Journal of Algebra **242,** 624–655 (2001)
doi:10.1006/jabr.2001.8838, available online at http://www.idealibrary.com on IDEAL®

Fusion Rules for the Charge Conjugation Orbifold

Toshiyuki Abe

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail: sm3002at@ecs.cmc.osaka-u.ac.jp

Communicated by Geoffrey Mason

Received July 17, 2000

We completely determine fusion rules for irreducible modules of the charge conjugation orbifold V_L^+ for a rank one even lattice L. © 2001 Academic Press

1. INTRODUCTION

The charge conjugation orbifold V_L^+ is the orbifold model which comes from the lattice vertex operator algebra V_L associated to a rank one even lattice L with the automorphism θ indexed from the -1-isometry of L (cf. [FLM, DVVV]). Any irreducible V_L-module or θ-twisted module is completely reducible as V_L^+-module. It is known that every irreducible V_L^+-module is equivalent to an irreducible component either of an irreducible V_L-module or of an irreducible θ-twisted module ([DN2]). In this paper we completely determine the fusion rules for the irreducible V_L^+-modules and construct nonzero intertwining operators which provide nontrivial fusion rules. The intertwining operators for V_L-modules constructed in [DL] basically give intertwining operators for untwisted type modules (which are irreducible V_L^+-modules derived from irreducible V_L-modules), whereas intertwining operators involving twisted type modules (which are irreducible V_L^+-modules derived from twisted V_L-modules) are obtained by modifying the twisted ones for $M(1)$ constructed in [FLM]. In determining the fusion rules for V_L^+, the fusion rules and explicit forms of intertwining operators for the free bosonic orbifold vertex operator algebra $M(1)^+$ determined in [A] play important roles.

The vertex operator algebra V_L^+ and its irreducible modules are constructed as follows. Let $L = \mathbb{Z}\alpha$ be a rank one even lattice with a $\mathbb{Z}$-bilinear

0021-8693/01 $35.00

form $\langle\cdot,\cdot\rangle$ defined by $\langle\alpha,\alpha\rangle = 2k$ for a positive integer k. Set $\mathfrak{h} = \mathbb{C}\otimes_{\mathbb{Z}} L$ and extend the $\mathbb{Z}$-bilinear form to a $\mathbb{C}$-bilinear form on $\mathfrak{h}$. Let $\hat{\mathfrak{h}} = \mathfrak{h}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$ be its affinization with the center K. Then the Fock space $M(1) = S(\mathfrak{h}\otimes t^{-1}\mathbb{C}[t^{-1}])$ is a simple vertex operator algebra. Let $\mathbb{C}[\mathfrak{h}] = \oplus_{\lambda\in\mathfrak{h}}\mathbb{C}e_\lambda$ be the group algebra of the abelian group $\mathfrak{h}$. For a subset M of $\mathfrak{h}$ set $\mathbb{C}[M] = \oplus_{\lambda\in M}\mathbb{C}e_\lambda$. Then $V_L = M(1)\otimes\mathbb{C}[L]$ is a simple vertex operator algebra, and $V_{\lambda+L} = M(1)\otimes\mathbb{C}[\lambda+L]$ is an irreducible V_L-module for all $\lambda\in L^\circ$, where L° is the dual lattice of L (see [FLM]). Let θ be the -1-isometry of L. The involution θ is lifted to an isomorphism of V_{L°, which induces automorphisms of $M(1)$ and V_L of order 2. For a θ-invariant subspace W of V_{L°, we denote the ±1-eigenspaces of W by $W^\pm$, respectively. Then V_L^+ and $M(1)^+$ are simple vertex operator algebras, and $V_L^\pm$, $V_{\alpha/2+L}^\pm$, and $V_{r\alpha/2k+L}$ for $1\le r\le k-1$ are irreducible V_L^+-modules (see [DN2]).

Let $\hat{\mathfrak{h}}[-1] = \mathfrak{h}\otimes t^{1/2}\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$ be the twisted affine Lie algebra, and set $M(1)(\theta) = S(\mathfrak{h}\otimes t^{-1/2}\mathbb{C}[t^{-1}])$. Then $M(1)(\theta)$ is an irreducible θ-twisted $M(1)$-module (see [FLM, D2]). The automorphism θ acts on $M(1)(\theta)$, and the ±1-eigenspaces $M(1)(\theta)^\pm$ become irreducible $M(1)^+$-modules (see [DN1]). Let T^1 and T^2 be irreducible $\mathbb{C}[L]$-modules on which e_α acts 1 and -1, respectively. Then the tensor products $V_L^{T^i} = M(1)(\theta)\otimes T^i (i=1,2)$ are irreducible θ-twisted V_L-modules, and their ±1-eigenspaces $V_L^{T^i,\pm}$ for θ become irreducible V_L^+-modules [DN2]. In [DN2], it is proved that any irreducible V_L^+-module is isomorphic to one of the irreducible modules $V_L^\pm$, $V_{\alpha/2+L}^\pm$, $V_{r\alpha/2k+L}$ for $1\le r\le k-1$ and $V_L^{T_i,\pm}$ for $i=1,2$.

For a vertex operator algebra V and its modules W^1, W^2, and W^3, the dimension of the vector space $I_V\binom{W^3}{W^1\ W^2}$ which consists of all intertwining operators of type $\binom{M^3}{M^1\ M^2}$ is called the fusion rule of corresponding type and denoted by $N_{W^1\ W^2}^{W^3}$. Fusion rules have the symmetry

$$I_V\begin{pmatrix} W^3 \\ W^1\ \ W^2\end{pmatrix} \cong I_V\begin{pmatrix} W^3 \\ W^2\ \ W^1\end{pmatrix} \cong I_V\begin{pmatrix} (W^2)' \\ W^1\ \ (W^3)'\end{pmatrix}, \tag{1.1}$$

where W' means the contragredient module of a V-module W (see [FHL, HL]). We give the correspondence between irreducible V_L^+-modules and their contragredient modules (see Proposition 2.8) by using Zhu's theory [Z]. Then one can use the symmetry of fusion rules (1.1) to reduce the amount of arguments in the process of determining the fusion rules for V_L^+.

Let us explain the method of determining the fusion rules for V_L^+ in more detail. Let W^1, W^2, and W^3 be V_L^+-modules, and let M^1 and M^2 be $M(1)^+$-submodules of W^1 and W^2, respectively. Then we have a cononical restriction map

$$I_{V_L^+}\begin{pmatrix} W^3 \\ W^1\ \ W^2\end{pmatrix} \to I_{M(1)^+}\begin{pmatrix} W^3 \\ M^1\ \ M^2\end{pmatrix}, \mathcal{Y}\mapsto \mathcal{Y}|_{M^1\otimes M^2}.$$

If W^1 and W^2 are irreducible, then the restriction map is injective (see [DL, Proposition 11.9]). Therefore we have

$$\dim I_{V_L^+}\begin{pmatrix} & W^3 & \\ W^1 & & W^2\end{pmatrix} \le \dim I_{M(1)^+}\begin{pmatrix} & W^3 & \\ M^1 & & M^2\end{pmatrix}. \tag{1.2}$$

We also prove that all irreducible V_L^+-modules are completely reducible as $M(1)^+$-modules and that the multiplicity of each irreducible $M(1)^+$-module is at most one (see Proposition 3.2). Using this fact, (1.2), and fusion rules for $M(1)^+$ (see Theorem 2.7), we are able to prove that fusion rules for V_L^+ are zero or one.

The formula (1.2) also shows that for irreducible V_L^+-modules W^1, W^2, and W^3, if there are $M(1)^+$-submodules M^1 of W^1 and M^2 of W^2 such that the fusion rule $N_{M^1\,M^2}^{W^3}$ for $M(1)^+$ is zero, then the fusion rule $N_{W^1\,W^2}^{W^3}$ for V_L^+ is zero. For almost all of the irreducible V_L^+-modules W^1, W^2, and W^3 for which the fusion rule $N_{W^1\,W^2}^{W^3}$ is zero, we can find such $M(1)^+$-submodules M^1 of W^1 and M^2 of W^2.

But there are irreducible V_L^+-modules W^1, W^2, and W^3 for which we cannot find $M(1)^+$-submodules M^1 of W^1 and M^2 of W^2 such that the fusion rule $N_{M^1\,M^2}^{W^3}$ is zero, though the fusion rule $N_{W^1\,W^2}^{W^3}$ is zero (for example, $W^1 = V_L^-$ and $W^2 = W^3 = V_{\alpha/2+L}^+$). In such cases, we first restrict an intertwining operator $\mathcal{Y}$ of corresponding type to $M^1 \otimes M^2$ for certain irreducible $M(1)^+$-submodules M^i of W^i for $i = 1, 2$, and view $\mathcal{Y}$ as an intertwining operator for $M(1)^+$. Next we show that $\mathcal{Y}$ is zero by using the explicit forms of intertwining operators for $M(1)^+$.

The nonzero fusion rules are provided by constructing nonzero intertwining operators explicitly as before mentioned. The construction is treated separately in the following two cases: one is the case that all modules are of untwisted types, and the other is the case that some modules are of twisted types.

Nontrivial intertwining operators for untwisted type modules were essentially given in [DL] (see Section 3.4). More precisely, a nontrivial interwining operator $\mathcal{Y}_{\lambda\mu}$ for V_L of type $\binom{V_{\lambda+\mu+L}}{V_{\lambda+L}\ V_{\mu+L}}$ for $\lambda, \mu \in L^\circ$ was constructed. The intertwining operator $\mathcal{Y}_{\lambda\mu}$ gives a nonzero intertwining operator for V_L^+ of type $\binom{V_{\lambda+\mu+L}}{V_{\lambda+L}\ V_{\mu+L}}$. Since θ induces a V_L^+-module isomorphism from $V_{\lambda+L}$ to $V_{-\lambda+L}$ for $\lambda \in L^\circ$, the operator $\mathcal{Y}_{\lambda,-\mu} \circ \theta$ defined by $\mathcal{Y}_{\lambda,-\mu} \circ \theta(u, z)v = \mathcal{Y}_{\lambda,-\mu}(u, z)\theta(v)$ for $u \in V_{\lambda+L}$ and $v \in V_{\mu+L}$ gives a nonzero intertwining operator of type $\binom{V_{\lambda-\mu+L}}{V_{\lambda+L}\ V_{\mu+L}}$. Then all nonzero intertwining for untwisted type modules are obtained as restrictions of $\mathcal{Y}_{\lambda\mu}$ or $\mathcal{Y}_{\lambda,-\mu} \circ \theta$.

Nonzero intertwining operators involving twisted type modules are constructed as follows (see Section 3.4). Let $\lambda \in L^\circ$. In [A], we construct a nonzero intertwining operator $\mathcal{Y}^\theta$ for $M(1)^+$ of type $\binom{M(1)(\theta)}{M(1,\lambda)\ M(1)(\theta)}$ following [FLM]. We define a linear isomorphism ψ_λ of $T^1 \oplus T^2$ which satisfies $e_\alpha \psi_\lambda = (-1)^{\langle \alpha, \lambda \rangle} \psi_\lambda e_\alpha = \psi_{\lambda+\alpha}$, and define $\widetilde{\mathscr{Y}}$ by $\widetilde{\mathscr{Y}}(u,z) = \mathscr{Y}^\theta(u,z) \otimes \psi_\mu$ for $\mu \in \lambda + L$ and $u \in M(1,\mu)$. Then for indices i and j subject to $(-1)^{\langle \lambda, \alpha \rangle + \delta_{i,j}+1} = 1$, $\widetilde{\mathscr{Y}}$ gives a nonzero intertwining operator of type $\binom{V_L^{T_j}}{V_{\lambda+L}\ V_L^{T_i}}$. All nonzero intertwining operators in this case are obtained by restricting $\widetilde{\mathscr{Y}}$ to irreducible V_L^+-modules and by using symmetry of fusion rules (1.1).

The organization of this paper is as follows: We review definitions of modules for a vertex operator algebra, fusion rules, and some related results in Section 2.1. We recall the vertex operator algebras $M(1)^+$ and V_L^+ and their irreducible modules in Section 2.2. In Section 2.3 we list the fusion rules for $M(1)^+$ and study the contragredient modules for V_L^+. In Section 3.1, we give the irreducible decompositions of irreducible V_L^+-modules as $M(1)^+$-modules and prove that the fusion rules for V_L^+ are zero or one. The main theorem is stated in Section 3.2 (Theorem 3.4). In Sections 3.3 and 3.4, we determine the fusion rules for untwisted type modules and ones involving twisted type modules, respectively.

2. PRELIMINARIES

In Section 2.1, we recall the definition of g-twisted modules for a vertex operator algebra and its automorphism g of finite order, intertwining operator, and fusion rule following [FLM, FHL, DMZ, DLM]. In Section 2.2, we review the constructions of vertex operator algebras $M(1)^+$, V_L^+ and their irreducible modules following [FLM, DL, DN1, DN2]. In Section 2.3, we state the fusion rules for $M(1)^+$ obtained in [A] (see Theorem 2.7) and discuss the correspondence between irreducible V_L^+-modules and their contragredient modules.

Throughout this paper, $\mathbb{N}$ is the set nonnegative integers and $\mathbb{Z}_+$ is the set of positive integers.

2.1. *Modules, Intertwining Operators, and Fusion Rules*

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and g an automorphism of V of order T. Then V is decomposed into the direct sum of eigenspaces for g:

$$V = \bigoplus_{r=0}^{T-1} V^r, V^r = \{a \in V \mid g(a) = e^{\frac{-2\pi i r}{T}} a\}.$$

A *g-twisted V-module* is a $\mathbb{C}$-graded vector space $M = \oplus_{\lambda \in \mathbb{C}} M(\lambda)$ such that each $M(\lambda)$ is finite dimensional and for fixed $\lambda \in \mathbb{C}$, $M(\lambda + n/T) = 0$

for a sufficiently small integer n, and equipped with a linear map

$$Y_M\colon V \to (\text{ End } M)\{z\},$$
$$a \mapsto Y_M(a,z) = \sum_{n\in\mathbb{Q}} a_n^M z^{-n-1}, (a_n^M \in \text{End } M)$$

such that these conditions hold for $0 \le r \le T-1$, $a \in V^r$, $b \in V$, *and* $u \in M$

$$Y_M(a,z) = \sum_{n\in r/T+\mathbb{Z}} a_n^M z^{-n-1},\ Y_M(a,z)v \in z^{-\frac{r}{T}} M((z)),$$
$$\times z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(a,z_1)Y_M(b,z_2)$$
$$- z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(b,z_2)Y_M(a,z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\left(\frac{z_1 - z_0}{z_2}\right)^{-\frac{r}{T}} Y_M(Y(a,z_0)b,z_2),$$
$$Y_M(\mathbf{1},z) = \mathrm{id}_M,$$
$$L(0)v = \lambda v \text{ for } v \in M(\lambda),$$

where we set $Y_M(\omega,z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$.

A g-twisted V-module is denoted by (M, Y_M) or simply by M. In the case g is the identity of V, a g-twisted V-module is called a *V-module*. An element $u \in M(\lambda)$ is called *a homogeneous element of weight* λ. We denote the weight by $\lambda = \mathrm{wt}(u)$. We write the component operator $a_n^M (a \in V$, $n \in \mathbb{Q})$ by a_n for simplicity.

For a V-module M, it is known that the restricted dual $M' = \oplus_{\lambda\in\mathbb{C}} M(\lambda)^*$ with the vertex operator $Y_M^*(a,z)$ for $a \in V$ defined by

$$\langle Y_M^*(a,z)u', v\rangle = \langle u', Y_M(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})v\rangle$$

for $u' \in M'$, $v \in M$ is a V-module (cf. [FHL, HL]). The V-module (M', Y_M^*) is called the *contragredient module of M*. The double contragredient module $(M')'$ of M is naturally isomorphic to M, and therefore if M is irreducible, then M' is also irreducible (see [FHL]).

Definition 2.1. Let V be vertex operator algebra and let (M^i, Y_{M^i}) $(i = 1,2,3)$ be *V modules*. An intertwining operator for V of type $\binom{M^3}{M^1\ M^2}$ is a linear map $\mathcal{Y}\colon M^1 \otimes M^2 \to M^3\{z\}$, or equivalently,

$$\mathcal{Y}\colon M^1 \to (\mathrm{Hom}(M^2, M^3))\{z\},$$
$$v \mapsto \mathcal{Y}(v,z) = \sum_{n\in\mathbb{C}} v_n z^n (v_n \in \mathrm{Hom}(M^2, M^3))$$

such that for $a \in V$, $v \in M^1$, and $u \in M^2$, following conditions are satisfied:

(1) For fixed $n \in \mathbb{C}$, $v_{n+k}u = 0$ for sufficiently large integer k,

(2) (The Jacobi identity)

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_{M^3}(a, z_1)\mathcal{Y}(v, z_2) \\ &\qquad - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\mathcal{Y}(v, z_2)Y_{M^2}(a, z_1) \\ &= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\mathcal{Y}(Y_{M^1}(a, z_0)v, z_2), \end{aligned} \tag{2.1}$$

(3) ($L(-1)$-*derivative property*)

$$\frac{d}{dz}\mathcal{Y}(v, z) = \mathcal{Y}(L(-1)v, z). \tag{2.2}$$

The identity (2.1) implies the following commutator formula for $n \in \mathbb{Z}$, $a \in V$ and $u \in M^1$:

$$[a_n, \mathcal{Y}(u, z)] = \sum_{i=0}^{\infty} \binom{n}{i} \mathcal{Y}(a_i u, z) z^{n-i}. \tag{2.3}$$

The vector space which consists of all intertwining operators of type $\binom{M^3}{M^1\ M^2}$ is denoted by $I_V\binom{M^3}{M^1\ M^2}$. The dimension of the vector space $I_V\binom{M^3}{M^1\ M^2}$ is called the *fusion rule* of corresponding type and denoted by $N_{M^1\ M^2}^{M^3}$. Fusion rules have the following symmetry (see [FHL, HL]).

PROPOSITION 2.2. *Let M^i $(i = 1, 2, 3)$ be V-modules. Then there exist natural isomorphisms*

$$I_V\begin{pmatrix} M^3 \\ M^1\ \ M^2 \end{pmatrix} \cong I_V\begin{pmatrix} M^3 \\ M^2\ \ M^1 \end{pmatrix} \text{ and } I_V\begin{pmatrix} M^3 \\ M^1\ \ M^2 \end{pmatrix} \cong I_V\begin{pmatrix} (M^2)' \\ M^1\ \ (M^3)' \end{pmatrix}.$$

The following lemma is often used in later sections.

LEMMA 2.3 ([DL, Proposition 11.9]). *Let V be a vertex operator algebra, and let M^1 and M^2 be irreducible V-modules and M^3 a V-module. If $\mathcal{Y}$ is a nonzero intertwining operator of type $\binom{M^3}{M^1\ M^2}$, then $\mathcal{Y}(u, z)v \neq 0$ for any nonzero vectors $u \in M^1$ and $v \in M^2$.*

As a direct consequence of Lemma 2.3, we have:

COROLLARY 2.4. *Let V, M^i $(i = 1, 2, 3)$ be as in Lemma 2.3, and let U be a vertex operator subalgebra of V with same Virasoro element, N^i a U-submodule of M^i for $i = 1, 2$. Then the restriction map*

$$I_V\begin{pmatrix} M^3 \\ M^1\ \ M^2 \end{pmatrix} \to I_U\begin{pmatrix} M^3 \\ N^1\ \ N^2 \end{pmatrix}, \mathcal{Y} \mapsto \mathcal{Y}|_{N^1 \otimes N^2},$$

is injective. In particular, we have

$$\dim I_V\binom{M^3}{M^1\quad M^2} \le \dim I_U\binom{M^3}{N^1\quad N^2}.$$

Let V, M^i $(i=1,2,3)$, U and N^i $(i=1,2)$ be as in Corollary 2.4. Suppose that M^3 is decomposed into a direct sum of U-modules as $M^3 = \oplus_{i\in I} L^i$. Then there is an isomorphism

$$I_U\binom{\oplus_{i\in I} L^i}{N^1\quad N^2} \cong \bigoplus_{i\in I} I_U\binom{L^i}{N^1\quad N^2}.$$

Therefore by Corollary 2.4, we have an inequality

$$\dim I_V\binom{M^3}{M^1\quad M^2} \le \sum_{i\in I} \dim I_U\binom{L^i}{N^1\quad N^2}. \tag{2.4}$$

Another consequence of Lemma 2.3 is:

LEMMA 2.5. *Let V be a simple vertex operator algebra, and let M^1 and M^2 be irreducible V-modules. If the fusion rule of type $\binom{M^2}{V\ M^1}$ is nonzero, then M^1 and M^2 are isomorphic to each other as V-modules.*

Proof. Let $\mathcal{Y}$ be an intertwining operator of type $\binom{M^2}{V\ M^1}$. Consider the operator $\mathcal{Y}(\mathbf{1}, z)$. By the $L(-1)$-derivative property (2.2), we see that $\mathcal{Y}(\mathbf{1}, z)$ is independent on z. Denote $f = \mathcal{Y}(\mathbf{1}, z) \in \mathrm{Hom}(M^1, M^2)$. Since V is simple and M^1 is irreducible, Lemma 2.3 implies that f is nonzero. By (2.3), we have a commutation relation

$$[a_n, f] = [a_n, \mathcal{Y}(\mathbf{1}, z)] = \sum_{i=0}^{\infty}\binom{n}{i}\mathcal{Y}(a_i\mathbf{1}, z)z^{n-i} = 0$$

for $a \in V$ and $n \in \mathbb{Z}$. Hence f is a nonzero V-module homomorphism from M^1 to M^2. Since M^1 and M^2 are irreducible, f is in fact an isomorphism. Therefore M^1 is ismorphic to M^2. ∎

2.2. *Vertex Operator Algebra V_L^+ and Its Irreducible Modules*

We review the construction of the vertex operator algebra V_L^+ and its irreducible modules following [FLM, DL, D1, D2, DN2]. We also refer to the vertex operator algebra $M(1)^+$ and its irreducible modules (see also [DN1]).

Let L be an even lattice of rank 1 with a nondegenerate positive definite $\mathbb{Z}$-bilinear form $\langle\cdot,\cdot\rangle$, and $\mathfrak{h} = \mathbb{C}\otimes_{\mathbb{Z}} L$. Then $\mathfrak{h}$ has the nondegenerate symmetric $\mathbb{C}$-bilinear form given by extending the form $\langle\cdot,\cdot\rangle$ of L. Let $\mathbb{C}[\mathfrak{h}]$ be the group algebra of $\mathfrak{h}$ with a basis $\{e_\lambda | \lambda \in \mathfrak{h}\}$. For a subset M of $\mathfrak{h}$, set $\mathbb{C}[M] = \oplus_{\lambda\in M}\mathbb{C}e_\lambda$.

Let $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be a Lie algebra with the commutation relation given by $[X \otimes t^m, X' \otimes t^n] = m\delta_{m+n,0}\langle X, X'\rangle K, [K, \hat{\mathfrak{h}}] = 0$ for $X, X' \in \mathfrak{h}$, and $m, n \in \mathbb{Z}$. Then $\hat{\mathfrak{h}}^+ = \mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}K$ is subalgebra of $\hat{\mathfrak{h}}$, and the group algebra $\mathbb{C}[\mathfrak{h}]$ becomes a $\hat{\mathfrak{h}}^+$-module by the action $\rho(X \otimes t^n)e_\lambda = \delta_{n,0}\langle X, \lambda\rangle e_\lambda$ and $\rho(K)e_\lambda = e_\lambda$ for $\lambda, X \in \mathfrak{h}$ and $n \in \mathbb{N}$. It is clear that for a subset M of $\mathfrak{h}$ the subspace $\mathbb{C}[M]$ is a $\hat{\mathfrak{h}}^+$-submodule of $\mathbb{C}[\mathfrak{h}]$. Set V_M the induced module of $\hat{\mathfrak{h}}$ by $\mathbb{C}[M]$

$$V_M = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^+)} \mathbb{C}[M] \cong S(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes \mathbb{C}[M]\,(\text{linearly}),$$

where $U(\mathfrak{g})$ means the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. Denote the action of $X \otimes t^n (X \in \mathfrak{h}, n \in \mathbb{Z})$ on $V_\mathfrak{h}$ by $X(n)$ and set $X(z) = \sum_{n\in\mathbb{Z}} X(n)z^{-n-1}$ for $X \in \mathfrak{h}$. For $\lambda \in \mathfrak{h}$, the vertex operator associated with e_λ is defined by

$$\mathcal{Y}^\circ(e_\lambda, z) = \exp\left(\sum_{n=1}^{\infty} \frac{\lambda(-n)}{n} z^n\right) \exp\left(-\sum_{n=1}^{\infty} \frac{\lambda(n)}{n} z^{-n}\right) e_\lambda z^{\lambda(0)}, \tag{2.5}$$

where e_λ in the right-hand side means the left multiplication of $e_\lambda \in \mathbb{C}[\mathfrak{h}]$ on the group algebra $\mathbb{C}[\mathfrak{h}]$, and $z^{\lambda(0)}$ is an operator on $V_\mathfrak{h}$ defined by $z^{\lambda(0)}u = z^{\langle\lambda,\mu\rangle}u$ for $\mu \in \mathfrak{h}$ and $u \in U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^+)} \mathbb{C}e_\mu$. For $v = X_1(-n_1)\cdots X_\ell(-n_\ell)e_\lambda \in V_\mathfrak{h}(X_i \in \mathfrak{h}$ and $n_i \in \mathbb{Z}_+)$, the corresponding vertex operator is defined by

$$\mathcal{Y}^\circ(v, z) = {}^\circ_\circ\, \partial^{(n_1-1)}X_1(z)\cdots\partial^{(n_\ell-1)}X_\ell(z)\mathcal{Y}^\circ(e_\lambda, z){}^\circ_\circ, \tag{2.6}$$

where $\partial^{(n)} = (\frac{1}{n!})(d/dz)^n$, and the normal ordering ${}^\circ_\circ \cdot {}^\circ_\circ$ is an operation which reorders the operators so that $X(n)(X \in \mathfrak{h}, n < 0)$ and $e_\lambda$to be placed to the left of $X(n)(X \in \mathfrak{h}, n \geq 0)$ and $z^{\lambda(0)}$. We extend $\mathcal{Y}^\circ$ to $V_\mathfrak{h}$ by linearity. We denote $Y(a, z) = \mathcal{Y}^\circ(a, z)$ when a is in V_L.

Set $L = \mathbb{Z}\alpha$ with $\langle\alpha, \alpha\rangle = 2k$ for $k \in \mathbb{Z}_+$, and $L^\circ = \{\lambda \in \mathfrak{h} \mid \langle\lambda, \alpha\rangle \in \mathbb{Z}\}$, the dual lattice of L. Let $h = \alpha/\sqrt{2k}$ be the orthonormal basis of $\mathfrak{h}$ and set $\mathbf{1} = 1 \otimes e_0$ and $\omega = (1/2)h(-1)^2 e_0$. Then $(V_L, Y, \mathbf{1}, \omega)$ is a simple vertex operator algebra with central charge 1 and for $\lambda \in L^\circ$, $(V_{\lambda+L}, Y)$ is an irreducible module for V_L. Set $M(1) = S(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes e_0 \subset V_L$, then $(M(1), Y, \mathbf{1}, \omega)$ is a simple vertex operator algebra. If we set $M(1, \lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^+)} \mathbb{C}e_\lambda$ for each $\lambda \in \mathfrak{h}$, then $(M(1, \lambda), Y)$ becomes an irreducible $M(1)$-module (see [D1, DL]).

Let θ be a linear isomorphism of $V_\mathfrak{h}$ defined by

$$\begin{aligned}&\theta(X_1(-n_1)X_2(-n_2)\cdots X_\ell(-n_\ell) \otimes e_\lambda)\\ &\qquad = (-1)^\ell X_1(-n_1)X_2(-n_2)\cdots X_\ell(-n_\ell) \otimes e_{-\lambda},\end{aligned}$$

for $X_i \in \mathfrak{h}$, $n \in \mathbb{Z}_+$, and $\lambda \in \mathfrak{h}$. Then θ induces automorphisms of V_L and $M(1)$. For a θ-invariant subspace W of $V_\mathfrak{h}$, we denote the ± 1-eigenspaces of

W for θ by $W^{\pm}$. Then $(V_L^+, Y, \mathbf{1}, \omega)$ and $(M(1)^+, Y, \mathbf{1}, \omega)$ are vertex operator algebras. Furthermore $M(1)^{\pm}$ and $M(1, \lambda)$ for $\lambda \neq 0$ are irreducible $M(1)^+$-modules, and θ induces an $M(1)^+$-module isomorphism between $M(1, \lambda)$ and $M(1, -\lambda)$ (see [DN1]). As to V_L^+-modules, $V_L^{\pm}$, $V_{\alpha/2+L}^{\pm}$, and $V_{r\alpha/2k+L}$ for $1 \leq r \leq k-1$ are irreducible modules and θ induces a V_L^+-module isomorphism between $V_{\lambda+L}$ and $V_{-\lambda+L}$ for $\lambda \in L^{\circ}$ (see [DN2]).

Next we review the construction of θ-twisted V_L-modules following [FLM, D2]. Let $\hat{\mathfrak{h}}[-1] = \mathfrak{h} \otimes t^{1/2}\mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be a Lie algebra with the commutation relation $[X \otimes t^m, X' \otimes t^n] = m\delta_{m+n,0}\langle X, X'\rangle K, [K, \hat{\mathfrak{h}}[-1]] = 0$ for $X, X' \in \mathfrak{h}$ and $m, n \in 1/2 + \mathbb{Z}$. Then there is a one-dimensional module for $\hat{\mathfrak{h}}[-1]^+ = \mathfrak{h} \otimes t^{1/2}\mathbb{C}[t] \oplus \mathbb{C}K$, which is identified with $\mathbb{C}$, by defining the representation ρ by $\rho(X \otimes t^n)1 = 0$ and $\rho(K)1 = 1$ for $X \in \mathfrak{h}$ and $n \in 1/2 + \mathbb{N}$. Set $M(1)(\theta)$ the induced $\hat{\mathfrak{h}}[-1]$-module:

$$M(1)(\theta) = U(\hat{\mathfrak{h}}[-1]) \otimes_{U(\hat{\mathfrak{h}}[-1]^+)} \mathbb{C} \cong S(\mathfrak{h} \otimes^{-\frac{1}{2}} \mathbb{C}[t^{-1}]) \quad \text{(linearly)}.$$

Denote the action of $X \otimes t^n (X \in \mathfrak{h}, n \in 1/2 + \mathbb{Z})$ on $M(1)(\theta)$ by $X(n)$, and set $X(z) = \sum_{n\in 1/2+\mathbb{Z}} X(n)z^{-n-1}$. For $\lambda \in L^{\circ}$ a twisted vertex operator associated with $e_\lambda \in V_{\mathfrak{h}}$ is defined by

$$\mathcal{Y}^{\theta}(e_\lambda, z) = 2^{-\langle\lambda,\lambda\rangle} z^{-\frac{\langle\lambda,\lambda\rangle}{2}} \exp\left(\sum_{n\in 1/2+\mathbb{N}} \frac{\lambda(-n)}{n} z^n\right) \times \exp\left(-\sum_{n\in 1/2+\mathbb{N}} \frac{\lambda(n)}{n} z^{-n}\right). \tag{2.7}$$

For $v = X_1(-n_1)\cdots X_\ell(-n_\ell)e_\lambda \in V_{L^{\circ}} (X_i \in \mathfrak{h}$ and $n_i \in \mathbb{Z}_+)$, set

$$W^{\theta}(v, z) = {}^{\circ}_{\circ}\, \partial^{(n_1-1)}X_1(z)\cdots\partial^{(n_\ell-1)}X_\ell(z)\mathcal{Y}^{\theta}(v_\lambda, z)\, {}^{\circ}_{\circ}, \tag{2.8}$$

and extend it to $V_{L^{\circ}}$ by linearity, where the normal ordering ${}^{\circ}_{\circ} \cdot {}^{\circ}_{\circ}$ is an operation which reorders so that $X(n)(X \in \mathfrak{h}, n < 0)$ to be placed to the left of $X(n)(X \in \mathfrak{h}, n > 0)$. Let $c_{mn} \in \mathbb{Q}$ be coefficients defined by the formal, power series expansion

$$\sum_{m,n\geq 0} c_{mn}x^m y^n = -\log\left(\frac{(1+x)^{1/2} + (1+y)^{1/2}}{2}\right),$$

and set $\Delta_z = \sum_{m,n\geq 0} c_{mn}h(m)h(n)z^{-m-n}$. Then the twisted vertex operator associated to $u \in V_{L^{\circ}}$ is defined by

$$\mathcal{Y}^{\theta}(u, z) = W^{\theta}(\exp(\Delta_z)u, z). \tag{2.9}$$

If we write $Y^{\theta}(a, z) = \mathcal{Y}^{\theta}(a, z)$ for $a \in M(1)$, the pair $(M(1)(\theta), Y^{\theta})$ is the unique irreducible θ-twisted $M(1)$-module.

Let T_1 and T_2 be irreducible $\mathbb{C}[L]$-modules which e_α acts as 1 and -1, respectively, and set $V_L^{T_i} = M(1)(\theta) \otimes_{\mathbb{C}} T_i$ for $i = 1, 2$. For $u \in M(1, \beta)$ ($\beta \in L$), the corresponding twisted vertex operator is defined by $Y^\theta(u, z) = \mathcal{Y}^\theta(u, z) \otimes e_\beta$. We extend Y^θ to V_L by linearity. Then $(V_L^{T_i}, Y^\theta)(i = 1, 2)$ are irreducible θ-twisted V_L-modules. Note that $V_L^{T_i}$ has a θ-twisted $M(1)$-module structure. Let t_i be a basis of T_i for $i = 1, 2$. Then we have a canonical θ-twisted $M(1)$-module isomorphism

$$\phi_i : M(1)(\theta) \to V_L^{T_i} : u \mapsto u \otimes t_i \qquad \text{for } i = 1, 2. \tag{2.10}$$

The action of the automorphism θ on $M(1)(\theta)$ is defined by

$$\theta(X_1(-n_1) \cdots X_\ell(-n_\ell)1) = (-1)^\ell X_1(-n_1) \cdots X_\ell(-n_\ell)1,$$

for $X_i \in \mathfrak{h}$, $n_i \in 1/2 + \mathbb{N}$. Set $M(1)(\theta)^\pm$ the ± 1-eigenspaces of $M(1)(\theta)$ for θ and $V_L^{T_i, \pm}$ the ± 1-eigenspaces of $V_L^{T_i}$ for $\theta \otimes 1$. Then $M(1)(\theta)^\pm$ and $V_L^{T_i, \pm}(i = 1, 2)$ become irreducible $M(1)^+$-modules and irreducible V_L^+-modules respectively (see [DN1, DN2]).

All irreducible $M(1)^+$-modules and all irreducible V_L^+-modules are classified in [DN1, DN2].

THEOREM 2.6. (1) ([DN1]) *The set*

$$\{M(1)^\pm, M(1)(\theta)^\pm, M(1, \lambda)(\cong M(1, -\lambda)) \mid \lambda \in \mathfrak{h} - \{0\}\} \tag{2.11}$$

gives all inequivalent irreducible $M(1)^+$*-modules.*

(2) ([DN2]) *The set*

$$\left\{V_L^\pm, V_{\alpha/2+L}^\pm, V_L^{T_i, \pm}, V_{r\alpha/2k+L} \mid i = 1, 2, 1 \le r \le k - 1\right\} \tag{2.12}$$

gives all inequivalent irreducible V_L^+*-modules.*

We call irreducible modules $V_L^\pm$, $V_{\alpha/2+L}^\pm$, and $V_{r\alpha/2k+L}$ *untwisted type* modules, and call $V_L^{T_i, \pm}(i = 1, 2)$ *twisted type* modules. Here and further we write $\lambda_r = r\alpha/2k$ for $r \in \mathbb{Z}$.

2.3. *Fusion Rules for $M(1)^+$ and Contragredient Modules for V_L^+*

First we list the fusion rules for $M(1)^+$ determined in [A]. The fusion rules play central roles in determining fusion rules for V_L^+.

THEOREM 2.7. ([A]) *Let M^1, M^2, and M^3 be irreducible $M(1)^+$-modules. Then the fusion rule $N_{M^1\, M^2}^{M^3}$ is zero or one, and the fusion rule $N_{M^1\, M^2}^{M^3}$ is zero if and only if $M^i(i = 1, 2, 3)$ satisfy the following cases*:

(i) $M^1 = M(1)^+$ *and* $M^2 \cong M^3$.

(ii) $M^1 = M(1)^-$ *and the pair* (M^2, M^3) *is one of the following*:

$$(M(1)^{\pm}, M(1)^{\mp}),\ (M(1)(\theta)^{\pm}, M(1)(\theta)^{\mp}),\ (M(1,\lambda), M(1,\mu))$$

$$\text{for } \lambda, \mu \in \mathfrak{h} - \{0\} \text{ such that } \langle\lambda, \lambda\rangle = \langle\mu, \mu\rangle.$$

(iii) $M^1 = M(1, \lambda)$ *for* $\lambda \in \mathfrak{h} - \{0\}$ *and the pair* (M^2, M^3) *is one of the following*:

$$(M(1)^{\pm}, M(1,\mu))(M(1,\mu), M(1)^{\pm}) \qquad \text{for } \mu \in \mathfrak{h} - \{0\}$$

$$\text{such that } \langle\lambda, \lambda\rangle = \langle\mu, \mu\rangle,$$

$$(M(1,\mu), M(1,\nu)) \text{ for } \mu, \nu \in \mathfrak{h} - \{0\} \text{ such that } \langle\nu, \nu\rangle = \langle\lambda \pm \mu, \lambda \pm \mu\rangle,$$

$$(M(1)(\theta)^{\pm}, M(1)(\theta)^{\pm}),\ (M(1)(\theta)^{\pm}, M(1)(\theta)^{\mp}).$$

(iv) $M^1 = M(1)(\theta)^+$ *and the pair* (M^2, M^3) *is one of the following*:

$$(M(1)^{\pm}, M(1)(\theta)^{\pm}),\ (M(1)(\theta)^{\pm}, M(1)^{\pm}),\ (M(1,\lambda), M(1)(\theta)^{\pm}),$$

$$(M(1)(\theta)^{\pm}, M(1,\lambda)) \qquad \text{for } \lambda \in \mathfrak{h} - \{0\}.$$

(v) $M^1 = M(1)(\theta)^-$ *and the pair* (M^2, M^3) *is one of the following*:

$$(M(1)^{\pm}, M(1)(\theta)^{\mp}),\ (M(1)(\theta)^{\pm}, M(1)^{\mp}),\ (M(1,\lambda), M(1)(\theta)^{\pm}),$$

$$(M(1)(\theta)^{\pm}, M(1,\lambda)) \qquad \text{for } \lambda \in \mathfrak{h} - \{0\}.$$

Next we study the contragredient modules of irreducible V_L^+-modules. We shall prove the following proposition.

PROPOSITION 2.8. (i) *If* k *is even, then all irreducible* V_L^+*-modules* W *are self-dual*; *that is*, $W \cong W'$ *as* V_L^+*-modules.*

(ii) *If* k *is odd, then*

$$(V_{\alpha/2+L}^{\pm})' \cong V_{\alpha/2+L}^{\mp},\ \left(V_L^{T_1,\pm}\right)' \cong V_L^{T_2,\pm},\ \left(V_L^{T_2,\pm}\right)' \cong V_L^{T_1,\pm}$$

and others are self-dual.

To prove this proposition, we recall the notion of Zhu's algebra (see[Z]). Let V be a vertex operator algebra. Zhu's algebra $A(V)$ associated with V is a quotient space of V modulo the subspace $O(V)$ spanned by

$$a \circ b = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}(a)}}{z^2} Y(a,z)b$$

for homogeneous element $a \in V$ and $b \in V$. The product of $A(V)$ is induced from the bilinear map $* : V \times V \to V$which is defined by

$$a * b = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}(a)}}{z} Y(a,z)b$$

TABLE I
Actions of $o(\omega)$, $o(J)$, and $o(E)$ on the Top Levels

	V_L^+	V_L^-	$V_{\lambda_r+L}(1 \le r \le k-1)$	$V_{\alpha/2+1}^+$	$V_{\alpha/2+L}^-$
ω	0	1	$r^2/4k$	$k/4$	$k/4$
J	0	-6	$(r^2/2k)^2 - r^2/4k$	$k^4/4 - k^2/4$	$k^4/4 - k^2/4$
E	0	0	0	1	-1

	$V_L^{T_1,+}$	$V_L^{T_1,-}$	$V_L^{T_2,+}$	$V_L^{T_2,-}$
ω	$1/16$	$9/16$	$1/16$	$9/16$
J	$3/128$	$-45/128$	$3/128$	$-45/128$
E	2^{-2k+1}	$-2^{-2k+1}(4k-1)$	-2^{-2k+1}	$2^{-2k+1}(4k-1)$

for homogeneous element $a \in V$ and $b \in V$. Let M be an irreducible V-module. Then there is a constant $h \in \mathbb{C}$ such that M has an eigenspace decomposition $M = \oplus_{n\in\mathbb{N}} M_n$, $M_n = \{v \in M \mid L(0)v = (h+n)v\}$ for $n \in \mathbb{N}$. We always assume that $M_0 \neq 0$. Define $o(a)u = a_{\mathrm{wt}(a)-1}u$ for $a \in V$ and $u \in M$. Then o induces an irreducible representation of $A(V)$ on the top level M_0. Furthermore, two irreducible V-modules M and N are isomorphic if and only if M_0 and N_0 are isomorphic $A(V)$-modules.

Now we return to our case: $V = V_L^+$. Suppose that $k \neq 1$. Then Zhu's algebra $A(V_L^+)$ is generated by three elements $[\omega]$, $[J]$, and $[E]$, where $[a] = a + O(V_L^+) \in A(V_L^+)$ for $a \in V_L^+$, and $J = h(-1)^4\mathbf{1} - 2h(-3)h(-1)\mathbf{1} + (3/2)h(-2)^2\mathbf{1}$ and $E = e_\alpha + e_{-\alpha}$ (see [DN2]). Hence for an irreducible V_L^+-module M, to find the irreducible module which is isomorphic to M', it is enough to see the actions of $[\omega]$, $[J]$, and $[E]$ on the top level of M'. Since the top level of an irreducible V_L^+-module is one-dimensional when $k \neq 1$, they act on the top level as scalar multiples. By the construction of irreducible V_L^+-modules, we have Table I.

Now we prove Proposition 2.8.

Proof of Proposition 2.8. First we consider the case $k \neq 1$. Let W be an irreducible V_L^+-module. Set the top level $W_0 = \mathbb{C}v$, and the top level of the contragredient module $W_0' = \mathbb{C}v'$. By the definition of a contragredient module, if $a \in V_L^+$ satisfies that $L(0)a = \mathrm{wt}(a)a$ and $L(1)a = 0$, we have $\langle o(a)v', v\rangle = (-1)^{\mathrm{wt}(a)}\langle v', o(a)v\rangle$, and hence

$$\begin{aligned} &\langle o(\omega)v', v\rangle = \langle v', o(\omega)v\rangle, \langle o(J)v', v\rangle = \langle v' o(J)v\rangle, \\ &\langle o(E)v', v\rangle = (-1)^k \langle v', o(E)v\rangle. \end{aligned} \tag{2.13}$$

Therefore by Table I, we have Proposition 2.8 for $k \neq 1$.

If $k = 1$, then the dimension of the top level $(V_L^-)_0$ is 2 and others are one. Hence we see that $(V_L^-)' \cong V_L^-$ because the dimension of $(V_L^-)_0'$ is 2. Since for irreducible V_L^+-modules except V_L^- Table I is valid, we may apply same arguments of the case $k \neq 1$ to such irreducible modules. Therefore Table I and (2.13) show that

$$(V_L^+)' \cong V_L^+,\ (V_{\alpha/2+L}^{\pm})' \cong V_{\alpha/2+L}^{\mp},\ \left(V_L^{T_1,\pm}\right)' \cong V_L^{T_2,\pm},\ \left(V_L^{T_2,\pm}\right)' \cong V_L^{T_1,\pm}.$$

This proves Proposition 2.8 for $k = 1$. ∎

3. FUSION RULES FOR V_L^+

In Section 3.1, we give irreducible decompositions of irreducible V_L^+-modules as $M(1)^+$-modules, and prove that every fusion rule for V_L^+ is zero or one with the help of fusion rules for $M(1)^+$. The main theorem is stated in Section 3.2. The rest of section is devoted to the proof of the theorem and it is divided into two cases: one is the case that all modules are of untwisted types (Section 3.3) and the other is the case that some irreducible modules are of twisted types (Section 3.4).

3.1. *Irreducible Decompositions of Irreducible V_L^+-Modules as $M(1)^+$-Modules*

Since $V_{\lambda+L} = \bigoplus_{m\in\mathbb{Z}} M(1, \lambda + m\alpha)$ for $\lambda \in L^\circ$ and $M(1, \mu)$ is an irreducible $M(1)^+$-module if $\mu \neq 0$, $V_{\lambda_r+L} (1 \leq r \leq k-1)$ is a completely reducible $M(1)^+$-module:

$$V_{\lambda_r+L} \cong \bigoplus_{m\in\mathbb{Z}} M(1, \lambda_r + m\alpha). \tag{3.1}$$

For a nonzero $\lambda \in \mathfrak{h}$, we consider the subspace $(M(1)^+ \otimes (e_\lambda \pm e_{-\lambda})) \oplus (M(1)^- \otimes (e_\lambda \mp e_{-\lambda}))$ of $M(1, \lambda) \oplus M(1, -\lambda)$. Since the action of $M(1)^+$ on $M(1, \lambda) \oplus M(1, -\lambda)$ commutes the action of θ, the subspaces $(M(1)^+ \otimes (e_\lambda \pm e_{-\lambda})) \oplus (M(1)^- \otimes (e_\lambda \mp e_{-\lambda}))$ are $M(1)^+$-submodules. In fact we have the following lemma.

LEMMA 3.1. *For a nonzero $\lambda \in \mathfrak{h}$, $M(1)^+$-submodules $(M(1)^+ \otimes (e_\lambda \pm e_{-\lambda})) \oplus (M(1)^- \otimes (e_\lambda \mp e_{-\lambda}))$ of $M(1, \lambda) \oplus M(1, -\lambda)$ are isomorphic to $M(1, \lambda)$.*

Proof. Define a linear map ϕ_λ by

$$\begin{aligned}\phi_\lambda\colon (M(1)^+ \otimes (e_\lambda + e_{-\lambda})) \oplus (M(1)^- \otimes (e_\lambda - e_{-\lambda})) &\to M(1, \lambda)\\ u \otimes (e_\lambda + e_{-\lambda}) + v \otimes (e_\lambda - e_{-\lambda}) &\mapsto (u + v) \otimes e_\lambda,\end{aligned} \tag{3.2}$$

for $u \in M(1)^+$ and $v \in M(1)^-$. Then the linear map ϕ_λ is an injective $M(1)^+$-module homomorphism. Since $M(1, \lambda)$ is irreducible for $M(1)^+$, the homomorphism is in fact an isomorphism. Hence $M(1)^+ \otimes (e_\lambda + e_{-\lambda}) \oplus$

$M(1)^- \otimes (e_\lambda - e_{-\lambda})$ is isomorphic to $M(1,\lambda)$ as $M(1)^+$-module. We can also prove that $M(1)^+ \otimes (e_\lambda + e_{-\lambda}) \oplus M(1)^- \otimes (e_\lambda + e_{-\lambda})$ is isomorphic to $M(1,\lambda)$ as $M(1)^+$-module in the same way. ∎

We give irreducible decompositions of irreducible V_L^+-modules for $M(1)^+$;

PROPOSITION 3.2. *Each irreducible V_L^+-module is decomposed into a direct sum of irreducible $M(1)^+$-modules as follows*:

$$V_L^{\pm} \cong M(1)^{\pm} \oplus \bigoplus_{m=1}^{\infty} M(1, m\alpha), \tag{3.3}$$

$$V_{\lambda_r+L} \cong \bigoplus_{m\in\mathbb{Z}} M(1, \lambda_r + m\alpha) \qquad \textit{for } 1 \le r \le k-1, \tag{3.4}$$

$$V_{\alpha/2+L}^{\pm} \cong \bigoplus_{m=0}^{\infty} M(1, \alpha/2 + m\alpha), \tag{3.5}$$

$$V_L^{T_i,\pm} \cong M(1)(\theta)^{\pm} \qquad \textit{for } i = 1, 2. \tag{3.6}$$

In particular, the multiplicity of an irreducible $M(1)^+$-module in any irreducible V_L^+-module is at most one.

Proof. The irreducible decompositions of $V_{\lambda r+L}$ $(1 \le r \le k-1)$ and $V_L^{T_i,\pm}$ $(i = 1, 2)$ have already given by (3.1) and (2.10), respectively. We see that $V_L^{\pm}$ and $V_{\alpha/2+L}^{\pm}$ have direct sum decompositions

$$V_L^{\pm} = \bigoplus_{m=0}^{\infty} ((M(1)^+ \otimes (e_{m\alpha} \pm e_{-m\alpha})) \oplus (M(1)^- \otimes (e_{m\alpha} \mp e_{-m\alpha}))),$$

$$V_{\alpha/2+L}^{\pm} = \bigoplus_{m=0}^{\infty} ((M(1)^+ \otimes (e_{\frac{\alpha}{2}+m\alpha} \pm e_{-\frac{\alpha}{2}-m\alpha})) \oplus (M(1)^- \otimes (e_{\frac{\alpha}{2}+m\alpha} \mp e_{-\frac{\alpha}{2}-m\alpha}))).$$

Hence Lemma 3.1 shows that these direct sum decompositions give irreducible decompositions of $V_L^{\pm}$ and $V_{\alpha/2+L}^{\pm}$. The second assertion is obvious by Theorem 2.6 (1). ∎

Using these irreducible decompositions (3.3)–(3.6), Theorem 2.7, and Corollary 2.4, we can show that all fusion rules for V_L^+ are at most one:

PROPOSITION 3.3. *Let W^1, W^2, and W^3 be irreducible V_L^+-modules. Then the following hold.*

(1) *The fusion rule $N_{W^1\,W^2}^{W^3}$ is zero or one.*

(2) *If all W^i $(i = 1, 2, 3)$ are twisted type modules, then the fusion rule $N_{W^1\,W^2}^{W^3}$ is zero.*

(3) *If one of* W^i $(i = 1, 2, 3)$ *is twisted type module and others are of untwisted types, then the fusion rule* $N_{W^1\ W^2}^{W^3}$ *is zero.*

Proof. Suppose that W^1 and W^2 have irreducible $M(1)^+$-submodules M and N respectively and that W^3 has an irreducible decomposition $W^3 = \oplus_{i\in I} M^i$ and $M(1)^+$-module, where I is a index set. By (2.4), we have an inequality

$$\dim I_{V_L^+}\begin{pmatrix} & W^3 & \\ W^1 & & W^2 \end{pmatrix} \le \sum_{i\in I} \dim I_{M(1)^+}\begin{pmatrix} & M^i & \\ M & & N \end{pmatrix}. \tag{3.7}$$

If W^1, W^2, and W^3 are of twisted types or if W^1 is of a twisted type and W^2 and W^3 are of untwisted types, then by Theorem 2.7 (iv), (v), and (3.3)–(3.6), we see that the fusion rule for $M(1)^+$ of type $\binom{M^i}{M\ N}$ is zero for any i. Hence (3.7) implies that the fusion rule $N_{W^1\ W^2}^{W^3}$ is zero. Since the contragredient module of an (un)twisted type module is of an (un)twisted type, (2) and (3) follow from Proposition 2.2.

By (2), (3), and Proposition 2.2, to show (1), it suffices to prove that for untwisted type modules W^1, W^2, and W^3 or for an untwisted type module W^1 and twisted type modules W^2 and W^3 the fusion rule $N_{W^1\ W^2}^{W^3}$ is zero or one.

If W^1 is of untwisted type module and W^2 and W^3 are of twisted types, then by Theorem 2.7 (i)–(iii) and irreducible decompositions (3.3)–(3.6), we see that the fusion rule for $M(1)^+$of type $\binom{W^3}{M\ W^2}$ is zero or one for any irreducible $M(1)^+$-submodules M of W^1. Hence Corollary 2.4 shows that the fusion rule $N_{W^1\ W^2}^{W^3}$ is zero or one.

Now we turn to the case that all $W^i (i = 1, 2, 3)$ are of untwisted types. We consider the following three cases separately; (i) $W^1 = V_L^{\pm}$, (ii) $W^1 = V_{\alpha/2+L}^{\pm}$, and (iii) $W^1 = V_{\lambda r+L}$ for $1 \le r \le k-1$. Let $W^3 = \oplus_i M^i$ be the irreducible decomposition of W^3 for $M(1)^+$. Then it suffices to prove that the right-hand side (3.7) is at most one for some $M(1)^+$-submodules M of W^1 and N of W^2.

(i) $W^1 = V_L^{\pm}$ cases: Take $M = M(1)^{\pm}$. By (3.3)–(3.5), we can take N to be isomorphic to $M(1, \lambda)$ for some $\lambda \in L^{\circ}$. Then by Theorem 2.7 (i) and (ii), the fusion rule for $M(1)^+$ of type $\binom{M^i}{M\ N}$ is one if and only if M^i is isomorphic to $M(1, \lambda)$. Since the multiplicity of $M(1, \lambda)$ in an untwisted type module is at most one, we see that the right-hand side of (3.7) is zero or one.

(ii) $W^1 = V_{\alpha/2+L}^{\pm}$ case: Take $M \cong M(1, \alpha/2)$. If N is isomorphic to $M(1, \lambda)$ for some $\lambda \in L^{\circ}$, then by Theorem 2.7 (iii), we see that the fusion rule for $M(1)^+$ of type $\binom{M^i}{M\ N}$ is one if and only if M^i is isomorphic to $M(1, \lambda + \alpha/2)$ or $M(1, \lambda - \alpha/2)$. If $W^2 = V_{\alpha/2+L}^{\pm}$, then by taking $\lambda = \alpha/2$, we see that the right-hand side of (3.7) is zero unless W^3 is V_L^+ or V_L^-. So these cases and the cases $W^2 = V_L^{\pm}$ reduce to the case (i) by means of Proposition 2.2. Therefore, to prove (1) in the case $W^1 = V_{\alpha/2+L}^{\pm}$, it is enough to consider the case $W^2 = V_{\lambda_r+L}$ for some $1 \le r \le k-1$. Then by taking $\lambda = \lambda_r$, we see that the right-hand side of (3.7) is zero unless W^3 is $V_{\lambda_{k-r}+L}$. By Corollary 2.4 and Proposition 2.8, the fusion rules of type $\binom{V_{\lambda_{k-r}+L}}{V_{\alpha/2+L}^{\pm}\ V_{\lambda_r+L}}$ are equal to those of types $\binom{(V_{\alpha/2+L}^{\pm})'}{V_{\lambda_r+L}\ V_{\lambda_{k-r}+L}}$. Hence we have to show that the right-hand side of (3.7) is at most one when $W^1 = V_{\lambda_r+L}$, $W^2 = V_{\lambda_{k-r}+L}$, and $W^3 = (V_{\alpha/2+L}^{\pm})'$. We take $M = M(1,\lambda_r)$ and $N = M(1,\lambda_{k-r})$. Since by Theorem 2.7 the fusion rule for $M(1)^+$ of type $\binom{M(1,\alpha/2+m\alpha)}{M\ \ N}$ is $\delta_{m,0}$ for $m \in \mathbb{N}$, (3.5) shows that the the right-hand side of (3.7) is at most one.

(iii) $W^1 = V_{\lambda_r+L}$ case for $1 \le r \le k-1$: By Proposition 2.2 and the cases (i) and (ii), to prove (1) in this case, it suffices to consider the case $W^2 = V_{\lambda_s+L}$ for $1 \le s \le k-1$. Then we can take $M = M(1,\lambda_r)$ and $N = M(1,\lambda_s)$. Hence by Theorem 2.7 (iii), we see that the fusion rule for $M(1)^+$ of type $\binom{M^i}{M\ N}$ is one if and only if M^i is isomorphic to $M(1,\lambda_r+\lambda_s)$ or $M(1,\lambda_r-\lambda_s)$. By (3.3)–(3.5), one sees that if both $M(1,\mu)$ and $M(1,\nu)$ $(\mu,\nu \in L^\circ)$ have multiplicity one in W^3, then $\mu+\nu \in L$ or $\mu-\nu \in L$. But $(\lambda_r+\lambda_s)+(\lambda_r-\lambda_s)$ and $(\lambda_r+\lambda_s)-(\lambda_r-\lambda_s)$ are not in L. Hence by (3.3)–(3.5), we see that the right-hand side of (3.7) is zero or one. ∎

3.2. *Main Theorem*

Here we state the main theorem. The proof is given in Sections 3.3 and 3.4:

THEOREM 3.4. *Let W^1, W^2, and W^3 be irreducible V_L^+-modules. Then* (1) *the fusion rule $N_{W^1\ W^2}^{W^3}$ is zero or one and* (2) *the fusion rule $N_{W^1\ W^2}^{W^3}$ is one if and only if $W^i (i = 1,2,3)$ satisfy following cases*:

(I) *The case that k is even.*

(i) *$W^1 = V_L^+$ and $W^2 \cong W^3$.*

(ii) *$W^1 = V_L^+$ and the pair (W^2, W^3) is one of the following pairs*:

$$(V_L^{\pm}, V_L^{\mp}), (V_{\alpha/2+L}^{\pm}, V_{\alpha/2+L}^{\mp}), \left(V_L^{T_1,\pm}, V_L^{T_1,\mp}\right), \left(V_L^{T_2,\pm}, V_L^{T_2,\mp}\right),$$
$$(V_{\lambda_r+L}, V_{\lambda_r+L}) \qquad \text{for } 1 \le r \le k-1.$$

(iii) $W^1 = V^+_{\alpha/2+L}$ *and the pair* (W^2, W^3) *is one of the following pairs*:

$$(V_L^{\pm}, V^{\pm}_{\alpha/2+L}), (V^{\pm}_{\alpha/2+L}, V_L^{\pm}), \left(V_L^{T_1,\pm}, V_L^{T_1,\pm}\right), \left(V_L^{T_2,\pm}, V_L^{T_2,\mp}\right),$$
$$(V_{\lambda_r+L}, V_{\alpha/2-\lambda_r+L}) \qquad \text{for } 1 \le r \le k-1.$$

(iv) $W^1 = V^-_{\alpha/2+L}$ *and the pair* (W^2, W^3) *is one of the following pairs*:

$$(V_L^{\pm}, V^{\mp}_{\alpha/2+L}), (V^{\pm}_{\alpha/2+L}, V_L^{\mp}), \left(V_L^{T_1,\pm}, V_L^{T_1,\mp}\right), \left(V_L^{T_2,\pm}, V_L^{T_2,\pm}\right),$$
$$(V_{\lambda_r+L}, V_{\alpha/2-\lambda_r+L}) \qquad \text{for } 1 \le r \le k-1.$$

(v) $W^1 = V_{\lambda_r+L}$ *for* $1 \le r \le k-1$ *and the pair* (W^2, W^3) *is one of the following pairs*:

$$(V_L^{\pm}, V_{\lambda_r+L}), (V_{\lambda_r+L}, V_L^{\pm}), (V^{\pm}_{\alpha/2+L}, V_{\alpha/2-\lambda_r+L}), (V_{\alpha/2-\lambda_r+L}, V^{\pm}_{\alpha/2+L}),$$
$$(V_{\lambda_s+L}, V_{\lambda_r\pm\lambda_s+L}) \qquad \text{for } 1 \le s \le k-1 \text{ such that } r \pm s \ne 0, k,$$
$$\left(V_L^{T_1,\pm}, V_L^{T_1,\pm}\right), \left(V_L^{T_1,\pm}, V_L^{T_1,\mp}\right), \left(V_L^{T_2,\pm}, V_L^{T_2,\pm}\right), \left(V_L^{T_2,\pm}, V_L^{T_2,\mp}\right) \text{ if } r \text{ is even},$$
$$\left(V_L^{T_1,\pm}, V_L^{T_2,\pm}\right), \left(V_L^{T_1,\pm}, V_L^{T_2,\mp}\right), \left(V_L^{T_2,\pm}, V_L^{T_1,\pm}\right), \left(V_L^{T_2,\pm}, V_L^{T_1,\mp}\right) \text{ if } r \text{ is odd}.$$

(vi) $W^1 = V_L^{T_1,+}$ *and the pair* (W^2, W^3) *is one of the following pairs*:

$$\left(V_L^{\pm}, V_L^{T_1,\pm}\right), \left(V_L^{T_1,\pm}, V_L^{\pm}\right), \left(V^{\pm}_{\alpha/2+L}, V_L^{T_1,\pm}\right), \left(V_L^{T_1,\pm}, V^{\pm}_{\alpha/2+L}\right),$$
$$\left(V_{\lambda_r+L}, V_L^{T_1,\pm}\right) \text{ and } \left(V_L^{T_1,\pm}, V_{\lambda_r+L}\right) \quad \text{for } 1 \le r \le k-1 \text{ such that } r \text{ is even},$$
$$\left(V_{\lambda_r+L}, V_L^{T_2,\pm}\right) \text{ and } \left(V_L^{T_2,\pm}, V_{\lambda_r+L}\right) \quad \text{for } 1 \le r \le k-1 \text{ such that } r \text{ is odd}.$$

(vii) $W^1 = V_L^{T_1,-}$ *and the pair* (W^2, W^3) *is one of the following pairs*:

$$\left(V_L^{\pm}, V_L^{T_1,\mp}\right), \left(V_L^{T_1,\pm}, V_L^{\mp}\right), \left(V^{\pm}_{\alpha/2+L}, V_L^{T_1,\mp}\right), \left(V_L^{T_1,\pm}, V^{\mp}_{\alpha/2+L}\right),$$
$$\left(V_{\lambda_r+L}, V_L^{T_1,\pm}\right) \text{ and } \left(V_L^{T_1,\pm}, V_{\lambda_r+L}\right) \quad \text{for } 1 \le r \le k-1 \text{ such that } r \text{ is even},$$
$$\left(V_{\lambda_r+L}, V_L^{T_2,\pm}\right) \text{ and } \left(V_L^{T_2,\pm}, V_{\lambda_r+L}\right) \quad \text{for } 1 \le r \le k-1 \text{ such that } r \text{ is odd}.$$

(viii) $W^1 = V_L^{T_2,+}$ *and the pair* (W^2, W^3) *is one of the following pairs*:

$\left(V_L^{\pm}, V_L^{T_2,\pm}\right)$, $\left(V_L^{T_2,\pm}, V_L^{\pm}\right)$, $\left(V_{\alpha/2+L}^{\pm}, V_L^{T_2,\mp}\right)$, $\left(V_L^{T_2,\pm}, V_{\alpha/2+L}^{\mp}\right)$,

$\left(V_{\lambda_r+L}, V_L^{T_2,\pm}\right)$ *and* $\left(V_L^{T_2,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is even*,

$\left(V_{\lambda_r+L}, V_L^{T_1,\pm}\right)$ *and* $\left(V_L^{T_1,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is odd*.

(ix) $W^1 = V_L^{T_2,-}$ *and the pair* (W^2, W^3) *is one of the following pairs*:

$\left(V_L^{\pm}, V_L^{T_2,\mp}\right)$, $\left(V_L^{T_2,\pm}, V_L^{\mp}\right)$, $\left(V_{\alpha/2+L}^{\pm}, V_L^{T_2,\pm}\right)$, $\left(V_L^{T_2,\pm}, V_{\alpha/2+L}^{\pm}\right)$,

$\left(V_{\lambda_r+L}, V_L^{T_2,\pm}\right)$ *and* $\left(V_L^{T_2,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is even*,

$\left(V_{\lambda_r+L}, V_L^{T_1,\pm}\right)$ *and* $\left(V_L^{T_1,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is odd*.

(II) *The case that* k *is odd*.

(i) $W^1 = V_L^{+}$ *and* $W^2 \cong W^3$.

(ii) $W^1 = V_L^{-}$ *and the pair* (W^2, W^3) *is one of the following pairs*:

$(V_L^{\pm}, V_L^{\mp})$, $(V_{\alpha/2+L}^{\pm}, V_{\alpha/2+L}^{\mp})$, $\left(V_L^{T_1,\pm}, V_L^{T_1,\mp}\right)$, $\left(V_L^{T_2,\pm}, V_L^{T_2,\mp}\right)$,

$(V_{\lambda_r+L}, V_{\lambda_r+L})$ *for* $1 \le r \le k-1$.

(iii) $W^1 = V_{\alpha/2+L}^{+}$ *and the pair* (W^2, W^3) *is one of the following pairs*:

$(V_L^{\pm}, V_{\alpha/2+L}^{\pm})$, $(V_{\alpha/2+L}^{\pm}, V_L^{\mp})$, $\left(V_L^{T_1,\pm}, V_L^{T_2,\pm}\right)$, $\left(V_L^{T_2,\pm}, V_L^{T_1,\mp}\right)$,

$(V_{\lambda_r+L}, V_{\alpha/2-\lambda_r+L})$ *for* $1 \le r \le k-1$.

(iv) $W^1 = V_{\alpha/2+L}^{-}$ *and the pair* (W^2, W^3) *is one of the following pairs*:

$(V_L^{\pm}, V_{\alpha/2+L}^{\mp})$, $(V_{\alpha/2+L}^{\pm}, V_L^{\pm})$, $\left(V_L^{T_1,\pm}, V^{T_2,\mp}\right)$, $\left(V_L^{T_2,\pm}, V_L^{T_1,\pm}\right)$,

$(V_{\lambda_r+L}, V_{\alpha/2-\lambda_r+L})$ *for* $1 \le r \le k-1$.

(v) $W^1 = V_{\lambda_r+L}$ *for* $1 \le r \le k-1$ *and the pair* (W^2, W^3) *is one of the following pairs*:

$(V_L^{\pm}, V_{\lambda_r+L}), (V_{\lambda_r+L}, V_L^{\pm}), (V_{\alpha/2+L}^{\pm}, V_{\alpha/2-\lambda_r+L}), (V_{\alpha/2-\lambda_r+L}, V_{\alpha/2+L}^{\pm})$,

$(V_{\lambda_s+L}, V_{\lambda_r\pm\lambda_s+L})$ *for* $1 \le s \le k-1$ *such that* $r \pm s \ne 0, k$,

$\left(V_L^{T_1,\pm}, V_L^{T_1,\pm}\right), \left(V_L^{T_1,\pm}, V_L^{T_1,\mp}\right), \left(V_L^{T_2,\pm}, V_L^{T_2,\pm}\right), \left(V_L^{T_2,\pm}, V_L^{T_2,\mp}\right)$ *if* r *is even*,

$\left(V_L^{T_1,\pm}, V_L^{T_2,\pm}\right), \left(V_L^{T_1,\pm}, V_L^{T_2,\mp}\right), \left(V_L^{T_2,\pm}, V_L^{T_1,\pm}\right), \left(V_L^{T_2,\pm}, V_L^{T_1,\mp}\right)$ *if* r *is odd*.

(vi) $W^1 = V_L^{T_1,+}$ *and the pair* (W^2, W^3) *is one of the following pairs*:

$\left(V_L^{\pm}, V_L^{T_1,\pm}\right)$, $\left(V_L^{T_2,\pm}, V_L^{\pm}\right)$, $\left(V_{\alpha/2+L}^{\pm}, V_L^{T_2,\pm}\right)$, $\left(V_L^{T_1,\pm}, V_{\alpha/2+L}^{\mp}\right)$,

$\left(V_{\lambda_r+L}, V_L^{T_1,\pm}\right)$ *and* $\left(V_L^{T_2,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is even*,

$\left(V_{\lambda_r+L}, V_L^{T_2,\pm}\right)$ *and* $\left(V_L^{T_1,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is odd*.

(vii) $W^1 = V_L^{T_1,-}$ *and the pair* (W^2, W^3) *is one of the following pairs*:

$\left(V_L^{\pm}, V_L^{T_1,\mp}\right)$, $\left(V_L^{T_2,\pm}, V_L^{\mp}\right)$, $\left(V_{\alpha/2+L}^{\pm}, V_L^{T_2,\mp}\right)$, $\left(V_L^{T_1,\pm}, V_{\alpha/2+L}^{\mp}\right)$,

$\left(V_{\lambda_r+L}, V_L^{T_1,\pm}\right)$ *and* $\left(V_L^{T_2,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is even*,

$\left(V_{\lambda_r+L}, V_L^{T_2,\pm}\right)$ *and* $\left(V_L^{T_1,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is odd*.

(viii) $W^1 = V_L^{T_2,+}$ *and the pair* (W^2, W^3) *is one of the following pairs*:

$\left(V_L^{\pm}, V_L^{T_2,\pm}\right)$, $\left(V_L^{T_1,\pm}, V_L^{\pm}\right)$, $\left(V_{\alpha/2+L}^{\pm}, V_L^{T_1,\mp}\right)$, $\left(V_L^{T_2,\pm}, V_{\alpha/2+L}^{\pm}\right)$,

$\left(V_{\lambda_r+L}, V_L^{T_2,\pm}\right)$ *and* $\left(V_L^{T_1,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is even*,

$\left(V_{\lambda_r+L}, V_L^{T_1,\pm}\right)$ *and* $\left(V_L^{T_2,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is odd*.

(ix) $W^1 = V_L^{T_2,-}$ *and the pair* (W^2, W^3) *is one of the following pairs*:

$\left(V_L^{\pm}, V_L^{T_2,\mp}\right)$, $\left(V_L^{T_1,\pm}, V_L^{\mp}\right)$, $\left(V_{\alpha/2+L}^{\pm}, V_L^{T_1,\pm}\right)$, $\left(V_L^{T_2,\pm}, V_{\alpha/2+L}^{\mp}\right)$,

$\left(V_{\lambda_r+L}, V_L^{T_2,\pm}\right)$ *and* $\left(V_L^{T_1,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is even*,

$\left(V_{\lambda_r+L}, V_L^{T_1,\pm}\right)$ *and* $\left(V_L^{T_2,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is odd*.

Since the assertion (1) of the main theorem has already been proved in Proposition 3.3, to complete the proof of the theorem, it is enough to show that for irreducible V_L^+-modules W^1, W^2, and W^3, the fusion rule $N_{W^1\,W^2}^{W^3}$ is nonzero if and only if the triple (W^1, W^2, W^3) satisfy indicated cases in the theorem. In Section 3.3, we prove this in the case that all W^i $(i = 1, 2, 3)$ are untwisted type modules, and in Section 3.4 we do in the case that some of W^i $(i = 1, 2, 3)$ are twisted type modules. To show the conditional part, we shall construct nonzero intertwining operators explicitly.

3.3. *Fusion Rules for Untwisted Type Modules*

We construct nonzero intertwining operators for untwisted type modules. For this purpose, we review intertwining operators for V_L following [DL].

As shown in [FLM Chap. 8], the operator $\mathcal{Y}^{\circ}$ satisfies Jacobi identity and $L(-1)$-derivative property on $V_{L^{\circ}}$ for $\beta \in L$, $\lambda \in L^{\circ}$, $a \in M(1, \beta)$, and $u \in M(1, \lambda)$:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y(a, z_1)\mathcal{Y}^{\circ}(u, z_2) - (-1)^{(\beta, \lambda)}z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)$$

$$\times \mathcal{Y}^{\circ}(u, z_2)Y(a, z_1) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\mathcal{Y}^{\circ}(Y(a, z_0)u, z_2),$$

$$\frac{d}{dz}\mathcal{Y}^{\circ}(u, z) = \mathcal{Y}^{\circ}(L(-1)u, z).$$

Let $\pi_\lambda (\lambda \in L^{\circ})$ be the linear endomorphism of $V_{L^{\circ}}$ defined by $\pi_\lambda(v) = e^{\langle\lambda, \mu\rangle\pi i}v$ for $\mu \in L^{\circ}$ and $v \in M(1, \mu)$. Set $\mathcal{Y}_{r,s}(u, z) = \mathcal{Y}^{\circ}(u, z)\pi_{\lambda_r|_{V_{\lambda_s+L}}}$ for $r, s \in \mathbb{Z}$ and $u \in V_{\lambda_r+L}$. Then the operator $\mathcal{Y}_{r,s}$ gives a nonzero intertwining operator for V_L of type $\binom{V_{\lambda_r+\lambda_s+L}}{V_{\lambda_r+L}\ V_{\lambda_s+L}}$ (see [DL]).

PROPOSITION 3.5. *The fusion rules for V_L^+ of the following types are nonzero*:

(i) $\binom{V_{(\lambda_r\pm\lambda_s)+L}}{V_{\lambda_r+L}\ V_{\lambda_s+L}}$ *for* $1 \le r, s \le k-1$,

(ii) $\binom{V_L^{\pm}}{V_L^{+}\ V_L^{\pm}}$, $\binom{V_L^{\mp}}{V_L^{-}\ V_L^{\pm}}$ *and* $\binom{V_{\lambda_r+L}}{V_L^{\pm}\ V_{\lambda_r+L}}$ *for* $0 \le r \le k-1$,

(iii) $\binom{V_{\alpha/2+L}^{\pm}}{V_L^{+}\ V_{\alpha/2+L}^{\pm}}$, $\binom{V_{\alpha/2+L}^{\mp}}{V_L^{-}\ V_{\alpha/2+L}^{\pm}}$ *and* $\binom{V_{(\alpha/2-\lambda_r)+L}}{V_{\alpha/2+L}^{\pm}\ V_{\lambda_r+L}}$ *for* $0 \le r \le k-1$.

Proof. Since (V_L, Y), $(V_{\alpha/2+L}, Y)$, and (V_{λ_r+L}, Y) $(1 \le r \le k-1)$ are irreducible V_L-modules, the vertex operator Y gives nonzero intertwining operators for V_L of types

$$\begin{pmatrix} V_L \\ V_L\ \ V_L \end{pmatrix}, \begin{pmatrix} V_{\alpha/2+L} \\ V_L\ \ V_{\alpha/2+L} \end{pmatrix}, \text{ and } \begin{pmatrix} V_{\lambda_r+L} \\ V_L\ \ V_{\lambda_r+L} \end{pmatrix}$$

for $1 \le r \le k-1$. Hence $Y(a, z)u$ is nonzero for any nonzero vectors $a \in V_L$ and $u \in V_{\lambda_s+L} (s \in \mathbb{Z})$ by Corollary 2.4. Therefore, since $\theta Y(a, z)\theta = Y(\theta(a), z)$ for $a \in V_L$, Y gives nonzero intertwining operators for V_L^+ of types

$$\begin{pmatrix} V_L^{\pm} \\ V_L^{+}\ \ V_L^{\pm} \end{pmatrix}, \begin{pmatrix} V_L^{\mp} \\ V_L^{-}\ \ V_L^{\pm} \end{pmatrix}, \begin{pmatrix} V_{\alpha/2+L}^{\pm} \\ V_L^{+}\ \ V_{\alpha/2+L}^{\pm} \end{pmatrix}, \begin{pmatrix} V_{\alpha/2+L}^{\mp} \\ V_L^{-}\ \ V_{\alpha/2+L}^{\pm} \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} V_{\lambda_r+L} \\ V_L^{\pm}\ \ V_{\lambda_r+L} \end{pmatrix}$$

for $1 \le r \le k-1$.

Next we show that fusion rules of types $\binom{V_{(\lambda_r\pm\lambda_s)+L}}{V_{\lambda_r+L}\ V_{\lambda_s+L}}$ for $r, s \in \mathbb{Z}$ are nonzero. Define $\mathcal{Y}_{r,-s}\circ\theta$ by $(\mathcal{Y}_{r,-s}\circ\theta)(u,z)v = \mathcal{Y}_{r,-s}(u,z)\theta(v)$ for $u \in V_{\lambda_r+L}$ and $v \in V_{\lambda_s+L}$. Then $\mathcal{Y}_{r,s}$ is a nonzero intertwining operator for V_L^+ of type $\binom{V_{(\lambda_r-\lambda_s)+L}}{V_{\lambda_r+L}\ V_{\lambda_s+L}}$ since θ commutes the action of V_L^+. This proves that fusion rules of types $\binom{V_{(\lambda_r\pm\lambda_s)+L}}{V_{\lambda_r+L}\ V_{\lambda_s+L}}$ are nonzero for any $r, s \in \mathbb{Z}$.

Finally we show that fusion rules of type $\binom{V_{(\alpha/2-\lambda_r)+L}}{V^{\pm}_{\alpha/2+L}\ V_{\lambda_r+L}}$ for $r \in \mathbb{Z}$ are nonzero. Since $V_{\alpha/2+L}$ is an irreducible V_L-module, Corollary 2.4 shows that $\mathcal{Y}_{k,r}(u,z)v$ is nonzero for any nonzero vectors $u \in V_{\alpha/2+L}$ and $v \in V_{\lambda_r+L}$. Hence $(\mathcal{Y}_{k,-r}\circ\theta)(u,z)v$ is also nonzero for any nonzero vectors $u \in V_{\alpha/2+L}$ and $v \in V_{\lambda_r+L}$. Therefore, $\mathcal{Y}_{k,-r}\circ\theta$ gives a nonzero intertwining operator of type $\binom{V_{(\alpha/2-\lambda_r)+L}}{V^{\pm}_{\alpha/2+L}\ V_{\lambda_r+L}}$. ∎

Next we show that if $W^i (i = 1, 2, 3)$ are of untwisted types and the fusion rule $N_{W^1W^2}^{W^3}$ is nonzero, then $\binom{W^3}{W^1\ W^2}$ is one of the types in Proposition 3.5. To prove this, by Proposition 2.2, it suffices to prove the following proposition.

PROPOSITION 3.6. *Let W^1, W^2, and W^3 be untwisted type modules. Then the fusion rule $N_{W^1W^2}^{W^3}$ is zero if $W^i (i = 1, 2, 3)$ satisfy the following cases:*

(i) *$W^1 = V_L^+$, and W^2, W^3 are not equivalent.*

(ii) *$W^1 = V_L^-$ and the pair (W^2, W^3) is one of the following pairs:*

$$(W^2, W^3) = (V_L^-, V_L^-), (V^{\pm}_{\alpha/2+L}, V^{\pm}_{\alpha/2+L}),$$
$$(V_{\lambda_r+L}, V_{\lambda_s+L}) \quad \textit{for } 1 \le r, s \le k-1 \textit{ such that } r \ne s,$$
$$(V_L^-, V_{\lambda_r+L}), (V^{\pm}_{\alpha/2+L}, V_{\lambda_r+L}) \quad \textit{for } 1 \le r \le k-1.$$

(iii) *$W^3 = V^{\pm}_{\alpha/2+L}$ and the pair (W^2, W^3) is one of the following pairs:*

$$(W^2, W^3) = (V_{\lambda_r+L}, V_{\lambda_s+L}) \quad \textit{for } 1 \le r, s \le k-1 \textit{ such that } r+s \ne k,$$
$$(V^{\pm}_{\alpha/2+L}, V_{\lambda_r+L}) \textit{ and } 1 \le r \le k-1.$$

(iv) *$W^1 = V_{\lambda_r+L}$ for $1 \le r \le k-1$ and the pair (W^2, W^3) is one of the following pairs:*

$$(W^2, W^3) = (V_{\lambda_s+L}, V_{\lambda_t+L}) \quad \textit{for } 1 \le s, t \le k-1 \textit{ such that } t \ne |r \pm s|.$$

Proof. Lemma 2.5 proves the proposition in the case (i).

Next we show the proposition in the cases (ii) except the pairs $(V^{\pm}_{\alpha/2+L}, V^{\pm}_{\alpha/2+L})$, (iii), and (iv). Let $W^3 = \oplus_i M^i$ be the irreducible decomposition of W^3 for $M(1)^+$. Then we can find irreducible $M(1)^+$-submodules M of W^1 and N of W^3 such that the fusion rule for $M(1)^+$

of type $\binom{M^i}{M\ N}$ is zero for any M^i. Hence (3.7) implies that the fusion rule $N_{W^1W^2}^{W^3}$ is zero; for example, in the case $W^1 = W^2 = W^3 = V_L^-$, we take $M = N = M(1)^-$, etc.

It remains to prove the assertion in the cases $(W^2, W^3) = (V_{\alpha/2+L}^{\pm}, V_{\alpha/2+L}^{\pm})$ of (ii). We show that fusion rule of type $\binom{V_{\alpha/2+L}^+}{V_L^-\ V_{\alpha/2+L}^+}$ is zero. The case of type $\binom{V_{\alpha/2+L}^-}{V_L^-\ V_{\alpha/2+L}^-}$ can be also proved in a similar way.

Set

$$V_{\alpha/2+L}^+[m]=M(1)^+\otimes(e_{\frac{\alpha}{2}+m\alpha}+e_{-(\frac{\alpha}{2}+m\alpha)})\oplus M(1)^-\otimes(e_{\frac{\alpha}{2}+m\alpha}-e_{-(\frac{\alpha}{2}+m\alpha)}), \tag{3.8}$$

for $m \in \mathbb{N}$. Note that $V_{\alpha/2+L}^+[m]$ is isomorphic to $M(1, \alpha/2 + m\alpha)$ as $M(1)^+$-module by Lemma 3.1. Let $\mathcal{Y}$ be an intertwining operator of type $\binom{(V_{\alpha/2+L}^+}{V_L^-\ V_{\alpha/2+L}^+)}$. By Theorem 2.7 (ii), we have $\mathcal{Y}(u, z)v \in V_{\alpha/2+L}^+[0]((z))$ for $u \in M(1)^-$ and $v \in V_{\alpha/2+L}^+[0]$. Recall the $M(1)^+$-module isomorphism $\phi_{\alpha/2}$: $V_{\alpha/2+L}^+[0] \to M(1, \alpha/2)$ defined in (3.2). For simplicity, we denote $\phi = \phi_{\alpha/2}$. Then the operator $\phi \circ \mathcal{Y} \circ \phi^{-1}$ defined by $(\phi \circ \mathcal{Y} \circ \phi^{-1})(u, z)v = \phi\mathcal{Y}(u, z)\phi^{-1}(v)$ for $u \in M(1)^-$ and $v \in M(1, \alpha/2)$ gives an intertwining operator of type $\binom{(M(1,\alpha/2)}{M(1)^-\ M(1,\alpha/2))}$. Since the dimension of $I_{M(1)^+}\binom{M(1,\alpha/2)}{M(1)^-\ M(1,\alpha/2)}$ is one and the corresponding intertwining operator is given by a scalar multiple of the vertex operator Y of the $M(1)$-module $(M(1, \alpha/2), Y)$, there exists a constant $d \in \mathbb{C}$ such that

$$\mathcal{Y}(u, z)v = d\phi^{-1}Y(u, z)\phi(v)$$

for all $u \in M(1)^-$ and $v \in V_{\alpha/2+L}^+[0]$. We write $\mathcal{Y}(u, z) = \sum_{n\in\mathbb{Z}} \tilde{u}(n)z^{-n-1}$, $\tilde{u}(n) \in \operatorname{End} V_{\alpha/2+L}^+$ for $u \in V_L^-$. Take $u = h(-1)\mathbf{1}$ and $v = e_{\alpha/2} + e_{-\alpha/2}$, then we have

$$\tilde{h}(0)(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}) = d\left\langle h, \frac{\alpha}{2}\right\rangle(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}), \tag{3.9}$$

$$\tilde{h}(-1)(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}) = d(h(-1)e_{\frac{\alpha}{2}} - h(-1)e_{-\frac{\alpha}{2}}), \tag{3.10}$$

where we denote $(h(\widetilde{-1})\mathbf{1})(n)$ by $\tilde{h}(n)$ for $n \in \mathbb{Z}$. By direct calculations, we see that

$$E_{k-1}(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}) = (e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}), \tag{3.11}$$

$$E_k(h(-1)e_{\frac{\alpha}{2}} - h(-1)e_{-\frac{\alpha}{2}}) = \langle h, \alpha\rangle(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}), \tag{3.12}$$

where $E = e_\alpha + e_{-\alpha} \in V_L^+$. Let $F = e_\alpha - e_{-\alpha} \in V_L^-$. Then by the commutator formula 2.3, we have a commutation relation

$$[E_m, \tilde{h}(n)] = -\langle h, \alpha\rangle\widetilde{F}(m + n) \tag{3.13}$$

for $m, n \in \mathbb{Z}$. Hence (3.9) and (3.11) imply that $\widetilde{F}(k-1)(e_{\alpha/2}+e_{-\alpha/2})=0$ (take $m=k-1$, $n=0$ in (3.13)). On the other hand, by (3.10) and (3.12) we have

$$-\langle h,\alpha\rangle \widetilde{F}(k-1)(e_{\frac{\alpha}{2}}+e_{-\frac{\alpha}{2}}) = [E_k, \tilde{h}(-1)](e_{\frac{\alpha}{2}}+e_{-\frac{\alpha}{2}})$$
$$= d\langle h,\alpha\rangle(e_{\frac{\alpha}{2}}+e_{-\frac{\alpha}{2}}),$$

(take $m=k$, $n=-1$ in (3.13)). Therefore $d=0$. This implies that $\mathcal{Y}(h(-1)\mathbf{1},z)(e_{\alpha/2}+e_{-\alpha/2})=0$, and then Lemma 2.3 shows $\mathcal{Y}=0$. Thus the fusion rule of type $\binom{V^{+}_{\alpha/2+L}}{V^{-}_{L}\ V^{+}_{\alpha/2+L}}$ is zero. ∎

Consequently, by Propositions 2.2, 2.8, 3.3, 3.5, and 3.6, we can determine fusion rules for untwisted type modules.

PROPOSITION 3.7. *Let W^1, W^2, and W^3 be untwisted type V_L^+-modules. Then the fusion rule $N_{W^1W^2}^{W^3}$ is zero or one. The fusion rule $N_{W^1W^2}^{W^3}$ is one if and only if $W^i (i=1,2,3)$ satisfy the following cases*:

(i) $W^1=V_L^+$, *and* $W^2\cong W^3$.

(ii) $W^1=V_L^-$ *and the pair* (W^2,W^3) *is one of the following pairs*:

$$(V_L^{\pm},V_L^{\mp}),(V^{\pm}_{\alpha/2+L},V^{\mp}_{\alpha/2+L}),(V_{\lambda_r+L},V_{\lambda_r+L})\quad \textit{for } 1\le r\le k-1.$$

(iii) $W^1=V^{+}_{\alpha/2+L}$ *and the pair* (W^2,W^3) *is one of the following pairs*:

$$(V_L^{\pm},V^{\pm}_{\alpha/2+L}),((V^{\pm}_{\alpha/2+L})',V_L^{\pm}),\ (V_{\lambda_r+L},V_{\alpha/2-\lambda_r+L})\quad \textit{for } 1\le r\le k-1.$$

(iv) $W^1=V^{-}_{\alpha/2+L}$ *and the pair* (W^2,W^3) *is one of the following pairs*:

$$(V_L^{\pm},V^{\mp}_{\alpha/2+L}),((V^{\pm}_{\alpha/2+L})',V_L^{\mp}),\ (V_{\lambda_r+L},V_{\alpha/2-\lambda_r+L})\quad \textit{for } 1\le r\le k-1.$$

(v) $W^1=V_{\lambda_r+L}$ *for* $1\le r\le k-1$ *and the pair* (W^2,W^3) *is one of the following pairs*:

$$(V_L^{\pm},V_{\lambda_r+L}),(V_{\lambda_r+L},V_L^{\pm}),\ (V^{\pm}_{\alpha/2+L},V_{\alpha/2-\lambda_r+L}),(V_{\alpha/2-\lambda_r+L},V^{\pm}_{\alpha/2+L}),$$
$$(V_{\lambda_s+L},V_{\lambda_r\pm\lambda_s+L})\qquad \textit{for } 1\le s\le k-1.$$

3.4. *Fusion Rules Involving Twisted Type Modules*

Set $\mathcal{P}_L=L^{\circ}\times\{1,2\}\times\{1,2\}$. We call $(\lambda,i,j)\in\mathcal{P}_L$ a *quasi-admissible triple if* λ, i, and j satisfy

$$(-1)^{\langle\lambda,\alpha\rangle+\delta_{i,j}+1}=1.$$

We denote the set of all quasi-admissible triples by $\mathcal{Q}_L$. For a quasi-admissible triple $(\lambda,i,j)\in\mathcal{Q}_L$, we first construct an intertwining operator for V_L^+ of type $\binom{V_L^{T_j}}{V_{\lambda+L}\ V_L^{T_i}}$.

As shown in [FLM, Chap. 9], the operator $\mathcal{Y}^\theta$ satisfies twisted Jacobi identity

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\mathcal{Y}^\theta(a,z_1)\mathcal{Y}^\theta(u,z_2) - (-1)^{\langle\beta,\lambda\rangle}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\mathcal{Y}^\theta(u,z_2)\mathcal{Y}^\theta(a,z_1) = \frac{1}{2}\sum_{p=0,1} z_2^{-1}\delta\left((-1)^p\frac{(z_1-z_0)^{1/2}}{z_2^{1/2}}\right)\mathcal{Y}^\theta(Y(\theta^p(a),z_0)u,z_2), \quad (3.14)$$

and $L(-1)$-derivative property

$$\frac{d}{dz}\mathcal{Y}^\theta(u,z) = \mathcal{Y}^\theta(L(-1)u,z) \quad (3.15)$$

for $\beta \in L$, $\lambda \in L^\circ$, $a \in M(1,\beta)$, and $u \in M(1,\lambda)$. Then we have the following lemma.

LEMMA 3.8. (1) *The intertwining operator $\mathcal{Y}^\theta$ gives nonzero intertwining operators of types*

$$\binom{M(1)(\theta)^\pm}{M(1,\lambda)\quad M(1)(\theta)^\pm}, \binom{M(1)(\theta)^\mp}{M(1,\lambda)\quad M(1)(\theta)^\pm} \quad \textit{for } \lambda \in L^\circ.$$

(2) *Define $\mathcal{Y}^\theta \circ \theta$ by $(\mathcal{Y}^\theta \circ \theta)(u,z) = \mathcal{Y}^\theta(\theta(u),z)$ for $u \in V_{L^\circ}$. Then $\mathcal{Y}^\theta \circ \theta$ gives nonzero intertwining operators for $M(1)^+$ of types $\binom{M(1)(\theta)}{M(1,\lambda)\quad M(1)(\theta)^\pm}$. Moreover restrictions of $\mathcal{Y}^\theta$ and $\mathcal{Y}^\theta \circ \theta$ to $M(1,\lambda)\otimes M(1)(\theta)^\pm$ form a basis of the vector space $I\binom{M(1)(\theta)}{M(1,\lambda)\quad M(1)(\theta)^\pm}$.*

Proof. The assertion (1) is proved in [A, Proposition 4.4]. Next we show (2). Clearly $\mathcal{Y}^\theta \circ \theta$ gives nonzero intertwining operators of types $\binom{M(1)(\theta)}{M(1,\lambda)\quad M(1)(\theta)^\pm}$. Since $\theta\mathcal{Y}^\theta(u,z)\theta(v) = \mathcal{Y}^\theta(\theta(u),z)v$ for $u \in M(1,\lambda)$ and $v \in M(1)(\theta)$, we have

$$p_\pm((\mathcal{Y}^\theta\circ\theta)(u,z)v) = \pm p_\pm(\mathcal{Y}^\theta(u,z)\theta(v)) \quad \text{for } u \in M(1,\lambda) \text{ and } v \in M(1)(\theta)^\pm,$$

where $p_\pm$ is the canonical projection from $M(1)(\theta)$ to $M(1)(\theta)^\pm$, respectively. Hence by Lemma 2.3 and (1), we see that $\mathcal{Y}^\theta$ and $\mathcal{Y}^\theta\circ\theta$ are linearly independent in the vector spaces $I\binom{M(1)(\theta)}{M(1,\lambda)\quad M(1)(\theta)^\pm}$. Since the fusion rules of types $\binom{M(1)(\theta)}{M(1,\lambda)\quad M(1)(\theta)^\pm}$ are 2 by Theorem 3.4, $\mathcal{Y}^\theta$ and $\mathcal{Y}^\theta\circ\theta$ in fact form a basis of $I\binom{M(1)(\theta)}{M(1,\lambda)\quad M(1)(\theta)^\pm}$. This proves (2). ∎

Set $T = T^1 \oplus T^2$ the direct sum of the irreducible $\mathbb{C}[L]$-modules T^1 and T^2, and define a linear isomorphism $\psi \in \operatorname{End} T$ by $\psi(t_1) = t_2$, $\psi(t_2) = t_1$, where t_i is a basis of T^i for $i = 1, 2$. For $\lambda \in L^\circ$, we write $\lambda = r\alpha/2k + m\alpha$ for $-k+1 \le r \le k$ and $m \in \mathbb{Z}$, and define $\psi_\lambda \in \operatorname{End} T$ by

$$\psi_\lambda = e_{m\alpha} \circ \underbrace{\psi \circ \cdots \circ \psi}_{r}.$$

Set $\widetilde{\mathcal{Y}}(u, z) = \mathcal{Y}^\theta(u, z) \otimes \psi_\lambda$ for $\lambda \in L^\circ$ and $u \in M(1, \lambda)$, and extend it to V_{L° by linearity. Then we have following proposition.

PROPOSITION 3.9. (1) *For $\lambda \in L^\circ$, the linear map ψ_λ has following properties*:

$$e_\beta \circ \psi_\lambda = (-1)^{\langle \beta, \lambda \rangle} \psi_\lambda \circ e_\beta = \psi_{\lambda+\beta} \qquad \text{for all } \beta \in L.$$

(2) *For $a \in V_L$ and $u \in V_{\lambda+L}$, we have*

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y^\theta(a, z_1)\widetilde{\mathcal{Y}}(u, z_2) - \delta\left(\frac{z_2 - z_1}{-z_0}\right)\widetilde{\mathcal{Y}}(u, z_2) Y^\theta(a, z_1)$$
$$= \frac{1}{2}\sum_{p=0,1} z_2^{-1}\delta\left((-1)^p \frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}}\right)\widetilde{\mathcal{Y}}(Y(\theta^p(a), z_0)u, z_2)$$

and

$$\frac{d}{dz}\widetilde{\mathcal{Y}}(u, z) = \widetilde{\mathcal{Y}}(L(-1)u, z).$$

Proof. Since $e_\alpha \circ \psi = -\psi \circ e_\alpha$, we have $e_{m\alpha} \circ \psi^r = (-1)^{mr} \psi^r \circ e_{m\alpha}$ for $m, r \in \mathbb{Z}$. Therefore $\psi_\lambda (\lambda \in L^\circ)$ satisfies $e_\beta \circ \psi_\lambda = (-1)^{\langle \beta, \lambda \rangle} \psi_\lambda \circ e_\beta$ and $e_\beta \circ \psi_\lambda = \psi_{\lambda+\beta}$ for $\beta \in L$. This proves (1). Then the assertion (2) follows from (3.14), (3.15), and (1). ∎

We note that for each quasi-admissible triple $(\lambda, i, j) \in \mathcal{Q}_L$, $\psi_\lambda(T^i) = T^j$. Thus we have:

PROPOSITION 3.10. *Let $(\lambda, i, j) \in \mathcal{Q}_L$ be an admissible triple. The restriction of $\widetilde{\mathcal{Y}}$ to $V_{\lambda+L} \otimes V_L^{T_i}$ gives an intertwining operator for V_L^+ of type $\binom{V_L^{T_j}}{V_{\lambda+L}\ V_L^{T_i}}$.*

Now we have some nonzero intertwining operators by restricting $\widetilde{\mathcal{Y}}$ to irreducible V_L^+-modules.

PROPOSITION 3.11. *The fusion rules for V_L^+ of following types are nonzero*:

(i) $\binom{V_L^{T_j,\pm}}{V_{\lambda_r+L}\ V_L^{T_i,\pm}}$, $\binom{V_L^{T_j,\mp}}{V_{\lambda_r+L}\ V_L^{T_i,\pm}}$ *for $r \in \mathbb{Z}$ and $(\lambda_r, i, j) \in \mathcal{Q}_L$,*

(ii) $\binom{V_L^{T_i,\pm}}{V_L^+ \ V_L^{T_i,\pm}}, \binom{V_L^{T_i,\mp}}{V_L^- \ V_L^{T_i,\pm}}$ *for* $i \in \{1,2\}$,

(iii) $\binom{V_L^{T_1,\pm}}{V_{\alpha/2+L}^+ \ (V_L^{T_1,\pm})'}, \binom{V_L^{T_2,\mp}}{V_{\alpha/2+L}^+ \ (V_L^{T_2,\pm})'}, \binom{V_L^{T_1,\mp}}{V_{\alpha/2+L}^- \ (V_L^{T_1,\pm})'}, \binom{V_L^{T_2,\pm}}{V_{\alpha/2+L}^- \ (V_L^{T_2,\pm})'}$.

Proof. By Lemma 3.8 and Proposition 3.10, we see that $\widetilde{\mathcal{Y}}$ gives nonzero intertwining operators of types $\binom{V_L^{T_j,\pm}}{V_{\lambda_r+L} \ V_L^{T_i,\pm}}$ and $\binom{V_L^{T_j,\mp}}{V_{\lambda_r+L} \ V_L^{T_i,\pm}}$ for $r \in \mathbb{Z}$ and $(\lambda_r, i, j) \in \mathcal{Q}_L$.

Next we show that fusion rules of types in (ii) and (iii) are nonzero. By Lemma 3.8 (2) and Corollary 2.4, $\mathcal{Y}^\theta(u \pm \theta(u), z)v = (\mathcal{Y}^\theta \pm \mathcal{Y}^\theta \circ \theta)(u, z)v$ are nonzero for any nonzero vectors $u \in M(1,\lambda)(\lambda \in L^\circ)$ and $v \in M(1)(\theta)^\pm$. Thus by Proposition 3.10, we see that $\widetilde{\mathcal{Y}}$ give nonzero intertwining operators of types

$$\begin{pmatrix} V_L^{T_i} \\ V_L^+ \quad V_L^{T_i,\pm} \end{pmatrix}, \begin{pmatrix} V_L^{T_i} \\ V_L^- \quad V_L^{T_i,\pm} \end{pmatrix}, \begin{pmatrix} V_L^{T_i} \\ V_{\alpha/2+L}^+ \quad \left(V_L^{T_i,\pm}\right)' \end{pmatrix},$$
$$\begin{pmatrix} V_L^{T_i} \\ V_{\alpha/2+L}^- \quad \left(V_L^{T_i,\pm}\right)' \end{pmatrix} \qquad \text{for } i \in \{1,2\}.$$

By the definition of $\psi_\lambda(\lambda \in L^\circ)$, we have

$$\psi_{-m\alpha} = \psi_{m\alpha}, \psi_{-(\alpha/2+m\alpha)} = e_{-\alpha}\psi_{\alpha/2+m\alpha} \qquad \text{for } m \in \mathbb{Z}. \tag{3.16}$$

Since $\theta\widetilde{\mathcal{Y}}(u,z)\theta = \mathcal{Y}^\theta(\theta(u), z) \otimes \psi_\lambda$ for $\lambda \in L^\circ$ and $u \in M(1,\lambda)$, by (3.16) we have

$$\theta\widetilde{\mathcal{Y}}(u,z)\theta = \widetilde{\mathcal{Y}}(\theta(u), z) \qquad \text{for } u \in V_L,$$
$$\theta\widetilde{\mathcal{Y}}(u,z)\theta = e_\alpha\widetilde{\mathcal{Y}}(\theta(u), z) \qquad \text{for } u \in V_{\alpha/2+L}.$$

This proves that $\widetilde{\mathcal{Y}}$ gives nonzero intertwining operators of types indicated in (ii) and (iii) in the proposition; for instance, for $u \in V_{\alpha/2+L}^+$ and $v \in \left(V_L^{T_2,-}\right)'$, we have

$$\begin{aligned} \theta\widetilde{\mathcal{Y}}(u,z)v &= e_\alpha\widetilde{\mathcal{Y}}(\theta(u), z)\theta(v) \\ &= \widetilde{\mathcal{Y}}(u,z)v. \end{aligned}$$

Hence $\widetilde{\mathcal{Y}}(u,z)v \in V_L^{T_2,+}\{z\}$. Thus $\widetilde{\mathcal{Y}}$ gives a nonzero intertwining operator of type $\binom{V_L^{T_2,+}}{V_{\alpha/2+L}^+ \ (V_L^{T_2,-})'}$. ∎

We shall show the following proposition. The proof is given after Proposition 3.13.

PROPOSITION 3.12. (1) *For $i, j \in \{1, 2\}$, the fusion rules of types*

$$\binom{V_L^{T_j}}{V_L^{\pm} \quad V_L^{T_i,\pm}} \text{ and } \binom{V_L^{T_j}}{V_L^{\pm} \quad V_L^{T_i,\mp}}$$

are zero if $i \neq j$.

(2) *For $1 \le r \le k-1$ and $i, j \in \{1, 2\}$, the fusion rules of types*

$$\binom{V_L^{T_j}}{V_{\lambda_r+L} \quad V_L^{T_i,\pm}}$$

are zero if $(-1)^{r+\delta_{i,j}+1} \neq 1$.

(3) *For $i, j \in \{1, 2\}$, the fusion rules of types*

$$\binom{V_L^{T_j}}{V_{\alpha/2+L}^{\pm} \quad V_L^{T_i,\pm}} \text{ and } \binom{V_L^{T_j}}{V_{\alpha/2+L}^{\pm} \quad V_L^{T_i,\mp}}$$

are zero if $(-1)^{k+\delta_{i,j}+1} \neq 1$.

To prove Proposition 3.12, we need the following proposition.

PROPOSITION 3.13. *Let W be an irreducible V_L^+-module and suppose that W contains an $M(1)^+$-submodule isomorphic to $M(1, \lambda)$ for some $\lambda \in L^o$. If $(\lambda, i, j) \in \mathscr{P}_L$ is not a quasi-admissible triple, then fusion rules of types $\binom{V_L^{T_j}}{W \; V_L^{T_i,\pm}}$ are zero.*

Proof. Let W be an irreducible V_L^+-module, let N be an $M(1)^+$-submodule of W isomorphic to $M(1, \lambda)$, and let f be an $M(1)^+$-isomorphism from $M(1, \lambda)$ to N. Consider an intertwining operator $\mathcal{Y} \in I_{V_L^+}\binom{V_L^{T_j}}{W \; V_L^{T_i,\epsilon}}$ for $i, j \in \{1, 2\}$ and $\epsilon \in \{\pm\}$. We have to prove that $\mathcal{Y} = 0$ if $(-1)^{\langle\alpha,\lambda\rangle+\delta_{i,j}+1} \neq 1$.

The restrictions of $\mathcal{Y}$ to $N \otimes V_L^{T_i,\epsilon}$ give an intertwining operator for $M(1)^+$ of type $\binom{V_L^{T_j}}{N \quad V_L^{T_i,\epsilon}}$. Set

$$\overline{\mathcal{Y}}(u, z) = \phi_j^{-1}\mathcal{Y}(f(u), z)\phi_i \qquad \text{for } u \in M(1, \lambda).$$

Then $\overline{\mathcal{Y}}$ is an intertwining operator for $M(1)^+$ of type $\binom{M(1)(\theta)}{M(1,\lambda) \; M(1)(\theta)^{\epsilon}}$. By Lemma 3.8 (2), for any $u \in M(1, \lambda)$, $\overline{\mathcal{Y}}(u, z)$ is a linear combination of $\mathcal{Y}^{\theta}(u, z)$ and $\mathcal{Y}^{\theta}(\theta(u), z)$. By (3.14), we have

$$\begin{aligned} z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\mathcal{Y}^{\theta}(E, z_1)\mathcal{Y}^{\theta}(e_{\pm\lambda}, z_2) - (-1)^{\langle\alpha,\lambda\rangle} z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right) \\ \times \mathcal{Y}^{\theta}(e_{\pm\lambda}, z_2)\mathcal{Y}^{\theta}(E, z_1) = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\mathcal{Y}^{\theta}(Y(E, z_0)e_{\pm\lambda}, z_2), \quad (3.17) \end{aligned}$$

where $E = e_\alpha + e_{-\alpha} \in V_L^+$. Hence one has

$$(z_1 - z_2)^M \mathcal{Y}^\theta(E, z_1)\mathcal{Y}^\theta(e_{\pm\lambda}, z_2) = (-1)^{\langle\alpha,\lambda\rangle}(z_1 - z_2)^M \mathcal{Y}^\theta(e_{\pm\lambda}, z_2)\mathcal{Y}^\theta(E, z_1)$$

for a sufficiently large integer M, and then

$$\begin{aligned}(z_1 - z_2)^M &\mathcal{Y}^\theta(E, z_1)\overline{\mathcal{Y}}(e_\lambda, z_2) \\ &= (-1)^{\langle\alpha,\lambda\rangle}(z_1 - z_2)^M \overline{\mathcal{Y}}(e_\lambda, z_2)\mathcal{Y}^\theta(E, z_1). \end{aligned} \tag{3.18}$$

(3.18) is an identity on $M(1)(\theta)^\epsilon$. We next derive an identity on $V_L^{T_i,\epsilon}$ from (3.18). Since $e_{\pm\alpha} \in \mathbb{C}[L]$ act on $V_L^{T_i}$ $(i = 1, 2)$ as the scalar $(-1)^{\delta_{1,2}}$, we have

$$e_{\pm\alpha}\phi_j\overline{\mathcal{Y}}(u, z)\phi_i^{-1} = (-1)^{\delta_{i,j}+1}\phi_j\overline{\mathcal{Y}}(u, z)\phi_i^{-1}e_{\pm\alpha}$$

for $u \in M(1, \lambda)$. And $Y^\theta(E, z)$ acts on $V_L^{T_i}(i = 1, 2)$ as $\mathcal{Y}^\theta(E, z) \otimes e_\alpha$. Hence by (3.18), we have

$$\begin{aligned}(z_1 - z_2)^M &Y^\theta(E, z_1)\mathcal{Y}(f(e_\lambda), z_2) \\ &= (-1)^{\langle\alpha,\lambda\rangle+\delta_{i,j}+1}(z_1 - z_2)^M \mathcal{Y}(f(e_\lambda), z_2)Y^\theta(E, z_1) \end{aligned} \tag{3.19}$$

for a sufficiently large integer M. On the other hand, since $\mathcal{Y}$ is an intertwining operator for V_L^+ of type $\binom{V_L^{T_j}}{W\ V_L^{T_i,\epsilon}}$, Jacobi identity (2.1) shows that

$$(z_1 - z_2)^M Y^\theta(E, z_1)\mathcal{Y}(f(e_\lambda), z_2) = (z_1 - z_2)^M \mathcal{Y}(f(e_\lambda), z_2)Y^\theta(E, z_1)$$

for a sufficiently large integer M. Therefore by (3.19) and (3.20), if $(-1)^{\langle\alpha,\lambda\rangle+\delta_{i,j}+1} \neq 1$, then

$$(z_1 - z_2)^M \mathcal{Y}(f(e_\lambda), z_2)Y^\theta(E, z_1)u = 0 \tag{3.20}$$

for a nonzero $u \in V_L^{T_i,\epsilon}$ and a sufficiently large integer M. Since there is an integer n_0 such that $E_{n_0}u \neq 0$ and $E_n u = 0$ for all $n > n_0$, by multiplying $z_1^{n_0}$ and taking Res_{z_1} on both sides of (3.20), we have $z_2^M\mathcal{Y}(f(e_\lambda), z_2)E_{n_0}u = 0$. Hence Lemma 2.3 implies that $\mathcal{Y} = 0$. ∎

Now we prove Proposition 3.12.

Proof of Proposition 3.12. By the irreducible decompositions (3.3)–(3.5), we see that V_{λ_r+L} contains $M(1, \lambda_r)$ for $1 \leq r \leq k - 1$, that $V^\pm_{\alpha/2+L}$ contains an $M(1)^+$-submodule isomorphic to $M(1, \alpha/2)$, and that $V_L^\pm$ contains an $M(1)^+$-submodule isomorphic to $M(1, \alpha)$. Hence Proposition 3.12 follows from Proposition 3.13. ∎

Finally, we prove the following proposition:

PROPOSITION 3.14. (1) *For $i \in \{1, 2\}$, the fusion rules of types*

$$\binom{V_L^{T_i,\mp}}{V_L^+ \quad V_L^{T_i,\pm}} \text{ and } \binom{V_L^{T_i,\pm}}{V_L^- \quad V_L^{T_i,\pm}}$$

are zero.

(2) *The fusion rules of types*

$$\binom{V_L^{T_1,\mp}}{V_{\alpha/2+L}^+ \quad \left(V_L^{T_1,\pm}\right)'}, \binom{V_L^{T_2,\pm}}{V_{\alpha/2+L}^+ \quad \left(V_L^{T_2,\pm}\right)'}, \binom{V_L^{T_1,\pm}}{V_{\alpha/2+L}^- \quad \left(V_L^{T_1,\pm}\right)'},$$
$$\binom{V_L^{T_2,\mp}}{V_{\alpha/2+L}^- \quad \left(V_L^{T_2,\pm}\right)'}$$

are zero.

Proof. Since $V_L^{\pm}$ contains the irreducible $M(1)^+$-module $M(1)^{\pm}$ and the fusion rules of types $\binom{M(1)(\theta)^{\mp}}{M(1)^+ \quad M(1)(\theta)^{\pm}}$ and $\binom{M(1)(\theta)^{\pm}}{M(1)^- \quad M(1)(\theta)^{\pm}}$ are zero by Theorem 2.7 (i) and (ii), (1) follows from Corollary 2.4.

Next we prove that the fusion rules of types $\binom{V_L^{T_1,\mp}}{V_{\alpha/2+L}^+ \quad (V_L^{T_1,\pm})'}$ and $\binom{V_L^{T_2,\pm}}{V_{\alpha/2+L}^+ \quad (V_L^{T_2,\pm})'}$ are zero. The assertion (2) for types $\binom{V_L^{T_1,\pm}}{V_{\alpha/2+L}^- \quad (V_L^{T_1,\pm})'}$ and $\binom{V_L^{T_2,\mp}}{V_{\alpha/2+L}^- \quad (V_L^{T_1,\pm})'}$ can be also proved in a similar way.

By Proposition 3.11 (iii), for $i \in \{1, 2\}$ and $\epsilon \in \{\pm\}$, there exists $\epsilon' \in \{\pm\}$ such that the fusion rule of type $\binom{V_L^{T_i,\epsilon'}}{V_{\alpha/2+L}^+ \quad (V_L^{T_i,\epsilon})'}$ is nonzero. Let $\{\tau, \epsilon'\} = \{\pm\}$. Then we have to prove that the fusion rule of type $\binom{V_L^{T_i,\tau}}{V_{\alpha/2+L}^+ \quad (V_L^{T_i,\epsilon})'}$ is zero. To show this, we prove that the projection

$$\binom{V_L^{T_i}}{V_{\alpha/2+L}^+ \quad \left(V_L^{T_i,\epsilon}\right)'} \to \binom{V_L^{T_i,\epsilon'}}{V_{\alpha/2+L}^+ \quad \left(V_L^{T_i,\epsilon}\right)'}, \mathcal{Y} \mapsto p_{\epsilon'} \circ \mathcal{Y}$$

is injective, where $p_{\epsilon'}$ are the canonical projections from $V_L^{T_i}$ to $V_L^{T_i,\epsilon'}$ and $p_{\epsilon'} \circ \mathcal{Y}$ is the intertwining operator defined by $(p_{\epsilon'} \circ \mathcal{Y})(u, z)v = p_{\epsilon'}(\mathcal{Y}(u, z)v)$ for $u \in V_{\alpha/2+L}^+$ and $v \in (V_L^{T_i,\epsilon})'$. To prove this, it is enough to prove that an arbitrary nonzero intertwining operator $\mathcal{Y}$ of type $\binom{V_L^{T_i}}{V_{\alpha/2+L}^+ \quad (V_L^{T_i,\epsilon})'}$ satisfies

$$\theta\mathcal{Y}(e_{\alpha/2} + e_{-\alpha/2}, z)\theta = (-1)^{\delta_{i,2}}\mathcal{Y}(e_{\alpha/2} + e_{-\alpha/2}, z). \tag{3.21}$$

Actually if $\mathcal{Y}$ is a nonzero intertwining operator of the indicated type satisfying (3.21), then $p_{\epsilon'}(\mathcal{Y}(e_{\alpha/2} + e_{-\alpha/2}, z)v) = \mathcal{Y}(e_{\alpha/2} + e_{-\alpha/2}, z)v$ for $v \in (V_L^{T_i,\epsilon})'$, and then $p_{\epsilon'} \circ \mathcal{Y}$ is nonzero by Corollary 2.4.

Let $V_L^{T_j} = (V_L^{T_i})'$, and let $V^+_{\alpha/2+L}[0]$ be as of (3.8). Then $\mathcal{Y}$ gives an intertwining operator for $M(1)^+$ of type $\binom{V_L^{T_i}}{V^+_{\alpha/2+L}[0]\ V_L^{T_j,\epsilon}}$. Thus by Lemma 3.8 (2), we see that $I_{M(1)^+}\binom{V_L^{T_j}}{V^+_{\alpha/2+L}[0]\ V_L^{T_i,\epsilon}}$ is spanned by intertwining operators $\mathcal{Y}^\pm$ defined by

$$\mathcal{Y}^\pm(u,z) = \phi_i \mathcal{Y}^\theta(\phi_{\pm\alpha/2}(u),z)\phi_j^{-1} \text{ for } u \in V^+_{\alpha/2+L}[0]. \tag{3.22}$$

Hence there exist constants $c_1, c_2 \in \mathbb{C}$ such that

$$\mathcal{Y}(u,z) = c_1\mathcal{Y}^+(u,z) + c_2\mathcal{Y}^-(u,z) \tag{3.23}$$

for all $u \in V^+_{\alpha/2+L}[0]$. Now for $\beta \in \mathfrak{h}$, set

$$\exp\left(\sum_{n=0}^\infty \frac{\beta(-n)}{n} z^n\right) = \sum_{n=0}^\infty p_n(\beta)z^n \in (\mathrm{End}V_{L^\circ})[[z]].$$

Then we have $E_0(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}) = p_{k-1}(\alpha)e_{\frac{\alpha}{2}} + p_{k-1}(-\alpha)e_{-\frac{\alpha}{2}} \in V^+_{\alpha/2+L}[0]$, and hence

$$\phi_{\frac{\alpha}{2}}(E_0(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}})) = p_{k-1}(\alpha)e_{\frac{\alpha}{2}},\ \ \phi_{-\frac{\alpha}{2}}(E_0(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}})) = p_{k-1}(-\alpha)e_{-\frac{\alpha}{2}}.$$

Thus by (3.22) and (3.23), we have

$$\begin{aligned}[E_0, \mathcal{Y}(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}, z)] &= \mathcal{Y}(E_0(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}), z) \\ &= \phi_i(c_1\mathcal{Y}^\theta(p_{k-1}(\alpha)e_{\frac{\alpha}{2}}, z) \\ &\quad + c_2\mathcal{Y}^\theta(p_{k-1}(-\alpha)e_{-\frac{\alpha}{2}}, z))\phi_j^{-1}.\end{aligned} \tag{3.24}$$

On the other hand, (3.17) shows that

$$\begin{aligned}[E_0, \phi_i\mathcal{Y}^\theta(e_{\pm\frac{\alpha}{2}}, z)\phi_j^{-1}] &= e_\alpha \phi_i \mathcal{Y}^\theta(E_0(e_{\pm\frac{\alpha}{2}}), z)\phi_j^{-1} \\ &= (-1)^{\delta_{i,2}}\phi_i\mathcal{Y}^\theta(p_{k-1}(\mp\alpha)e_{\mp\frac{\alpha}{2}}, z)\phi_j^{-1}.\end{aligned}$$

Hence by (3.22) and (3.23) again, we have

$$\begin{aligned}[E_0, \mathcal{Y}(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}, z)] &= (-1)^{\delta_{i,2}}c_1([E_0, \mathcal{Y}^+(E_0(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}), z)] \\ &\quad + c_2[E_0, \mathcal{Y}^-(E_0(e_{\frac{\alpha}{2}} + e_{-\frac{\alpha}{2}}), z)]) \\ &= (-1)^{\delta_{i,2}}\phi_i(c_1\mathcal{Y}^\theta(p_{k-1}(-\alpha)e_{-\frac{\alpha}{2}}, z) \\ &\quad + c_2\mathcal{Y}^\theta(p_{k-1}(\alpha)e_{\frac{\alpha}{2}}, z))\phi_j^{-1}.\end{aligned} \tag{3.25}$$

Subtracting (3.24) from (3.25) gives the identity

$$\begin{aligned}&(c_1 - (-1)^{\delta_{i,2}}c_2)\phi_i[\mathcal{Y}^\theta(p_{k-1}(\alpha)e_{\frac{\alpha}{2}}, z) \\ &\quad - (-1)^{\delta_{i,2}}\mathcal{Y}^\theta(p_{k-1}(-\alpha)e_{-\frac{\alpha}{2}}, z)]\phi_j^{-1} = 0.\end{aligned}$$

Then Lemma 3.8 shows that $c_1 = (-1)^{\delta_{i,2}}c_2$. Since $\theta\mathcal{Y}^\pm(u,z)\theta = \mathcal{Y}^\mp(u,z)$ for $u \in V^+_{\alpha/2+L}[0]$, we have (3.21). ∎

Now the following proposition follows from Propositions 2.2, 3.3, 3.11, 3.12, and 3.14.

PROPOSITION 3.15. *Let W^1, W^2, and W^3 be irreducible V_L^+-modules and suppose that some of them are of twisted types. Then the fusion rule $N_{W^1W^2}^{W^3}$ is zero or one. Assume that W^1 is a twisted type module, then the fusion rule $N_{W^1W^2}^{W^3}$ is one if and only if W^i $(i = 1, 2, 3)$ satisfy the following cases*:

(i) $W^1 = \left(V_L^{T_1,+}\right)'$ *and the pair* (W^2, W^3) *is one of pairs*

$\left(V_L^{\pm}, \left(V_L^{T_1,\pm}\right)'\right)$, $\left(V_L^{T_1,\pm}, V_L^{\pm}\right)$, $\left(V_{\alpha/2+L}^{\pm}, V_L^{T_1,\pm}\right)$, $\left(\left(V_L^{T_1,\pm}\right)', \left(V_{\alpha/2+L}^{\pm}\right)'\right)$,

$\left(V_{\lambda_r+L}, \left(V_L^{T_1,\pm}\right)'\right)$ *and* $\left(V_L^{T_1,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is even*,

$\left(V_{\lambda_r+L}, \left(V_L^{T_2,\pm}\right)'\right)$ *and* $\left(V_L^{T_2,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is odd*.

(ii) $W^1 = (V_L^{T_1,-})'$ *and the pair* (W^2, W^3) *is one of pairs*

$\left(V_L^{\pm}, \left(V_L^{T_1,\mp}\right)'\right)$, $\left(V_L^{T_1,\pm}, V_L^{\mp}\right)$, $\left(V_{\alpha/2+L}^{\pm}, V_L^{T_1,\mp}\right)$, $\left(\left(V_L^{T_1,\pm}\right)', \left(V_{\alpha/2+L}^{\mp}\right)'\right)$,

$\left(V_{\lambda_r+L}, \left(V_L^{T_1,\pm}\right)'\right)$ *and* $\left(V_L^{T_1,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is even*,

$\left(V_{\lambda_r+L}, \left(V_L^{T_2,\pm}\right)'\right)$ *and* $\left(V_L^{T_2,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is odd*.

(iii) $W^1 = (V_L^{T_2,+})'$ *and the pair* (W^2, W^3) *is one of pairs*

$\left(V_L^{\pm}, \left(V_L^{T_2,\pm}\right)'\right)$, $\left(V_L^{T_2,\pm}, V_L^{\pm}\right)$, $\left(V_{\alpha/2+L}^{\pm}, V_L^{T_2,\mp}\right)$, $\left(\left(V_L^{T_2,\pm}\right)', \left(V_{\alpha/2+L}^{\mp}\right)'\right)$,

$\left(V_{\lambda_r+L}, \left(V_L^{T_2,\pm}\right)'\right)$ *and* $\left(V_L^{T_2,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is even*,

$\left(V_{\lambda_r+L}, \left(V_L^{T_1,\pm}\right)'\right)$ *and* $\left(V_L^{T_2,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is odd*.

(iv) $W^1 = (V_L^{T_2,-})'$ *and the pair* (W^2, W^3) *is one of pairs*

$\left(V_L^{\pm}, \left(V_L^{T_2,\mp}\right)'\right)$, $\left(V_L^{T_2,\pm}, V_L^{\mp}\right)$, $\left(V_{\alpha/2+L}^{\pm}, V_L^{T_2,\pm}\right)$, $\left(\left(V_L^{T_2,\pm}\right)', \left(V_{\alpha/2+L}^{\pm}\right)'\right)$,

$\left(V_{\lambda_r+L}, \left(V_L^{T_2,\pm}\right)'\right)$ *and* $\left(V_L^{T_2,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is even*,

$\left(V_{\lambda_r+L}, \left(V_L^{T_1,\pm}\right)'\right)$ *and* $\left(V_L^{T_1,\pm}, V_{\lambda_r+L}\right)$ *for* $1 \le r \le k-1$ *such that* r *is odd*.

By using Propositions 2.2, 3.7, and 3.15, we the fusion rule $N_{W^1W^2}^{W3}$ for arbitrary irreducible V_L^+-modules W^1, W^2, and W^3. Then by using Proposition 2.8 and by writing the types for which the fusion rule is one without the expression of contragredient module, we have Theorem 3.4.

ACKNOWLEDGMENT

I thank Professor Kiyokazu Nagatomo for helpful advice.

REFERENCES

[A] T. Abe, Fusion rules for the free bosonic orbifold vertex operator algebra, *J. Algebra* **229** (2000), 333–374.

[D1] C. Dong, Vertex algebras associated with even lattices, *J. Algebra* **160** (1993), 245–265.

[D2] C. Dong, Twisted modules for vertex algebras associated with even lattices, *J. Algebra* **165** (1994), 90–112.

[DG] C. Dong and R. L. Griess, Rank one lattice type vertex operator algebras and their automorphism groups, *J. Algebra* **208** (1998), 262–275.

[DL] C. Dong and J. Lepowsky, Generalized vertex algebras and relative vertex operators, *in* "Progress in Mathematics," Vol. 112, Birkhäuser Boston, Cambridge, MA, 1993.

[DLM] C. Dong, H.-S. Li, and G. Mason, Twisted representations of vertex operator algebras, *Math. Ann.* **310** (1998), 571–600.

[DM] C. Dong and G. Mason, On quantum Galois theory, *Duke Math. J.* **86**, No. 2 (1997), 305–321.

[DMZ] C. Dong, G. Mason and Y.-C. Zhu, Discrete series of the Virasoro algebra and the moonshine module, *in* "Proc. Sympos. Pure Math.," Vol. 56, Chap. II, pp. 295–316, Am. Math. Soc., Providence, 1994.

[DN1] C. Dong and K. Nagatomo, Classification of irreducible modules for the vertex operator algebra $M(1)^+$, *J. Algebra* **216** (1999), 384–404.

[DN2] C. Dong and K. Nagatomo, Representations of vertex operator algebra V_L^+ for rank one lattice L, *Comm. Math. Phys.* **202** (1999), 169–195.

[DVVV] R. Dijkgraff, C. Vafa, E. Verlinde, and H. Verlinde, The operator algebra of orbifold models, *Comm. Math. Phys.* **123** (1989), 485–526.

[FHL] I. Frenkel, Y.-Z. Huang, and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, *Mem. Amer. Math. Soc.* **104** (1993).

[FLM] I. Frenkel, J. Lepowsky, and A. Meurman, Vertex operator algebras and the monster, *in* "Pure and Applied Mathematics," Vol. 134, Academic Press, Boston, 1988.

[FZ] I. Frenkel and Y.-C. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J.* **66** (1992), 123–168.

[HL] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra I, II, *Selecta Math.* (N.S.) **1** (1995), 757–786.

[Z] Y.-C. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237–302.