violated. Again, using the numerical equation, described in Eq. (45), here we choose $a_{221} = 1$ for convenience. The results for the three cases are given below.

*Case (A).* Let $a_{211} = 1, a_{230} = -\frac{1}{9}$ which results in $9a_{230} + a_{211}^2 = 0$. Executing the Maple program yields the simplest normal form:

$$\begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= u_1u_2 - \frac{1}{9}u_1^3 + u_1^2u_2 + u_1^4 + \boxed{u_1^3u_2} + \frac{2}{3}u_1^5 \\ &\quad + \frac{3621}{448}u_1^7 - \frac{24939007}{376320}u_1^8 + \frac{333914934217}{541900800}u_1^9 \\ &\quad - \frac{269347581147289}{34139750400}u_1^{10} + \frac{416637981737123969}{5608022999040}u_1^{11} \\ &\quad - \frac{1338191366489037465555259}{158626936258560000}u_1^{12}. \end{aligned} \tag{50}$$

Note that the 4th order term $u_1^3u_2$ is an extra term retained due to the violation of the first condition of (49). In other words, if $9a_{230} + a_{211}^2 \neq 0$, then this 4th order term can be removed from the simplest normal form using an $h$ coefficient.

*Case (B).* Let $a_{211} = 1, a_{230} = -\frac{3}{62}$, then $62a_{230} + 3a_{211}^2 = 0$. Our matching pursuit technique program produces the simplest normal form given by

$$\begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= u_1u_2 - \frac{3}{62}u_1^3 + u_1^2u_2 + u_1^4 + \frac{2}{3}u_1^5 - \frac{14249}{5425}u_1^6 \end{aligned}$$

$$
\begin{aligned}
&- \frac{328918}{875}\boxed{u_1^5u_2} - \frac{102353455517}{59810625}u_1^8 \\
&- \frac{2111755396570657}{382189893750}u_1^9 - \frac{32177532310230110717}{28893555967500}u_1^{10} \\
&+ \frac{464736490815052588637611013}{29579777921728125000}u_1^{11} \\
&- \frac{628486844636952471726823764521}{825958414275946875000}u_1^{12}.
\end{aligned} \tag{51}
$$

Similarly, if the condition $62a_{230} + 3a_{211}^2 \neq 0$ is held, then the 6th order term $u_1^5u_2$ can be removed.

*Case (C).* Let $a_{211} = 1$, $a_{230} = \frac{229+\sqrt{44881}}{630}$, which renders $315a_{230}^2 - 229a_{230}a_{211}^2 - 6a_{211}^4 = 0$. The simplest normal form for this case is found by using the matching pursuit technique as

$$
\begin{aligned}
\dot{u}_1 &= u_2, \\
\dot{u}_2 &= u_1u_2 + \frac{229+\sqrt{60001}}{630}u_1^3 + u_1^2u_2 + u_1^4 + \frac{2}{3}u_1^5 + \frac{43\sqrt{60001}-1790}{9450}u_1^6 \\
&\quad + \frac{38921872 - 138287\sqrt{60001}}{782775}u_1^7 + \frac{290685973\sqrt{60001} - 68546927567}{328765500}u_1^8 \\
&\quad + \frac{3355418332083737 - 13698517799633\sqrt{60001}}{6904075500}\boxed{u_1^7u_2} \\
&\quad + \frac{2663452386309233068 - 10873082633724827\sqrt{60001}}{6524351347500}u_1^{10} \\
&\quad + \frac{436651948790906635720110052517 - 1782608491453734639408295583\sqrt{60001}}{4435977661838451225000}u_1^{11} \\
&\quad + \frac{7258395195718581514659263443917\sqrt{60001} - 1777951073104100318846081480159243}{3220519782494715589350000}u_1^{12},
\end{aligned} \tag{52}
$$

where an extra term $u_1^7u_2$ cannot be eliminated due to the third non-algebraic number condition of (49) being violated.

It can be seen from this example that the Maple program developed in this paper can be used to compute the simplest normal form of the systems containing not only rational coefficients, but also irrational coefficients. In fact, the program can be executed for any combinations of numerical numbers and symbolic notations.

## 5.5. Example 5

From the previous examples, we have observed that, in general, the two terms of the conventional normal form at each order may be eliminated by one, two, or none. Thus one may expect that no simplest normal forms may have more terms at any order than that of the conventional normal form. However, this is not always true. Now we shall give an example to demonstrate that the general rule is not applicable if the conventional normal form looks sufficiently “irregular”.

For a more clear investigation, consider the following 15th order conventional normal form:

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= y_1^2 y_2 + y_1^4 + y_1^3 y_2 + \boxed{y_1^4 y_2} + \tfrac{1}{2} y_1^6 + 5y_1^7 + 3y_1^7 y_2 + \tfrac{3}{7} y_1^9 + \tfrac{2}{9} y_1^{10} \\ &\quad - \tfrac{5}{9} \boxed{y_1^9 y_2} + \tfrac{1}{7} \boxed{y_1^{11}} + \tfrac{2}{11} y_1^{10} y_2 + 3y_1^{12} + \tfrac{2}{3} \boxed{y_1^{11} y_2} + 3y_1^{13} + 7 \boxed{y_1^{12} y_2} \\ &\quad + 9y_1^{14} + y_1^{13} y_2 + 5y_1^{15} + 11 \boxed{y_1^{14} y_2} \end{aligned} \tag{53}$$

which satisfies

$$a_{220} = a_{211} = a_{230} = a_{250} = a_{251} = a_{261} = a_{280} = a_{281} = 0. \tag{54}$$

The box notation given in Eq. (53) is marked for the comparison with the simplest normal form obtained below. Note that here $a_{221} \neq 0$ and $a_{240} \neq 0$, suggesting that this case may belong to $\mu = 3$, $\nu = 2 (\mu < 2\nu)$. However, since more higher order $a$ coefficients vanish, it does not follow the “rule” of the case. Executing our Maple program yields the following simplest normal form up to 15th order:

$$\begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= u_1^2 u_2 + u_1^4 + u_1^3 u_2 + \frac{1}{2} u_1^6 - \frac{1}{9} \boxed{u_1^5 u_2} + 5u_1^7 + \frac{41}{42} u_1^8 - u_1^9 + \frac{50453}{74088} \boxed{u_1^8 u_2} \\ &\quad + \frac{7963}{37044} u_1^{10} + \frac{3914237}{33006204} u_1^{10} u_2 - \frac{448499369}{24004512} u_1^{12} + \frac{82102121}{432081216} u_1^{13} \\ &\quad - \frac{45215814840251}{634592280924} u_1^{14} - \frac{561243855596423502097}{928799415836861184} u_1^{13} u_2 \\ &\quad - \frac{2464725735875010107}{25396859026789173} u_1^{15}. \end{aligned} \tag{55}$$

Comparing the above simplest normal form with the conventional normal form given by Eq. (53) shows that (paying particular attention to the terms marked by the boxes):

(a) The simplest normal form and conventional normal form have the *same* number of terms up to 3rd, 6th, 7th, 8th or 10th order.

(b) The conventional normal form has *one* 5th order term while the simplest normal form has *no* 5th order term.

(c) The conventional normal form has *one* 6th order term but the simplest normal form has *two* 6th order terms.

(d) The conventional normal form has *one* 9th order term but the simplest normal form has *two* 9th order terms.

(e) The conventional normal form has *two* 10th order terms while the simplest normal form has *one* 10th order term.

(f) From the 11th order on, the simplest normal form resumes the normal simplification process.

It can be seen from this “irregular” example that the simplest normal form is simpler than the conventional normal form up to 5th order, while the conventional normal form is simpler than the simplest normal form up to 9th order. They have same terms up to 6th order and 10th order. Starting from 11th order terms, the simplification process in finding the simplest normal form resumes normally, i.e., the simplest normal form simplifies the conventional normal form at any order $k \geq 11$.

## 6. Conclusions

A matching pursuit technique has been developed for computing the simplest normal form of the Takens–Boganov dynamical singularity. It has been shown that this approach is indeed computationally efficient. From the computational point of view, the method completely solves the simplest normal form of the Takens–Bogdanov dynamical singularity. It does not need any non-algebraic number conditions or requirements as other approaches do. “Automatic” symbolic computation programs written in Maple have been developed. Examples are presented to show the advantages of the matching pursuit method. It has been observed from the five examples that in general the process of simplification is carried out order by order. However, for “irregular” systems like example 5 there may exist an “upper boundary” order (which is 10 for example 5). When the order of the simplest normal form is smaller than the boundary, the conventional normal form contains no fewer terms than the simplest normal form (as we would expect). Although the simplest normal form is simpler than the conventional normal form for sufficiently high order, the conventional normal form may actually be simpler than the simplest normal form for some lower orders. When the order is greater than the boundary, the simplification process resumes normally, i.e., the simplest normal form simplifies the conventional normal form at any order after the “boundary”.

It should be pointed out that the five examples presented in this paper for computing the simplest normal form do not contain perturbation parameters (unfolding). In fact, it has been noted that no single example has been given to show the real application of the simplest normal form in bifurcation analysis, since a physical or engineering system always contains perturbation parameters. Thus, for real applications, the theory and methodology for computing the simplest normal form with unfolding needs to be developed. Such simplest normal form for single zero dynamical singularity can be found in Yu (2002), and that for Hopf bifurcation has also been obtained (Yu and Leung, 2003). It is expected that the matching pursuit technique can be extended to consider the simplest normal form with perturbation parameters.

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