Nonconvex Minimization Problems for Functionals Defined on Vector Valued Functions

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We consider the minimization problem

\[
\min_{v \in W^{1,1}(B^n, \mathbb{R}^m)} \int_{B^n} \left[ f(|\nabla v(x)|) + h(v(x)) \right] dx,
\]

where \( B^n \) is the ball of \( \mathbb{R}^n \) centered at the origin and with radius \( R > 0 \), \( f \) is a lower semicontinuous function, and \( h \) is a convex function. We give sufficient conditions for the existence and uniqueness of minimizers. Our technique relies on a detailed knowledge of the properties of the solutions to the convexified problem, obtained using the corresponding Euler–Lagrange inclusions.

Key Words: calculus of variations; existence; uniqueness; Euler–Lagrange inclusions; radially symmetric solutions; nonconvex problems; noncoercive problems.

1. INTRODUCTION

The study of the elastostatic equilibrium of a body can be reduced, under suitable assumptions on its shape and on the deformation produced,
to a minimization problem of the form

$$ \min_{v \in W^{1,1}(B^n_R, \mathbb{R}^n)} \int_{B^n_R} \left[ f(|\nabla v(x)|) + h(v(x)) \right] dx = \min_{v \in W^{1,1}(B^n_R, \mathbb{R}^n)} J(v), \quad (1) $$

where $n, m \geq 1$ and $B^n_R$ is the ball of $\mathbb{R}^n$ centered at the origin and with radius $R > 0$. In general the function $f$ may be not convex (see [1, 11, 17, 18] and the references therein).

Furthermore, nonconvex variational problems of the form (1) were also introduced in order to describe mechanical models related to phase transitions (see [2]) and shape optimization problems (see [13, 14]).

In order to establish the existence of a solution to (1), it is the custom to take into account the convexified problem

$$ \min_{v \in W^{1,1}(B^n_R, \mathbb{R}^m)} \int_{B^n_R} \left[ f^**(|\nabla v(x)|) + h(v(x)) \right] dx = \min_{v \in W^{1,1}(B^n_R, \mathbb{R}^m)} \bar{J}(v), \quad (2) $$

where $f^{**}$ is the bidual function of the map $s \mapsto f(|s|)$.

In this paper, we consider the radially symmetric problems (1) and (2), where the functions $f$ and $h$ satisfy the following assumptions.

(H1) $f: [0, +\infty[ \to \mathbb{R} \cup \{\infty\}$ is a lower semicontinuous function, not identically $+\infty$, such that $M = \lim_{s \to +\infty} f^**(s)/s > 0$.

(H2) $h: \mathbb{R}^m \to \mathbb{R}$ is a convex function, and there exists $H \in ]0, M n / R[$ such that $\partial h(u) \subset \overline{B^n_R}$ for every $u \in \mathbb{R}^m$.

Under these assumptions, the sublevel sets of the functional $\bar{J}$ are not necessarily compact in the weak topology of $W^{1,1}_0(B^n_R, \mathbb{R}^m)$; hence the direct method of the calculus of variations does not apply. Nevertheless, the existence of solutions to (2) was proved in [10]. More precisely, it is known that problem (2) admits at least one radially symmetric solution, which satisfies a system of differential inclusions of the Euler–Lagrange type.

These Euler–Lagrange inclusions are used in order to gather detailed information about the solutions to (2). Adding to (H1) and (H2) one of the following assumptions:

(H3) $f$ has a minimum at the origin;

(H4) the (possibly empty) convex set $A = \operatorname{argmin} h = \{u \in \mathbb{R}^m; 0 \in \partial h(u)\}$ is not a singleton,

this information allows us to prove that there exists a radially symmetric minimizer $v_0 \in W^{1,1}_0(B^n_R, \mathbb{R}^m)$ of $\bar{J}$ such that the set $\{x \in B^n_R; f^**(|\nabla v_0(x)|) < f(|\nabla v_0(x)|)\}$ has Lebesgue measure zero. Henceforth $v_0$ is a solution to (1) and $\min J = \min \bar{J}$.  

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We remark that the hypotheses (H1) and (H2) are not enough in order to obtain an existence result for the nonconvex problem. Namely, if \( n \neq \pi \), \( f(s) = (1 - s^2)^2 \), and \( h(u) = |u| \), then \( \inf J = 0 < J(v) \) for every \( v \in W^{1,1}_0((-1,1)) \) (see [4, 12]).

The existence of solutions to (1) in the scalar one-dimensional coercive case, i.e., \( n = m = 1 \) and \( M = +\infty \), was recently studied in [12, 15]. In particular, it was proved that, if \( h \) is a lower semicontinuous function, bounded from below, then the conditions (H1) and (H3) are sufficient for the existence of minimizers.

In the last section of this paper we give sufficient conditions for the uniqueness of the solution. More precisely, we show that, if \( h \) is smooth, (H1), (H2), and at least one of the following assumptions hold:

(H3') \( f \) has a strict minimum at the origin,

(H4') \( h \) does not have minimizers,

then problems (1) and (2) admit one and only one solution.

We remark that (H3') and (H4') generalize to the case \( m > 1 \) the assumptions used in [3, 8] in order to prove the uniqueness of the solution.

2. PRELIMINARIES

In what follows \( \langle \cdot, \cdot \rangle \) and \( | \cdot | \) will denote respectively the standard scalar product and the Euclidean norm in \( \mathbb{R}^d \), \( d \geq 1 \), while \( B_r^d \subset \mathbb{R}^d \) will denote the open ball centered at the origin and with radius \( r > 0 \). We shortly write \( \mathbb{R} = \mathbb{R}^1 \), and \( \mathbb{R} = [-\infty, +\infty] \). The norm of a matrix \( Q = (q_{ij})_{i,j=1}^n \) is defined by \( |Q| = (\sum_{i,j} q_{ij}^2)^{1/2} \).

We shall denote by \( \overline{A} \) and \( \text{int} \ A \) respectively the closure and the interior of a set \( A \). If \( A \subset \mathbb{R}^d \) is a convex set, then its relative interior \( \text{ri} \ A \) is defined as the interior of \( A \) regarded as a subset of its affine hull. The set of the extremal points of \( A \) will be denoted by \( \text{ext} \ A \). We recall that \( \xi \in \text{ext} \ A \) if \( \lambda x + (1 - \lambda) y = \xi \) for \( x, y \in A \) and \( \lambda \in ]0, 1[ \) implies that \( x = y = \xi \).

As customary, \( L^p(\Omega, \mathbb{R}^d) \) and \( W^{1, p}_0(\Omega, \mathbb{R}^d) \), \( 1 \leq p \leq +\infty \), will denote the Lebesgue and Sobolev spaces of functions defined in an open set \( \Omega \) and with values in \( \mathbb{R}^d \). If \( d = 1 \) we shortly write \( L^p(\Omega) \equiv L^p(\Omega, \mathbb{R}) \) and \( W^{1, p}_0(\Omega) \equiv W^{1, p}_0(\Omega, \mathbb{R}) \). The usual norm in \( L^p(\Omega, \mathbb{R}^d) \) will be denoted by \( \| \cdot \|_{L^p} \). We shall denote by \( AC([0, R], \mathbb{R}^d) \) the set of all absolutely continuous functions from \([0, R] \) to \( \mathbb{R}^d \), while \( AC_{\text{loc}}(I, \mathbb{R}^d) \) will be the set of all functions \( u \) defined on an interval \( I \subset [0, R] \) such that \( u \in AC(J, \mathbb{R}^d) \) for every compact interval \( J \subset I \).
If \( v \in W_{0}^{1,1}(B_{R}^{n}, \mathbb{R}^{n}) \) is a radially symmetric function, that is, \( v(x) = u(|x|) \) for some function \( u: [0, R] \to \mathbb{R}^{n} \), then \( u \) belongs to the set

\[
W := \{ u \in \text{AC}_{\text{loc}}([0, R], \mathbb{R}^{n}); u(R) = 0, t \mapsto t^{n-1}|u'(t)| \in L^{1}(0, R) \}.
\]

Namely, it can be easily checked that \( u \) belongs to \( \text{AC}_{\text{loc}}([0, R], \mathbb{R}^{n}) \), and that, denoting by \( \alpha_{n} \) the \( (n-1) \)-dimensional Hausdorff measure of \( \partial B_{1}^{n} \),

\[
\alpha_{n} \int_{0}^{R} t^{n-1}|u'(t)| \, dt = \int_{B_{R}^{n}} \left| \frac{\nabla V(x)}{|x|} \right| \, dx \leq \int_{B_{R}^{n}} |\nabla V(x)| \, dx,
\]

which implies that \( t^{n-1}|u'(t)| \in L^{1}(0, R) \). Finally, since \( v(\omega R) = 0 \) for a.e. with \( |\omega| = 1 \), we obtain \( u(R) = 0 \); hence \( u \in W \).

Given a function \( \psi: \mathbb{R}^{d} \to \mathbb{R} \), we shall denote by \( \text{Dom} \psi \) its effective domain, that is, the set \( \{ \xi \in \mathbb{R}^{d}; \psi(\xi) \in \mathbb{R} \} \), and by \( \psi^{*} \) its dual function, defined by \( \psi^{*}(p) = \sup_{\xi \in \mathbb{R}^{d}} \langle p, \xi \rangle - \psi(\xi) \) for every \( p \in \mathbb{R}^{d} \). As customary, we denote by \( \psi^{\ast\ast} \) the bidual function \( (\psi^{*})^{*} \). We recall that \( \psi^{\ast\ast} \) is the greatest convex lower semicontinuous function which is pointwise less than \( \psi \).

If \( \psi \) is a convex function, we define its subgradient at \( \xi \in \text{Dom} \psi \) by

\[
\partial \psi(\xi) := \{ p \in \mathbb{R}^{d}; \psi(\eta) \geq \psi(\xi) + \langle p, \eta - \xi \rangle, \text{ for every } \eta \in \mathbb{R}^{d} \}.
\]

By definition, we set \( \partial \psi(\xi) = \emptyset \) for every \( \xi \not\in \text{Dom} \psi \). We recall that, if \( \text{Dom} \psi^{*} \neq \emptyset \), then \( p \in \partial \psi(\xi) \) if and only if \( \xi \in \partial \psi^{*}(p) \). Furthermore, if \( \psi \) is differentiable at \( \xi \), then \( \partial \psi(\xi) = \{ \nabla \psi(\xi) \} \). If \( \psi \) is a scalar convex function, then \( \partial \psi(\xi) = [\psi_{-}'(\xi), \psi_{+}'(\xi)] \), where \( \psi_{-}' \) and \( \psi_{+}' \) denote respectively the left and right derivatives of \( \psi \).

3. STUDY OF THE EULER–LAGRANGE INCLUSIONS

In the first part of this section we gather the basic results, proved in [10], concerning the convexified problem (2). Afterwards, we shall investigate some additional properties of the solutions to (2) that will be fundamental in order to prove existence and uniqueness of the solution to the non-convex problem (1).

The following proposition shows that the convexified problem (2) can be reduced to a one-dimensional problem (see [10, Lemma 3.6]).

**Proposition 3.1.** Problem (2) has a solution if and only if it admits a radially symmetric solution. Furthermore, \( v \in W_{0}^{1,1}(B_{R}^{n}, \mathbb{R}^{n}) \) is a radially
symmetric solution to (2) if and only if the function \( u \in W \) defined by \( v(x) = u(|x|) \) is a solution to the minimization problem

\[
\min_{w \in W} \int_{-1}^{1} t^{n-1} \left[ f^{**}(|w'(t)|) + h(w(t)) \right] dt,
\]

where \( W \) is the set defined in (3).

The turning point due to the introduction of problem (5) is that we can associate to it a system of differential inclusions of the Euler–Lagrange type (see [10, Theorem 4.1]).

**Proposition 3.2.** Assume that (H1) and (H2) hold. Let us define the set

\[
W^* = \{ p \in AC([0, R], \mathbb{R}^m); p(0) = 0, t^{1-n}|p'(t)| \in L^1(0, R) \}.
\]

Then, a function \( u \in W \) is a solution to (5) if and only if there exists \( p \in W^* \) such that the pair \( (u, p) \) is a solution of the Euler–Lagrange inclusions

\[
p'(t) \in t^{n-1} \partial h(u(t)),
\]

\[
p(t) \in t^{n-1} \partial f^{**}(|u'(t)|) \frac{u'(t)}{|u'(t)|}, \quad \text{if } |u'(t)| \neq 0,
\]

\[
p(t) \in t^{n-1} \overline{B}_{r_0}, \quad \text{if } |u'(t)| = 0,
\]

for a.e. \( t \in [0, R] \), where \( r_0 = (f^{**})_{+}(0) \geq 0 \).

In [10, Theorem 3.16], a radially symmetric solution to (2) was obtained as limit of solutions to “smooth” approximating problems.

**Proposition 3.3.** Assume that (H1) and (H2) hold. Then there exists a radially symmetric solution \( v \in W_0^{1,1}(B^+_0, \mathbb{R}^m) \) to problem (2). More precisely, if \( u \in W \) is such that \( v(x) = u(|x|) \), then \( u \) is a solution to (5) obtained by approximation; that is, there exist three sequences \( (h_k)_k, (u_k)_k, \) and \( (p_k)_k \) such that

(a) \( \lim_k u_k(t) \) for a.e. \( t \in [0, R] \);

(b) \( (h_k)_k \) is a sequence of convex functions of class \( C^2(\mathbb{R}^m) \), converging pointwise to \( h \);

(c) for every \( k \in \mathbb{N} \), the pair \( (u_k, p_k) \in W \times W^* \) is a solution of the Euler–Lagrange inclusions

\[
p'_k(t) = t^{n-1} \nabla h_k(u_k(t)),
\]
\begin{align}
p_k(t) & \in t^{n-1} \partial f^*(|u_k'(t)|) \frac{u_k'(t)}{|u_k'(t)|}, \quad \text{if } |u_k'(t)| \neq 0, \\
p_k(t) & \in t^{n-1} \overline{B}_{\rho_0}, \quad \text{if } |u_k'(t)| = 0,
\end{align}

for a.e. \( t \in [0, R] \).

Remark 3.4. In [10] it was proved that every solution \( u \in W \) to (5) obtained by approximation is actually a solution to (5); hence, by Proposition 3.2, there exists \( p \in W^* \) such that \((u, p)\) is a solution of the Euler–Lagrange inclusions (7) and (8). If \( h \in C^1(\mathbb{R}^m) \), then \( p \) is uniquely determined by (7) and the initial condition \( p(0) = 0 \). In the general case, if \((h_k), (u_k), \) and \((p_k)\) are the sequences defined in Proposition 3.3, we can always choose \( p \) in such a way that the sequence \((p_k)\) converges pointwise to \( p \). A function \( p \) with these properties will be called a coextremal of \( u \).

We collect here some a priori bounds concerning the solutions to the Euler–Lagrange inclusions. In particular, the estimate (12) below implies that every radially symmetric solution to (2) belongs to \( W^{1,*}(\mathbb{R}^m, \mathbb{R}) \).

Lemma 3.5. Assume that (H1) and (H2) hold. Then there exists a positive constant \( U \) such that, if \((u, p) \in W \times W^* \) is a solution of the Euler–Lagrange inclusions (7) and (8), then

\begin{align}
|p'(t)| & \leq t^{n-1}H, \\
|p(t)| & \leq \frac{Ht^n}{n}, \\
|u'(t)| & \leq U,
\end{align}

for a.e. \( t \in [0, R] \).

Proof. The first estimate in (11) follows from (7) and (H2), whereas the second one is obtained by integration taking into account the initial condition \( p(0) = 0 \). In particular we get

\begin{align}
t^{1-n} |p(t)| & \leq M_0 \doteq \frac{HR}{n}, \quad \text{for every } t \in [0, R].
\end{align}

From (H2) and (H1) we have that \( M_0 \in ] - M, M[ \subset \text{Dom } f^* \). Applying the duality property of the subgradient to (8) we infer that

\begin{align}
|u'(t)| & \in \partial f^*(t^{1-n}|p(t)|), \quad \text{a.e. } t \in [0, R].
\end{align}

Hence, from (13), (14), and the monotonicity of the subgradient, we obtain that (12) is fulfilled choosing \( U \doteq (f^*)_e(M_0) \).
The next result involves one-dimensional non-autonomous functionals, which generalize the ones appearing in (5). We recall that a function \( g: [0, R] \times [0, +\infty) \to \mathbb{R} \) is said to be a normal integrand if the map \( g(t, \cdot) \) is lower semicontinuous for a.e. \( t \in [0, R] \), and there exists a Borel function \( \hat{g}: [0, R] \times [0, +\infty) \to \mathbb{R} \) such that \( \hat{g}(t, \cdot) = g(t, \cdot) \) for a.e. \( t \in [0, R] \).

**Proposition 3.6.** Let \( g: [0, R] \times [0, +\infty) \to \mathbb{R} \) be a normal integrand satisfying
\[
g(t, 0) = \min_{s \geq 0} g(t, s), \quad \text{for a.e. } t \in [0, R],
\]
and let \( h: \mathbb{R}^{m} \to \mathbb{R} \) be a convex function. Let \( u \in W \) be a solution to the problem
\[
\min_{w \in W} \int_{0}^{R} t^{n-1} \left[ g(t, |w'(t)|) + h(w(t)) \right] dt = \min_{w \in W} G(w).
\]

Then the map \( \varphi(t) \equiv h(u(t)) \) is monotone non-decreasing in \([0, R]\).

**Proof.** Assume by contradiction that there exist \( t_1, t_2 \in [0, R], \quad t_1 < t_2 \), such that \( \varphi(t_1) > \varphi(t_2) \). Since \( \varphi \) is a continuous function in \([0, R]\), then there exist \( 0 \leq a < b \leq t_2 \) such that the interval \([a, b] \) is the connected component of the open set \( \{ t \in [0, t_2]; \varphi(t) > \varphi(t_2) \} \) containing the point \( t_1 \). Clearly we have that \( \varphi(b) = \varphi(t_2) \). If either \( a = 0 \) or \( a \in [0, b] \) and \( u(a) = u(b) \), then the function
\[
\hat{u}(t) = \begin{cases} u(b), & \text{if } t \in [a, b], \\
u(t), & \text{otherwise}, \end{cases}
\]
belongs to \( W \) and
\[
h(\hat{u}(t)) = \varphi(b) = \varphi(t_2) < \varphi(t) = h(u(t)), \quad \forall t \in [a, b]. \tag{16}
\]
By (15) and (16) we have that \( G(\hat{u}) < G(u) \), contradicting the minimality of \( u \).

Assume now that \( a \in [0, b] \) and \( u(a) \neq u(b) \). Let \( \xi \equiv \frac{u(b) - u(a)}{u(b) - u(a)} \) and \( \chi(t) \equiv \int_{a}^{t} |u'(s)| ds \). Since \( \chi \in AC([a, b]) \), \( \chi(a) = 0 \), and \( \chi(b) \geq |u(b) - u(a)| \), then there exists \( \tau \in [a, b] \) such that \( \chi(\tau) = |u(b) - u(a)| \). Let us define the map
\[
\hat{u}(t) = \begin{cases} u(t), & \text{if } t \in [0, a] \cup [b, R], \\
u(a) + \chi(t) \xi, & \text{if } t \in [a, \tau], \\
u(b), & \text{if } t \in [\tau, b]. \end{cases}
\]
Then \( \hat{u} \in W \) and
\[
|\hat{u}'(t)| = \begin{cases} 0, & \text{for a.e. } t \in [\tau, b], \\ |u'(t)|, & \text{otherwise}, \end{cases}
\]
so that, by (15),
\[
\int_{0}^{R} t^{n-1} g(t, |\hat{u}'(t)|) \, dt \leq \int_{0}^{R} t^{n-1} g(t, |u'(t)|) \, dt. \tag{17}
\]
On the other hand, by the very definition of \( a \) and \( b \), we have that
\[
h(u(t)) > h(u(a)) = h(u(b)) \quad \forall t \in ]a, b[. \tag{18}
\]
Moreover, for every \( t \in ]a, \tau[ \) one has \( \lambda(t) = \frac{x(t)}{|x(t) - x(a)|} \in [0, 1] \) and, by the convexity of \( h \),
\[
h(\hat{u}(t)) = h(\lambda(t) u(b) + (1 - \lambda(t)) u(a))
\leq \lambda(t) h(u(b)) + (1 - \lambda(t)) h(u(a)) = h(u(b)) < h(u(t)). \tag{19}
\]
Then, by (18) and (19) we obtain
\[
\int_{0}^{R} t^{n-1} h(\hat{u}(t)) \, dt < \int_{0}^{R} t^{n-1} h(u(t)) \, dt,
\]
which contradicts, together with (17), the minimality of \( u \).}

In the following lemma we collect some properties of a coextremal of a solution that will be used in the proof of Lemma 3.8 below. We recall that \( r_{0} = (f^{\ast \ast})_{+}(0) \geq 0 \).

**Lemma 3.7.** Assume that (H1) and (H2) hold, and that \( h \in C^{2}(\mathbb{R}^m) \). Let \( u \in W \) be a solution to (5), let \( p \in W^* \) be the coextremal of \( u \), and let \( \nu : [0, R] \to [0, +\infty[ \) be the function defined by
\[
\nu(t) = \begin{cases} \frac{|p(t)|}{t^{n-1}}, & t \in ]0, R[, \\ 0, & t = 0. \end{cases}
\tag{20}
\]
Then the following properties hold.

(i) \( p \in W^{1, \gamma}([0, R], \mathbb{R}^m) \);

(ii) \( \nu(t) \geq r_{0} \) for a.e. \( t \in [0, R] \) such that \( u'(t) \neq 0 \);
(iii) $u'(t) \neq 0$ and $\langle p'(t), p(t) \rangle \geq 0$ for a.e. $t \in [0, R]$ such that $\nu(t) > r_0$.

Proof. (i) Since $h \in C^2(\mathbb{R}^m)$ we have that (7) becomes
\begin{equation}
 p'(t) = t^{n-1} \nabla h(u(t)), \quad \text{a.e. } t \in [0, R].
\end{equation}
By (12), the map $t \mapsto \nabla h(u(t))$ belongs to $W^{1,\infty}([0, R], \mathbb{R}^m)$; hence the conclusion follows from (21).

(ii) We can assume that $r_0 > 0$; otherwise there is nothing to prove. Since $f^{**}$ is a convex scalar function, for a.e. $t \in [0, R]$ such that $u'(t) \neq 0$ we have that $\partial f^{**}([u'(t)]) \cap [0, r_0] = \emptyset$, and hence, by (8), $|p(t)| \geq t^{n-1}r_0$.

(iii) For a.e. fixed $t \in [0, R]$ such that $\nu(t) > r_0$, by (8) we have that $u'(t) \neq 0$ and
\begin{equation}
 \frac{p(t)}{|p(t)|} = \frac{u'(t)}{|u'(t)|},
\end{equation}
which gives, together with (21),
\begin{align*}
\langle p'(t), p(t) \rangle &= \frac{|p(t)|}{|u'(t)|} \langle p'(t), u'(t) \rangle \\
&= t^{n-1} \frac{|p(t)|}{|u'(t)|} \langle \nabla h(u(t)), u'(t) \rangle \geq 0,
\end{align*}
where the last inequality follows from Proposition 3.6.

The main tool needed in the next sections is a monotonicity result for $\nu$.

**Lemma 3.8.** Assume that (H1) and (H2) hold. Let $u$ be a solution of (5) obtained by approximation, let $p$ be a coextremal of $u$, and let $\nu$ be defined by (20). Then $\nu$ is a monotone non-decreasing absolutely continuous function in $[0, R]$. Furthermore, there exists $t_0 \in [0, R]$ such that $\nu(t) \leq r_0$ for every $t \in [0, t_0]$, and $\nu(t) > r_0$ for every $t \in [t_0, R]$. More precisely:

(i) If $t_0 < R$ then $\nu(t) > r_0$ for every $t \in [t_0, R]$ and $\nu$ is a strictly increasing function in $[t_0, R]$.

(ii) If $t_0 > 0$ and $r_0 = 0$ then $p(t) = 0$ for every $t \in [0, t_0]$.

(iii) If $t_0 > 0$ and $r_0 > 0$, then $u(t) = u(t_0)$ for every $t \in [0, t_0]$ and there exists $v \in \partial h(u(t_0))$ with $|v| \leq nr_0/t_0$ such that $p(t) = (t^n/n)v$ for every $t \in [0, t_0]$.

Proof. By (11) we deduce that $0 \leq \nu(t) \leq tH/n$ for every $t \in [0, R]$; hence the function $\nu$ is continuous on $[0, R]$. We shall divide the proof...
into two steps. In the first step we shall analyze the case of smooth $h$, whereas the second step will deal with the general case.

**Step 1.** Assume that $h \in C^2(\mathbb{R}^m)$. Let $t_0 \in [0, R]$ be defined by

$$ t_0 \doteq \sup\{t \in [0, R]; \nu(s) \leq r_0 \ \forall s \in [0, t]\}, $$

and suppose that $t_0 > 0$. If $r_0 = 0$, then $\nu(t) = 0$ and hence $p(t) = 0$ for every $t \in [0, t_0]$, proving (ii). If $r_0 > 0$, let us define

$$ \tau \doteq \sup\{t \in [0, R]; \nu(s) < r_0 \ \forall s \in [0, t]\}. $$

Since $\nu(0) = 0$, then $0 < \tau \leq t_0$. Moreover, by Lemma 3.7(ii) we have that $u'(t) = 0$ a.e. in $[0, \tau]$. If we set $\nu = h(u(\tau))$, then by (21) we obtain $p'(t) = t^{n-1}v$ and $p(t) = (t^n/n)v$ for every $t \in [0, \tau]$. In addition, by the continuity of $\nu$ we have that $\nu(\tau) \leq r_0$, which implies $|v| \leq nr_0/\tau$.

In order to complete the proof of (iii), it remains to show that $\tau = t_0$. Since $\tau \leq t_0$, it is enough to prove the opposite inequality, which is trivially satisfied if $\tau = R$. On the other hand, if $\tau < R$ then $\nu(\tau) = r_0 > 0$; hence $\nu$ is differentiable in $\tau$. Moreover, since $\nu(t) = t|v|/n$ for every $t \in [0, \tau]$, we have that $|v| = nr_0/\tau > 0$, and $\nu'(\tau) = |v|/n > 0$. This implies that there exists $\varepsilon > 0$ such that $\nu(t) > \nu(\tau) = r_0$ for every $t \in ]\tau, \tau + \varepsilon[$, so that $t_0 = \tau$, and

$$ [t_0, t_0 + \varepsilon] \subset B \doteq \{t \in [0, R]; \nu(t) > r_0\}. $$

(23)

Concerning the proof of (i), assume that $t_0 < R$. We want to prove that $\nu$ is strictly monotone increasing in $[t_0, R]$. This result will be achieved by proving that the set $B$, defined in (23), coincides with the interval $[t_0, R]$ and that, for every $0 \leq \delta < 1$, the function $\nu_\delta: [0, R] \to \mathbb{R}$ defined by

$$ \nu_\delta(t) \doteq \begin{cases} \frac{|p(t)|}{t^{n-1+\delta}}, & t \in ]0, R[, \\ 0, & t = 0, \end{cases} $$

is monotone and non-decreasing in $[t_0, R]$. From Lemma 3.7(i) we have that $p \in C^1([0, R], \mathbb{R}^m)$. Since $p(t) \neq 0$ for every $t \in B$, we deduce that $\nu_\delta$ is differentiable in $B$, and

$$ \nu_\delta'(t) = \frac{1}{t^{n+\delta}} \chi(t), \quad \text{for every } t \in B, $$

(25)

where the function $\chi: B \to \mathbb{R}$ is defined by

$$ \chi(t) \doteq \left( p'(t), \frac{p(t)}{|p(t)|} \right) - (n - 1 + \delta)|p(t)|. $$
Claim 1. The function $\chi$ is continuous and monotone non-decreasing in every connected component of $B$.

Proof. From Lemma 3.7(i) and the fact that $p(t) \neq 0$ for every $t \in B$, we infer that $\chi$ is differentiable almost everywhere in $B$. Moreover, recalling (21), we obtain that

$$
\chi'(t) = nt^{n-1} \left( \nabla h(u(t)) \cdot \frac{p(t)}{|p(t)|} \right) + t^n \frac{d}{dt} \left( \nabla h(u(t)) \cdot \frac{p(t)}{|p(t)|} \right) - (n - 1 + \delta) \left( p'(t), \frac{p(t)}{|p(t)|} \right) - (1 - \delta) \left( p'(t), \frac{p(t)}{|p(t)|} \right) + t^n \frac{d}{dt} \left( \nabla h(u(t)) \cdot \frac{p(t)}{|p(t)|} \right). \tag{26}
$$

Let us prove that

$$
\zeta(t) = \frac{d}{dt} \left( \nabla h(u(t)) \cdot \frac{p(t)}{|p(t)|} \right) \geq 0, \quad \text{for a.e. } t \in B. \tag{27}
$$

By (22) and (23) we have

$$
\zeta(t) = \frac{d}{dt} \left( \nabla h(u(t)) \cdot \frac{p(t)}{|p(t)|} \right) + \left( \nabla h(u(t)) \cdot \frac{p'(t)}{|p(t)|} - \frac{p(t)}{|p(t)|} \right) \left( p'(t), p(t) \right) + \frac{1}{t^{n-1} |p(t)|} \left( p'(t), p'(t) \right) \left( p(t) \right) - p(t) \left( p'(t), p(t) \right). \tag{28}
$$

Since $h$ is a convex function, it follows that $\langle \nabla^2 h(u(t)) \cdot u'(t), u'(t) \rangle \geq 0$ for a.e. $t \in [0, R]$. Finally, thanks to Cauchy–Schwartz inequality,

$$
\left( p'(t), p'(t) \right) \left( p(t) \right) \geq 0,
$$

so that $\zeta(t) \geq 0$. Hence, by (26), (27), and Lemma 3.7(iii), we deduce that $\chi'(t) \geq 0$ for a.e. $t \in B$. Finally, since $\chi \in AC_{\text{loc}}(I)$ for every connected
component \( I \) of \( B \), then \( \chi \) is continuous and monotone non-decreasing on \( I \).

By \((11)\) we have that there exists a constant \( c > 0 \) such that
\[
|\chi(t)| \leq ct^n, \quad \forall t \in B. \tag{29}
\]

Let us assume that \( r_0 > 0 \), and let \( t_1 = \sup \{ t \in [l_0, R] : I \in B \} \). From (23), we deduce that \( t_1 \in [t_0 + \varepsilon, R] \). An easy consequence of Claim 1 and (29) is that the restriction of \( \chi \) to \([l_0, t_1] \) can be extended continuously on \([t_0, t_1] \). The next task will be to prove that \( \chi(t_0) \geq 0 \).

**Claim 2.** If \( r_0 > 0 \), then \( \chi(t_0) \geq 0 \).

**Proof.** If \( t_0 = 0 \), then by (29) we get \( \chi(0) = 0 \). If \( t_0 \in [0, R] \), then by (iii) there exists \( v \in \mathbb{R}^m \) such that \( p'(t_0) = t_0^{n-1}v \) and \( p(t_0) = \frac{n}{n+1}t_0^nu \), which implies that \( \chi(t_0) = t_0^n[1 - \frac{n+1}{n+2}]|v| \geq 0 \); hence Claim 2 is proved.

By (11) we have that \( 0 \leq v_\delta(t) \leq \frac{t^{n-\delta}}{n-\delta} t^{1-\delta} \) for every \( t \in [0, R] \); hence \( v_\delta \) is continuous in \([0, R]\). Furthermore, from (25) and (29), we deduce that \( |v_\delta(t)| \leq ct^{\delta} \) for a.e. \( t \in [t_0, t_1] \); hence \( v_\delta \in L^1(t_0, t_1) \). Since \( v_\delta \in AC_{\text{loc}}([t_0, t_1]) \), we conclude that \( v_\delta \) belongs to \( AC_B(t_0, t_1] \).

Finally, by Claims 1 and 2, we infer that \( \chi(t) \geq 0 \) for every \( t \in [t_0, t_1] \). By (25), one gets \( v_\delta(t) \geq 0 \) for every \( t \in [t_0, t_1] \); hence \( v_\delta \) is monotone non-decreasing in \([t_0, t_1]\). Since \( v(t) = t^{n-\delta}v_\delta(t) \) and \( v_\delta \) is positive in \([t_0, t_1] \), then \( v \) is a strictly monotone function in \([t_0, t_1] \); hence \( t_1 = R \). Namely, if \( t_1 < R \), then, by the continuity of \( v \), one has \( v(t_1) = r_0 = \chi(t_0) \), which contradicts the strict monotonicity of \( v \).

It remains to prove (i) when \( r_0 = 0 \). In this case, since \( t_0 < R \), then the set \( B \) is not empty. Let \( I \) be a connected component of \( B \). From the continuity of \( v \) we deduce that \( B \) is an open set in the relative topology of \([0, R]\); hence we have either \( I = [a, b] \), with \( b < R \), or \( I = [a, R] \). By Claim 1 and (29), the restriction of \( \chi \) to \( I \) can be extended continuously to \( \tilde{I} \). Since \( p(a) = 0 \) and, by Lemma 3.7(iii), \( \langle p'(t), p(t) \rangle \geq 0 \) in \( I \), we deduce that \( \chi(a) = \lim_{t \to a^+} \chi(t) \geq 0 \). From Claim 1 we infer that \( \chi(t) \geq 0 \) for every \( t \in \tilde{I} \), which implies that \( v \) is strictly monotone increasing in \( \tilde{I} \). Using the same argument given above for the case \( r_0 > 0 \), we conclude that \( \sup I = R \), so that \( B = [a, R] \). Since, by the definition of \( B \), \( v(t) = 0 \) for every \( t \in [0, a] \), we deduce that \( t_0 = a \), completing the proof of (i).

We remark that, from the continuity of \( v \) and the analysis above, it easily follows that \( v \) is monotone non-decreasing on \([0, R]\).

**Step 2.** Let \( h \) satisfy (H2), and let \( u \) be a solution of (5) obtained by approximation. Let \( (h_k)_k, (u_k)_k, (p_k)_k \) be the sequences defined in Proposition 3.3, let \( p \) be a coextremal of \( u \), and let \( \nu \) and \( \nu_\delta \) be the functions
defined in (20) and (24), respectively. For every \( k \in \mathbb{N} \) and every \( \delta \in [0, 1] \) let us define the function

\[
\nu_{k, \delta}(t) = \begin{cases} \frac{|p_k(t)|}{t^{n-1+\delta}}, & t \in ]0, R], \\ 0, & t = 0. \end{cases}
\]

(30)

Since the sequence \( (p_k)_k \) converges pointwise to \( p \) in \([0, R]\) (see Remark 3.4), then for every \( \delta \in [0, 1] \) we have that \( \lim_{k} \nu_{k, \delta} = v_0 \). By Step 1, for every \( \delta \in [0, 1] \) \( \nu_{k, \delta} \) is a monotone non-decreasing function in \([0, R]\) for every \( k \in \mathbb{N} \); hence \( v_0 \) is a monotone non-decreasing function in \([0, R]\). Let \( t_0 = \inf\{t \in [0, R]; v(t) > r_0\} \) and \( t_k = \inf\{t \in [0, R]; v_k(t) > r_0\} \). Since \( v \) is a monotone function, we have that the set \( \{t \in [0, R]; v(t) > r_0\} \) coincides with \([t_0, R]\); hence (i) and (ii) are fulfilled.

**Claim 3.** If \( t_0 > 0 \) and \( r_0 > 0 \), then \( \lim_{k} t_k = t_0 \), and

\[
\lim_{k \to +\infty} u_k(t_k) = u(t_0). \tag{31}
\]

**Proof.** Suppose, by contradiction, that \( \limsup_k t_k = \tilde{t} > t_0 \). Then for every \( t \in ]t_0, \tilde{t}[ \) one has

\[
r_0 < v(t) = \lim_{k \to +\infty} v_k(t) \leq \limsup_{k \to +\infty} v_k(t_k) \leq r_0.
\]

Hence \( \limsup_k t_k \leq t_0 \). Assume now that \( \liminf_k t_k = \hat{t} < t_0 \). Let us choose \( t \in ]t, \hat{t}[ \), and let \( \delta \in ]0, 1[ \). From the monotonicity of \( \nu_k \) one has

\[
r_0 \geq \nu(t_0) \geq \left( \frac{t_0}{t} \right)^{\delta} v(t) = \left( \frac{t_0}{t} \right)^{\delta} \lim_{k \to +\infty} v_k(t) \geq \left( \frac{t_0}{t} \right)^{\delta} \liminf_{k \to +\infty} v_k(t_k) = \left( \frac{t_0}{t} \right)^{\delta} r_0 > r_0.
\]

Then \( \liminf_k t_k \geq t_0 \), so that \( \lim_k t_k = t_0 \). Finally, from (12) we have that \( \|u'_k\|_{L^r} \leq U \) for every \( k \in \mathbb{N} \); hence (31) holds, and Claim 3 is proved.

Now we are in a position to prove (iii). Namely, for every \( t \in ]0, t_0[ \) there exists \( k_0 \in \mathbb{N} \) such that \( t \in [0, t_k] \) for every \( k > k_0 \). Hence, thanks to the first step, \( u_k(t) = u_k(t_k) \) for every \( k > k_0 \). Passing to the limit as \( k \) goes to \( +\infty \), from (31) we obtain that \( u(t) = u(t_0) \) for every \( t \in ]0, t_0[ \). The other conclusions of (iii) follow from (7).

Finally, from the continuity of \( \nu \), properties (ii) and (iii), and the absolute continuity of \( \nu \) in \([t_0, R]\), we infer that \( \nu \in AC([0, R], \mathbb{R}^m) \).
4. EXISTENCE

Let $E_0$ be the set of all $s \geq 0$ such that $(s, f^{**}(s))$ is an extremal point of the epigraph of $f^{**}$, and let

$$s_0 = \sup \{ s \geq 0 ; f^{**}(s) = f^{**}(0) \}. \quad (32)$$

Notice that, by (H1), $s_0$ is finite and $\{ s \geq 0 ; f^{**}(s) = f^{**}(0) \} = [0, s_0]$. Moreover, $s_0 \in E_0 \subseteq [s_0, +\infty]$ and, since $f$ is lower semicontinuous,

$$f^{**}(s) = f(s), \quad \text{for every } s \in E_0 \quad (33)$$

(see [8, Remark 5.3]).

**Remark 4.1.** Under the assumptions of Lemma 3.8, from the Euler–Lagrange inclusions (7) and (8) we have that, if $t_0 > 0$ and $r_0 = 0$, then $|u'(t)| \leq s_0$ and $0 \in \partial h(u(t))$ for a.e. $t \in [0, t_0]$.

**Lemma 4.2.** Under the assumptions of Lemma 3.8, there exists $t_1 \in [t_0, R]$ such that

$$|u'(t)| \leq s_0, \quad \text{a.e. } t \in [0, t_0],$$

$$|u'(t)| = s_0, \quad \text{a.e. } t \in [t_0, t_1],$$

$$|u'(t)| > s_0, \quad \text{a.e. } t \in [t_1, R]. \quad (34)$$

**Proof.** Notice that, since $\nu(t) \in \partial f^{**}(|u'(t)|)$ for a.e. $t \in [0, R]$, then by the monotonicity of the subgradient we deduce that, for a.e. $t \in [0, R]$,

$$\text{if } \nu(t) < r_0 \text{ then } |u'(t)| \leq s_0, \quad (35)$$

$$\text{if } \nu(t) > r_0 \text{ then } |u'(t)| > s_0. \quad (36)$$

If $r_0 > 0$, then $s_0 = 0$ and, by Lemma 3.8, the function $\nu$ is either identically zero if $t_0 = R$ and $\nu = 0$ or it is strictly monotone increasing in $[0, R]$. In both cases the conclusion follows from (35) and (36), choosing $t_1 = t_0$.

Assume now that $r_0 = 0$, and let $\theta = (f^{**})_\nu(s_0)$. We have that, for a.e. $t \in [0, R]$,

$$\text{(i) if } \nu(t) = 0 \text{ then } |u'(t)| \leq s_0,$$

$$\text{(ii) if } 0 < \nu(t) < \theta \text{ then } |u'(t)| = s_0,$$

$$\text{(iii) if } \nu(t) > \theta \text{ then } |u'(t)| > s_0.$$

If $\theta = 0$, then the conclusion follows choosing $t_1 = t_0$. If $\theta > 0$, we can assume without loss of generality that $s_0 > 0$. Then $\nu(t) = 0$ for every $t \in [0, t_0]$, and $\nu$ is positive, continuous, and strictly monotone increasing.
in $]r_0, R]$. Hence, if $\nu(R) < \theta$ then it is enough to choose $t_1 = R$; otherwise $t_1$ will be defined as the unique point in $]r_0, R]$ such that $\nu(t_1) = \theta$.

From Lemma 4.2 we deduce that, if $s_0 = 0$, then $t_1 = t_0$, and $t_0 \in [0, R]$ is uniquely determined by the conditions $u'(t) = 0$ for a.e. $t \in [0, t_0]$, $u'(t) \neq 0$ for a.e. $t \in [r_0, R]$.

The properties shown in Lemma 3.8 are now used in order to find out when $f$ and $f^{**}$ coincide along a solution of (2).

**Lemma 4.3.** Assume that (H1) and (H2) hold. Let $u$ be a solution of (5) obtained by approximation, and let $t_0 \in [0, R]$ be defined as in Lemma 3.8. Then

(i) $|u(t)| \in E_0$ for a.e. $t \in [t_0, R]$;

(ii) if, in addition, either (H3') or (H4') holds, then $|u'(t)| \in E_0$ for a.e. $t \in [0, R]$, and $u'(t) = 0$ for a.e. $t \in [0, t_0]$. In particular, if $s_0 > 0$ then $t_0 = 0$.

**Proof.** Let $p$ be the coextremal of $u$ defined in Lemma 3.8, and let $\nu$ be defined by (20). In order to prove the first part of the lemma, we can assume that $t_0 < R$. Assume by contradiction that there exist $\alpha > 0$ and $\emptyset \neq [a, b[C0, +\infty[$ such that $(f^{**})(s) = \alpha$ for every $s \in [a, b]$ and the set $S = \{t \in [t_0, R]; |u'(t)| \in [a, b]\}$ has positive Lebesgue measure. Then from (8) we obtain that $|p(t)| = \alpha t^{n-1}$ and hence $\nu(t) = \alpha$ for a.e. $t \in S$, which contradicts the strict monotonicity of the function $\nu$ in $[t_0, R]$ proved in Lemma 3.8(i).

Concerning the proof of (ii), it is not restrictive to assume that $t_0 > 0$. Assume that (H3') holds; that is, $s_0 = 0$. By Lemma 4.2, we have that $|u'(t)| = 0 \in E_0$ for a.e. $t \in [0, t_0]$.

Assume now that (H4') holds. Since $t_0 > 0$, from Remark 4.1 we deduce that necessarily $r_0 > 0$; hence $s_0 = 0$, which is the case considered above.

We are now in a position to prove the existence of a solution to the nonconvex problem (1).

**Theorem 4.4.** Let $f$ and $h$ satisfy (H1) and (H2). Suppose in addition that either (H3) or (H4) holds. Then there exists at least a radially symmetric solution to problem (1).

**Proof.** Let $u$ be a solution of (5) obtained by approximation, let $t_0$ be defined as in Lemma 3.8, and let $s_0$ be the constant defined in (32).
If $t_0 = 0$, then by Lemma 4.3(i) and (33), we have that $f^{**}(|u'(t)|) = f(|u'(t)|)$ for a.e. $t \in [0, R]$; hence $u$ is a solution to
\[ \min_{w \in W} \int_0^R t^{n-1} \left[ f(|w'(t)|) + h(w(t)) \right] dt, \tag{37} \]
and $v(x) = u(|x|)$ is a solution to (1). From now on we shall assume that $t_0 > 0$.

Assume that (H3) holds. If $s_0 = 0$, then the assumption (H3') is satisfied and, from Lemma 4.3(ii) and (33), we deduce that $f^{**}(|u'(t)|) = f(|u'(t)|)$ for a.e. $t \in [0, R]$. Hence $u$ is a solution to (37) and $v(x) = u(|x|)$ provides a solution to problem (1). Consider now the case $s_0 > 0$. Let $u_0 \in W$ be the function defined by
\[ u_0(t) = \begin{cases} u(t_0), & \text{if } t \in [0, t_0], \\ u(t), & \text{otherwise}. \end{cases} \]
From Remark 4.1 we have that $|u'(t)| \leq s_0$ and $0 \in \partial h(u(t))$ for a.e. $t \in [0, t_0]$; hence $f^{**}(|u'(t)|) = f^*(0) = f(|u_0'(t)|)$ for a.e. $t \in [0, t_0]$. Moreover both $u(t)$ and $u_0(t)$ belong to $\text{argmin} \ h$ for every $t \in [0, t_0]$, so that $h(u_0(t)) = h(u(t))$ for every $t \in [0, t_0]$. On the other hand, by Lemma 4.3(i) and (33), we have that $f^{**}(|u'(t)|) = f(|u'(t)|)$ for a.e. $t \in [t_0, R]$; hence $u_0$ is a solution to (37) and $v_0(x) = u_0(|x|)$ is a solution to (1).

Suppose now that (H4) holds. Clearly we can assume, without loss of generality, that (H3) does not hold, so that $s_0 > 0$ and $r_0 = 0$. If $A = \emptyset$, then by Remark 4.1 we have that $t_0 = 0$, which is the case considered at the beginning. If $0 \neq A$, then by Remark 4.1 we have that $u(t_0) \in A$. Furthermore, there exist $\xi \in \mathbb{R}^m$, with $|\xi| = s_0$, and $r > 0$ such that $u(t_0) + \lambda \xi \in A$ for every $\lambda \in [0, r]$. Let us define $u_0 \in W$ such that $u_0 = u$ in $[t_0, R]$, and
\[ u_0(t) = \begin{cases} \xi, & \text{if } t \in [0, t_0] \cap \left( \bigcup_{k \geq 0} [t_0 - (2k + 1)r, t_0 - 2kr] \right), \\ -\xi, & \text{if } t \in [0, t_0] \cap \left( \bigcup_{k \geq 1} [t_0 - 2kr, t_0 - (2k - 1)r] \right). \end{cases} \]
Then $u_0(t) \in A$ and $|u_0'(t)| = s_0$ for a.e. $t \in [0, t_0]$. Hence
\[ f(|u_0'(t)|) = f(s_0) = f^{**}(s_0) \leq f^{**}(|u'(t)|), \quad \text{a.e. } t \in [0, t_0], \]
and $u_0(t), u(t) \in \text{argmin} \ h$ for every $t \in [0, t_0]$, which easily imply that $v_0(x) = u_0(|x|)$ is a solution to (1).
5. UNIQUENESS

In this section we shall show that, if (H1) and (H2) hold, \( h \) is smooth, and at least one of the assumptions (H3') or (H4') holds, then the solution to (1) is unique. We remark that (H3') and (H4') imply (H3) and (H4), respectively; hence the existence of a solution follows from Theorem 4.4. 

**Theorem 5.1.** Assume that (H1) and (H2) hold. Suppose in addition that \( h \) is of class \( C^2(\mathbb{R}^m) \) and that either (H3') or (H4') holds. Then problem (1) admits one and only one solution.

Before proving the uniqueness of the solution, we need some preliminary results. We mention that Lemma 5.3 below in the scalar, coercive non-autonomous case was proved in [3, Theorem 1].

**Remark 5.2.** If \( v \) and \( \tilde{v} \) are two solutions of problem (2), then for every \( \lambda \in [0, 1] \) the function \( v_{\lambda} = \lambda v + (1 - \lambda)\tilde{v} \) is a solution of (2) and

\[
\begin{align*}
f^{**}(|\nabla v_{\lambda}(x)|) &= f^{**}(\lambda|\nabla v(x)| + (1 - \lambda)|\nabla \tilde{v}(x)|) \\
&= \lambda f^{**}(|\nabla v(x)|) + (1 - \lambda) f^{**}(|\nabla \tilde{v}(x)|)
\end{align*}
\]

for a.e. \( x \in B^n_R \). In particular, for every \( x \in B^n_R \) such that (38) is satisfied and \(|\nabla v(x)| \geq s_0\) for some \( \lambda \in [0, 1] \), we have that \(|\nabla v(x)| \geq s_0\) and \(|\nabla \tilde{v}(x)| \geq s_0\). Namely, if \(|\nabla v(x)| < s_0\), then from (38) we infer that \(|\nabla \tilde{v}(x)| \leq s_0\); hence \(|\nabla v(x)| \leq \lambda|\nabla v(x)| + (1 - \lambda)|\nabla \tilde{v}(x)| < s_0\).

**Lemma 5.3.** Under the assumptions of Theorem 5.1, every solution \( v \) to problem (2) is radially symmetric, and \(|\nabla v(x)| \geq s_0\) for a.e. \( x \in B^n_R \).

**Proof.** Let \( v \in W^{1,1}_0(B^n_R, \mathbb{R}^m) \) be a solution to (2), and let \( \tilde{v} \) be the radially symmetric function defined by

\[
\tilde{v}(x) = \frac{1}{\alpha_n} \int_{|\omega|=1} v(\omega|x|) \, d\omega, \quad x \in B^n_R.
\]

By [10, Lemma 3.6], the function \( \tilde{v} \) belongs to \( W^{1,1}_0(B^n_R, \mathbb{R}^m) \) and provides a radially symmetric solution to (2).

Let us define the function \( w = (v + \tilde{v})/2 \), and let \( \Omega = \{ x \in B^n_R, |\nabla w(x)| > s_0 \} \). From Remark 5.2 we have that \(|\nabla v(x)| \geq s_0\) and \(|\nabla \tilde{v}(x)| \geq s_0\) for a.e. \( x \in \Omega \). Then, since \( f^{**} \) is a strictly increasing function in \([s_0, +\infty)\), from (38) we deduce that

\[
|\nabla w(x)| = \frac{1}{2}|\nabla v(x)| + \frac{1}{2}|\nabla \tilde{v}(x)|, \quad \text{a.e. } x \in \Omega.
\]
From the strict convexity of the Euclidean norm we infer that there exists a measurable function $\mu: \Omega \to [0, +\infty]$ such that

$$\nabla v(x) = \mu(x) \nabla \tilde{v}(x), \quad \text{a.e. } x \in \Omega.$$  \hspace{1cm} (40)

Now it is enough to show that $\text{meas}(B_R^n \setminus \Omega) = 0$. Indeed, in this case, from (40) we obtain that $v$ is radially symmetric.

If (H3') holds, then $s_0 = 0$; hence $\text{meas}(B_R^n \setminus \Omega) = 0$, and the proof is complete.

Assume that (H4') holds. Without loss of generality, we can assume that $s_0 > 0$. If $u \in W$ is defined by $\tilde{v}(x) = u(|x|)$, then by Lemmas 4.2 and 4.3(ii), there exists $t_1 \in [0, R]$, such that $\nabla \tilde{v}(x) > s_0$ for a.e. $x \in B_R^n \setminus \bar{B}_{t_1}$, and $|\nabla \tilde{v}(x)| = s_0$ for a.e. $x \in B_{t_1}^n$. Since $|\nabla w(x)| \geq s_0$ if $|\nabla \tilde{v}(x)| > s_0$, from (40) we obtain

$$\nabla w(x) = \frac{1}{2} [\mu(x) + 1] \nabla \tilde{v}(x), \quad \text{a.e. } x \in B_R^n \setminus \bar{B}_{t_1},$$

which implies that $w$ is radially symmetric in $B_R^n \setminus \bar{B}_{t_1}$. On the other hand, we have that

$$\tilde{v}(x) = \frac{1}{\alpha_n} \int_{|\omega| = 1} w(\omega|x|) \, d\omega, \quad x \in B_R^n,$$ \hspace{1cm} (41)

which allows us to conclude that

$$w = \tilde{v}, \quad \nabla w = \nabla \tilde{v}, \quad \text{in } B_R^n \setminus \bar{B}_{t_1};$$ \hspace{1cm} (42)

hence $|\nabla w(x)| > s_0$ for a.e. $x \in B_R^n \setminus \bar{B}_{t_1}$. In order to prove that $\text{meas}(B_R^n \setminus \Omega) = 0$, it is enough to show that, if $t_1 > 0$, then $|\nabla w(x)| \geq s_0$ for a.e. $x \in B_{t_1}^n$. Assume by contradiction that the set $\{x \in B_{t_1}^n; |\nabla w(x)| < s_0\}$ has positive Lebesgue measure. Since, by (41),

$$s_0 = |\nabla \tilde{v}(x)| \leq \frac{1}{\alpha_n} \int_{|\omega| = 1} |\nabla w(\omega|x|)| \, d\omega, \quad \text{a.e. } x \in B_{t_1}^n,$$

we have that also the set $S = \{x \in B_{t_1}^n; |\nabla w(x)| > s_0\}$ has positive Lebesgue measure. Moreover, by the strict monotonicity of $f^{**}$ in $[s_0, +\infty]$, we have that

$$\int_{B_{t_1}^n} f^{**}(|\nabla w(x)|) \, dx = \int_{S} f^{**}(|\nabla w(x)|) \, dx$$

$$+ \int_{B_{t_1}^n \setminus S} f^{**}(|\nabla w(x)|) \, dx$$

$$> \int_{B_{t_1}^n} f^{**}(s_0) \, dx = \int_{B_{t_1}^n} f^{**}(|\nabla \tilde{v}(x)|) \, dx.$$ \hspace{1cm} (43)
Finally, from (41) and the Jensen’s inequality we obtain that $\int h(w(x)) \, dx \geq \int h(\tilde{v}(x)) \, dx$; hence, by (42) and (43), we infer that $\tilde{J}(w) > \tilde{J}(\tilde{v})$, which contradicts the fact that, as underlined in Remark 5.2, the function $w$ is a minimum point of $\tilde{J}$.

Proof of Theorem 5.1. Since $\min J = \min \tilde{J}$, it is enough to prove that the convex problem (2) admits one and only one solution.

Let $v_0$ and $v_1$ be two solutions of (2). By Remark 5.2, for every $\lambda \in [0, 1]$ the function $v_\lambda = \lambda v_1 + (1 - \lambda)v_0$ is a solution to (2). From Lemmas 5.3 and 4.3 we have that for every $\lambda \in [0, 1]$ there exists a solution $u_\lambda \in W$ to (5), such that $v_\lambda(x) = u_\lambda(|x|)$ and

$$|u'_\lambda(t)| \in E_0, \quad \text{a.e. } t \in [0, R].$$

Hence, from (38), (44), and the very definition of extremal point we conclude that $u'_0(t) = u'_1(t)$ for a.e. $t \in [0, R]$, which implies that $v_0 = v_1$.

Remark 5.4. It is easy to see that, under the assumptions of Theorem 4.4, the uniqueness of the solution may fail. For example, in the case $m = n = 1$, if we consider the maps $f(s) \doteq \{0, s^2 - 1\}$, $h(u) \doteq \max\{0, |u| - 1\}$, then every function $u \in AC([0, R])$ satisfying $u(R) = 0$, $|u(t)| \leq 1$, $|u'(t)| \leq 1$ for a.e. $t \in [0, R]$, is a solution to (37).

REFERENCES