A Family of Quadratic Forms Associated to Quadratic Mappings of Spheres

JoAnn S. Turisco
Mathematics Department
U.S. Naval Academy
Annapolis, Maryland 21402

Submitted by Olga Taussky Todd

ABSTRACT

The general form of a real quadratic mapping of spheres can be determined by studying the diagonalization of each form in an associated family of quadratic forms. In particular, the eigenvalues provide a means for detecting maps which are of the Hopf type. When the eigenvalues are nonzero for every form in the family, the forms associated to $f: S^n \to S^m$ give rise to a quadratic form on the tangent bundle of the unit sphere S^n. If f is of the Hopf type, nondegeneracy of each form occurs only when $n = 1,3,7,15$.

1. PRELIMINARIES

We will follow the notations and definitions which appeared in our previous work [9]. The symbols Z, R, and C denote the ring of integers and the fields of real and complex numbers, respectively. Let (U, q_U) and (V, q_V) be real positive definite quadratic spaces of dimension n and m, respectively, with $n, m \geqslant 2$. Let $S_U = \{x \in U: q_U(x) = 1\}$, $S_V = \{x \in V: q_V(x) = 1\}$, and $S(U,V) = \{f: U \to V: f$ is a quadratic map and $q_V(f(x)) = q_U(x)^2$ for all $x \in U\}$. Clearly, $f: S_U \to S_V$. For every $f \in S(U,V)$ we define a family of quadratic forms f_e, $e \in S_U$, as follows: $f_e(z) = \langle f(z), f(e)\rangle_V$, where $\langle\ ,\ \rangle_V$ denotes the symmetric bilinear form on V corresponding to q_V. Since (U, q_U) is a nondegenerate quadratic space, there is a linear self-adjoint map $F_e: U \to U$ such that $\langle f(z), f(e)\rangle_V = \langle F_e(z), z\rangle_U$ for $z \in U$, where $\langle\ ,\ \rangle_U$ denotes the corresponding symmetric bilinear form on (U, q_U). All of the eigenvalues of F_e are real. Let p_e be the multiplicity of the eigenvalue 1. We have shown [9, p. 262] that $F_e(e) = e$ $(p_e \geqslant 1)$. The following theorems show the relationship

52 Vanderbilt Ave., New York, NY 10017 0024-3795/85/$3.30

between the form f_e and the general form of the map f [9], p. 262, Theorem 1.

THEOREM 1.1. *Let $f \in S(U,V)$, $e \in S_U$, $\varepsilon = f(e) \in S_V$. The form f_e and F_e are defined as above. Let $X = \{z \in U: F_e(z) = z\}$ and $U = X \perp Y$, $V = R\varepsilon \perp V_1$. The quadratic map $\beta: U \to V_1$ is given by*

$$\beta(z) = f(z) - f_e(z)\varepsilon, \qquad \textit{where} \quad z = (x,y) \in U,$$

and $B: X \times Y \to V_1$ is the bilinear map given by

$$B(x,y) = \tfrac{1}{2}[\beta(x+y) - \beta(x) - \beta(y)] = \tfrac{1}{2}[\beta(x+y) - \beta(y)].$$

Then f has the following form:

$$f(z) = f(x,y) = [q_X(x) + f_e(y)]\varepsilon + [2B(x,y) + \beta(y)]$$

where $q_X = q_U | X$, and the following are true:

(1) $q_{V_1}(\beta(y)) + f_e(y)^2 = q_Y(y)^2$, *where*

$$q_{V_1} = q_V | V_1, \qquad q_Y = q_U | Y,$$

(2) $\langle B(x,y), \beta(y) \rangle_{V_1} = 0$,
(3) $2q_{V_1}(B(x,y)) + q_X(x) f_e(y) = q_X(x) q_Y(y)$.

COROLLARY. *There is a basis of U over R with respect to which $f \in S(U,V)$ has the form*

$$f(z) = \begin{bmatrix} x_1^2 + \cdots + x_{p_e}^2 + \lambda_1 y_1^2 + \cdots + \lambda_{q_e} y_{q_e}^2 \\ 2B(x,y) + \beta(y) \end{bmatrix},$$

where $e \in S_U$, $z = (x,y)$, $x = (x_1, \ldots, x_{p_e}) \in X_e = \{x \in U: F_e(x) = x\}$, $y = (y_1, \ldots, y_{q_e}) \in X_e^{\perp} = Y_e$, and $-1 \leqslant \lambda_i < 1$ for all i, $1 \leqslant i \leqslant q_e$, with $n = p_e + q_e$ = dimension of U over R.

2. HOPF MAPS OVER R

We will define various subsets of $S(U,V)$ using the value of p_e, for $e \in S_U$. We begin with the following theorem (see [6], p. 166, Proposition 5):

THEOREM 2.1. *For $e \in S_U$ and $n, m, p = p_e$, $q = q_e$ defined as above, we have $p_e \geqslant n - m + 1$.*

Proof. Let $f(z) = f(x, y) = [q_x(x) + f_e(y)]\varepsilon + [2B(x, y) + \beta(y)] \in S(U, V)$ and $z = (x, y) \in U = X \perp Y$, with X defined as above and $\varepsilon = f(e) \in S_V$. We have $B(x, y) = \lambda(x)y$, where $\lambda(x): Y \to V_1$ is linear. By statement (3) of Theorem 1.1, we see that $\lambda(x)y = 0$ implies that $q_X(x)f_e(y) = q_X(x)q_Y(y)$. If $x \neq 0$, then $q_x(x) \neq 0$. Therefore, $f_e(y) = q_Y(y) = y_1^2 + \cdots + y_q^2 = \lambda_1 y_1^2 + \cdots + \lambda_q y_q^2$. Since $\lambda_i \neq 1$ for all i, we must have $y_i = 0$, $1 \leqslant i \leqslant q$, and hence $y = 0$. Thus, if $x \neq 0$, $\lambda(x)$ is injective and the dimension of Y over R is $q \leqslant m - 1 =$ dimension of V_1 over R, i.e., $n - p_e = q_e \leqslant m - 1$, and $p_e \geqslant n - m + 1$ for all $e \in S_U$. ∎

Let $S_0(U, V)$ denote the subset of "constant" maps, i.e., $S_0(U, V) = \{ f \in S(U, V): f(z) = q_U(z)\varepsilon,\ \varepsilon \in S_U,\ z \in U \}$. It is easy to see that for $f \in S_0(U, V)$, $p_e = n$ for all $e \in S_U$. The following was shown by R. Wood [10, p. 163, Theorem 2]: If n is a power of 2, then all polynomial mappings of the unit spheres $S^n \to S^{n-1}$ are constant. A consequence of this is the following: for $n \geqslant 2r$ all polynomial maps $S^n \to S^r$ are constant. We can prove this result, for quadratic maps, as follows:

THEOREM 2.2. $S(R^{n+1}, R^{r+1}) = S_0(R^{n+1}, R^{r+1})$ *if* $n \geqslant 2r$.

Proof. $f \in S_0(R^{n+1}, R^{r+1}) \Leftrightarrow p_e = n + 1$ for some $e \in S^n \Leftrightarrow p_e = n + 1$ for all $e \in S^n$.

Assume $f \in S(R^{n+1}, R^{r+1}) - S_0(R^{n+1}, R^{r+1})$. Therefore we have $1 \leqslant p_e = p \leqslant n$, $q_e = q = n + 1 - p$, $f_e(z) = \langle f(z), f(e) \rangle = \langle F_e(z), z \rangle$, p is the dimension over R of $X = \{ x \in R^{n+1}: F_e(x) = x \}$, $R^{n+1} = X \perp Y$, and $R^{r+1} = R\varepsilon \perp V$, with $\varepsilon = f(e)$.

By Theorem 1.1, with $z = (x, y) \in R^{n+1}$, we have $f(z) = f(x, y) = [q_x(x) + f_e(y)]\varepsilon + [2B(x, y) + \beta(y)]$. As above, we write $B(x, y) = \lambda(x)y = \mu(y)x$, where $\lambda(x): Y \hookrightarrow V$, $\mu(y): X \hookrightarrow V$ if $x \neq 0 \neq y$. Therefore, $p \leqslant r$, $q \leqslant r$, and $n + 1 - r \leqslant n + 1 - q = p \leqslant r$, i.e., $n + 1 \leqslant 2r$, $n < 2r$. ∎

We use the value of p_e to study mappings of spheres whose form resembles that of the classical Hopf fibrations [1, 2]. We make the following

DEFINITION. A map $f: U \to V$ is a Hopf map if there exist orthogonal decompositions $U = X \perp Y$ and $V = R\varepsilon \perp V_1$, with $\varepsilon \in S_V$, and a bilinear map $B: X \times Y \to V_1$ with $q_{V_1}(B(x, y)) = q_X(x)q_Y(y)$ where $q_X = q_U | X$, $q_Y =$

$q_U | Y$, and $q_{V_1} = q_V | V_1$, such that f has the form $f(z) = f(x, y) = [q_X(x) - q_Y(y)]\varepsilon + 2B(x, y)$. Let $H(U, V)$ denote the set of Hopf maps $U \to V$.

Clearly, $H(U, V) \subseteq S(U, V)$.

It has been shown by R. Wood that every map in $S(U, V)$ is homotopic to a map in $H(U, V)$ [10, p. 163, Theorem 3].

We state the following results [6, p. 164, Theorem 3]:

THEOREM 2.3. *$H(U, V) = \{ f \in S(U, V) : |\lambda_i| = 1$ for all eigenvalues λ_i of F_e, for some $e \in S_U \}$.*

Define

$$H_0(U, V) = \{ f \in S(U, V) : \lambda_i = 1 \text{ for all eigenvalues } \lambda_i \text{ of } F_e \},$$

$$S_1(U, V) = \{ f \in S(U, V) : p_e = n - m + 1 \text{ for some } e \in S_U \},$$

$$H_1(U, V) = \{ f \in H(U, V) : 1 \leqslant p_e,\ q_e = m - 1 \text{ for some } e \in S_U \}.$$

It follows that

$$S_0(U, V) \cap S_1(U, V) = \varnothing,$$

$$H_0(U, V) \cap H_1(U, V) = \varnothing,$$

$$S_0(U, V) = H_0(U, V).$$

The result of R. Wood shows that if $f \in S_1(U, V)$, then f is homotopic to a map $g \in H_1(U, V)$. Therefore, $S_1(U, V) \neq \varnothing \Leftrightarrow H_1(U, V) \neq \varnothing$.

There is a close connection between the existence of a map in $H_1(U, V)$ and the existence of orthonormal vector fields on the unit sphere. If there is a bilinear map $B: R^{p_e} \times R^{q_e} \to R^{q_e}$ such that $|B(x, y)| = |x|\,|y|$, then there are $p_e - 1$ orthonormal vector fields on $S^{q_e - 1}$. However, there are at most $\rho(m-1) - 1$ orthonormal vector fields on S^{m-2} [4, p. 225, Theorem 13.10], where $\rho(n)$ is the Radon-Hurwitz number ($n = 2^{4a+b} n_0$, n_0 odd; $0 \leqslant b \leqslant 3 \Rightarrow \rho(n) = 8a + 2^b$). Furthermore, a bilinear map $B: R^p \times R^q \to R^q$ with $|B(x, y)| = |x|\,|y|$ exists if and only if $p \leqslant \rho(q)$ [6, p. 208, Theorem 7].

We have the following:

THEOREM 2.4. $H_1(U, V) \neq \varnothing \Leftrightarrow 1 \leqslant n - m + 1 \leqslant \rho(m-1)$.

It was shown above that if $S(R^n, R^m) - S_0(R^n, R^m) \neq \varnothing$, then $n = p_e + q_e$, with $p_e \leqslant m-1$, $q_e \leqslant m-1$, i.e., $n \leqslant 2m-2$. Write $H_i(R^n, R^m) = H_i$, for $i = 0, 1$. We have the following theorems see [6, p. 177, Theorem 8],

THEOREM 2.5.

(1)

$$\begin{array}{lll} n = 2m-2 & (m = 2,3,5,9) & \Rightarrow \quad H = H_0 \cup H_1; \\ n = 2m-2 & (\textit{all other } m) & \Rightarrow \quad H = H_0. \end{array}$$

(2)

$$\begin{array}{lll} n = 2m-3 & (m = 3,5,9) & \Rightarrow \quad H = H_0 \cup H_1; \\ n = 2m-3 & (\textit{all other } m) & \Rightarrow \quad H = H_0 \end{array}$$

Proof. (1): $H \supseteq H_0$. $H - H_0 \neq \varnothing \Rightarrow n = 2m-2 = p+q$, with $p, q \leqslant m-1$. Therefore, $n - m + 1 = m - 1 = p = q$. A map $f \in H$ corresponds to a bilinear map $B: R^p \times R^p \to R^p$ with $|B(x,y)| = |x||y|$ for all $x, y \in R^p$. By a theorem of Hurwitz [3], $p = 1$, 2, 4, or 8 if and only if such a bilinear map exists.

(2): $H - H_0 \neq \varnothing \Rightarrow 2m - 3 = p + q$, with $p, q \leqslant m-1$. Therefore, $(p,q) = (m-2, m-1)$, or $(p,q) = (m-1, m-2)$. Now $n - m + 1 = m - 2 = p \Rightarrow f(x,y) = [q_X(x) - q_Y(y)]\varepsilon + 2B(x,y) \in H_1$. However, $f(x,y) = [q_Y(y) - q_X(x)](-\varepsilon) + 2B(x,y) \in H_1$ if $q = m-2$. A map $f \in H_1$ corresponds to a bilinear map $B: R^{m-2} \times R^{m-1} \to R^{m-1}$, with $|B(x,y)| = |x||y|$. This map exists if and only if $m - 2 \leqslant p(m-1)$. But $m - 2 \leqslant \rho(m-1)$ if and only if $m = 3,5,9$. ∎

Note that $H(R^n, R^n) \neq \varnothing$, since we have the following map $f \in H_1(R^n, R^n)$:

$$f(x, y_1, \ldots, y_{n-1}) = \begin{bmatrix} x^2 - y_1^2 - \cdots - y_{n-1}^2 \\ 2xy_1 \\ \vdots \\ 2xy_{n-1} \end{bmatrix}.$$

We have

$$H(R^2, R^2) = H_0 \cup H_1,$$

$$H(R^3, R^3) = H_0 \cup H_1.$$

$H(R^4, R^4) \supsetneqq H_0 \cup H_1$, since the map

$$f(x, y) = \begin{bmatrix} x_1^2 + x_2^2 - y_1^2 - y_2^2 \\ 2(x_1y_1 + x_2y_2) \\ 2(x_1y_2 - x_2y_1) \\ 0 \end{bmatrix} \in H - (H_0 \cup H_1).$$

3. ACTIONS BY THE ORTHOGONAL GROUP

There is a double action by the orthogonal group on the set $S(U, V)$. Let $O(U), O(V)$ be the orthogonal groups on U, V with respect to q_U, q_V respectively. Let $\phi: O(V) \times S(U, V) \times O(U) \to S(U, V)$ be defined by $\phi(\tau, f, \sigma) = {}^{\tau}f^{\sigma}$, and ${}^{\tau}f^{\sigma}(z) = \tau(f(\sigma(z)))$ for all $z \in U$. This action defines an equivalence relation on $S(U, V)$. Denote by $\bar{S}(U, V)$ the resulting set of double cosets:

$$\bar{S}(U, V) = O(V) \backslash S(U, V) / O(U).$$

While it is clear that for any map $f \in S(U, V)$ we have $f_e = {}^{\tau}f_e$, the forms $\{f_e\}_{e \in S_U}$ are independent of the class of f in $\bar{S}(U, V)$ $\Leftrightarrow$ $f_{\sigma e}(\sigma z) = f_e(z)$ for all $z \in U$, $\sigma \in O(U)$ $\Leftrightarrow$ ${}^t\sigma F_{\sigma e}\sigma = F_e$, where $F_e \in \mathrm{Sym}(n)$, with $\mathrm{Sym}(n)$ the set of real symmetric matrices of order n, is the matrix of f_e with respect to a fixed basis. This leads to the following definitions:

(1) A map $f \in S(U, V)$ induces a map $f^{\#}: S_U \to \mathrm{Sym}(n)$ given by $f^{\#}(e) = F_e$.

(2) We have the following action by $O(U)$ on $\mathrm{Sym}(n)$:

$$\tau: O(U) \times \mathrm{Sym}(n) \to \mathrm{Sym}(n),$$

$$\tau(\sigma, A) = \sigma A \sigma^{-1} = {}^{\sigma}A$$

(3) A map $f \in S(U, V)$ is *invariant* with respect to $\sigma \in O(U)$ if $f^{\#}(\sigma e) = {}^{\sigma}F_e$.

We would like to determine those maps $f \in S(U, V)$ which are invariant with respect to every $\sigma \in G$, where G is a subgroup of $O(U)$ which acts transitively on S_U. We have the following results, the proofs of which are straightforward.

THEOREM 3.1. *Maps in* $S(U, V)$ *are invariant with respect to* $\sigma \in G$ *in the following cases*:

(1) $m = 1$, $n \geqslant 2$, $f \in S(U, V)$, $\sigma \in G = O(U)$.
(2) $n, m \geqslant 2$, $f \in S_0(U, V)$, $\sigma \in G = O(U)$.
(3) $n = m = 2$, $f(z) = z^2$, *where* $z \in C$ *and* $\sigma \in G = \{\sigma \in O(U) : \det \sigma = 1\} \approx \{\sigma \in C : N\sigma = 1\}$.

Let H denote the Hamiltonian quaternion algebra over R and $H^{(1)} = \{z \in H : Nz = 1\}$. $H^{(1)}$ acts transitively on itself by right multiplication. Write $\sigma(z) = z \cdot \sigma = \hat{\sigma} z$ for $z, \sigma \in H^{(1)}$ and $\hat{\sigma} \in O(R^4)$. We identify H as a vector space with R^4, and the norm $N : H \to R$ with $Nz = \bar{z}z$ with the quadratic form $q_4(z) = {}^t zz$. The corresponding bilinear form is $\langle x, y \rangle = \frac{1}{2} T(\bar{x} y)$, where the trace $T : H \to R$ is given by $T(z) = \bar{z} + z$.

THEOREM 3.2. *Let* $f(z) = \bar{z} i z$, $z \in H$, $i \in H$ *with* $\bar{i} = -i$. *Then* $f_{\sigma e}(\sigma z) = f_e(z)$ *for any* $e \in H^{(1)}$, *where* $\sigma(z) = z \cdot \sigma$ *with* $\sigma \in H^{(1)}$.

Proof. $f_e(z) = \langle f(z), f(e) \rangle = \frac{1}{2} T(\overline{f(z)} f(e)) = -\frac{1}{2} T((\bar{z} i z)(\bar{e} i e))$ and $f_{\sigma e}(\sigma z) = -\frac{1}{2} T(\overline{z\sigma} i (z\sigma)(\overline{e\sigma}) i (e\sigma)) = -\frac{1}{2} T((\bar{\sigma} \bar{z} i z)(\sigma \bar{\sigma}) \bar{e} i e \sigma) = \frac{1}{2} T(\sigma^{-1} \overline{f(z)} f(e) \sigma) = f_e(z)$. ∎

The map $f(z) = \bar{z} i z \in S(R^4, R^3)$ is the classical Hopf fibration $S^3 \to S^2$. We suspect that the other classical Hopf fibrations $f : S^{2n-1} \to S^n$, $n = 4, 8$, also are invariant with respect to a subgroup of the orthogonal group.

The next section deals with a more general problem, namely, that of determining when the forms $\{f_e\}_{e \in S_U}$ are always nondegenerate.

4. THE NONDEGENERACY OF $\{f_e\}_{e \in S^{n-1}}$

Fix orthonormal bases in (U, q_U) and (V, q_V). For $f \in S(U, V)$, $e \in S_U \approx S^{n-1}$, we write $f_e(z) = \langle f(z), f(e) \rangle_V = \sum_{i=1}^m f_i(z) f_i(e) = {}^t z F_e z$. Therefore, F_e

$=(f_{ij}(e))$, where $f_{ij}(e)$ is a quadratic form in $e=(e_1,\dots,e_n)\in S^{n-1}$. The map $s: e \to F_e$ is a continuous map $R^n \to R^m$, with $m = n(n+1)/2$. There is a nonsingular matrix T_e such that

$$^tT_eF_eT_e = \begin{bmatrix} I_{p_e} & 0 \\ 0 & I_{q_e} \end{bmatrix}$$

if the determinant $|F_e| \neq 0$, where I_k is the $k \times k$ identity matrix.

If we assume that f_e is a nondegenerate form for every $e \in S^{n-1}$, then it follows from the continuity of the map s that $q_e = q_{e_0}$ for all $e, e_0 \in S^{n-1}$. That is, if the forms are all nondegenerate, then they must be isometric over R.

LEMMA 4.1. *$f \in H(R^n, R^m)$. The quadratic forms $\{f_e\}_{e\in S^{n-1}}$ are all nondegenerate only if n is even.*

Proof. There exist decompositions $R^n = X \perp Y$, $R^m = R\varepsilon \perp V_1$, with $\varepsilon = f(e) \in S^{m-1}$, $e \in S^{n-1}$, and a bilinear map $B: X \to Y \to V_1$ with $|B(x,y)| = |x||y|$ such that f has the form $f(z) = f(x,y) = (x_1^2 + \cdots + x_{p_e}^2 - y_1^2 - \cdots - y_{q_e}^2)\varepsilon + 2B(x,y)$. We have $f_e(z) = x_1^2 + \cdots + x_{p_e}^2 - y_1^2 - \cdots - y_{q_e}^2$. If $\{f(e)\}_{e\in S^{n-1}}$ are all nondegenerate, then there is a map $f^{\#}: S^{n-1} \to \mathrm{Sym}_q(n)$, where $\mathrm{Sym}_q(n)$ denotes the set of real symmetric matrices of order n with q negative eigenvalues; i.e., q is independent of $e \in S^{n-1}$. Therefore, $\{f_e\}_{e\in S^{n-1}}$ are all nondegenerate $\Rightarrow$ $f^{\#}(e) = F_e \in \mathrm{Sym}_q(n)$, $q = q_e$. Let α be the vector

$$(0,\dots,0,\underset{\substack{\uparrow \\ p_e+1}}{1},0,\dots,0).$$

We have $f(\alpha) = -\varepsilon$ and $f_\alpha(z) = -(x_1^2 + \cdots + x_{p_e}^2) + y_1^2 + \cdots + y_{q_e}^2$ with $f^{\#}(\alpha) = F_\alpha \in \mathrm{Sym}_{p_e}(n)$. Therefore, $p_e = p = q = q_e$. ■

LEMMA 4.2. *$f \in S(R^n, R^m)$. If the family of forms $\{f_e\}_{e\in S^{n-1}}$ are all nondegenerate, then S^{n-1} admits a quadratic form of signature k.*

Proof. $f \in S(R^n, R^m)$ induces a continuous map $f^{\#}: S^{n-1} \to \mathrm{Sym}_k(n)$, $f^{\#}(e) = F_e$, where k is the number of negative eigenvalues of F_e, which is by definition the "signature" of the quadratic form f_e [8, p. 204]. The map $f^{\#}$

gives rise to a section of the product bundle:

$$\begin{array}{cc} S^{n-1} \times \mathrm{Sym}_k(n) & \\ \downarrow \quad\quad \uparrow & \\ S^{n-1} \quad\quad s & \end{array} \qquad s(e) = (e, \phi_e), \quad \phi_e(x, y) = {}^t x F_e y.$$

We claim that we can construct a section of the bundle of quadratic forms of signature k over S^{n-1}, i.e., a section of the bundle

$$\begin{array}{c} \mathrm{Sym}_k(TS^{n-1}) \\ \downarrow \\ S^{n-1} \end{array}$$

where TS^{n-1} is the tangent bundle of S^{n-1}. We have the following:

$$F_e(e) = e, \qquad e \in S^{n-1},$$

$$T_e S^{n-1} \cong (Re)^{\perp},$$

$$TS^{n-1} \underset{i}{\hookrightarrow} S^{n-1} \times R^n \underset{j}{\rightarrow} S^{n-1} \times Re,$$

where j is the projection along the vector e. Therefore, $S^{n-1} \times R^n \cong TS^{n-1} \oplus (S^{n-1} \times Re)$ and

$$\begin{array}{c} \mathrm{Sym}_k(TS^{n-1}) \\ \downarrow \quad\quad \uparrow \\ S^{n-1} \quad\quad i^*s \end{array} \qquad \text{where} \quad (i^*s)_e(u_e, v_e) = s_e(u_e, v_e).$$

Note that the signature of the restricted symmetric bilinear form i^*s is k, since at each $e \in S^{n-1}$ there is an orthogonal matrix T_e such that

$${}^t T_e F_e T_e = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & G_e \end{array}\right] \qquad \text{with} \quad G_e \in \mathrm{Sym}_k(n-1). \qquad \blacksquare$$

Theorem 4.1. *$f \in H(R^n, R^m)$. The family of forms $\{f_e\}_{e \in S^{n-1}}$ are all nondegenerate only if $n = 2, 4, 8,$ or 16.*

Proof. By Lemma 4.1, the forms $\{f_e\}_{e \in S^{n-1}}$ are all nondegenerate $\Rightarrow$ $n = 2k$ $(p = q = k)$. By Lemma 4.2, S^{n-1} admits a quadratic form of signature k. Therefore, S^{n-1} admits a continuous field of tangent k-planes [8, p. 207, Theorem 40.11]. By duality, S^{n-1} admits a continuous field of tangent $(n-1-k = k-1)$-planes. Since $2(k-1) = 2k-2 = n-2 < n-1$, S^{n-1} admits $k-1$ continuous linearly independent vector fields [8, p. 144, Theorem 27.16]. Therefore, $k - 1 \leqslant \rho(n) - 1$, where $\rho(n)$ is the Randon-Hurwitz number; i.e., $k = n/2 \leqslant \rho(n)$. This is true only when $n = 2$, 4, 8, or 16. ■

We can improve this result if we restrict ourselves to maps defined over certain R-lattices. Write $f(x) = \sum_{i,j=1}^{n} x_i x_j s_{ij}$, where $s_{ij} \in R^m$, $f \in S(R^n, R^m)$. Let $\Lambda \subset V = R^m$ be an R-lattice. Define

$$S_{n,m}(\Lambda) = \left\{ f \in S(R^n, R^m) : f = \sum_{i,j=1}^{n} x_i x_j s_{ij}, \right.$$

$$\left. \text{where } s_{ij} \in \Lambda \text{ for all } i, j,\ 1 \leqslant i, j \leqslant n \right\},$$

$$H_{n,m}(\Lambda) = H(R^n, R^m) \cap S_{n,m}(\Lambda).$$

It has been shown by T. Ono [5, p. 158] that $H_{n,m}(\Lambda) = S_{n,m}(\Lambda)$ if Λ is an R-lattice such that $q_U(z) \in Z$ for all $z \in \Lambda$.

THEOREM 4.2. *Let $\Lambda \subset V \approx R^m$ be an R-lattice such that $\Lambda = \sigma A^m$, where $\sigma \in O(V)$ and A^m is the standard lattice in R^m with integral coordinates. For $f \in S_{n,m}(\Lambda) = H_{n,m}(\Lambda)$, the forms $\{f_e\}_{e \in S^{n-1}}$ are all nondegenerate if and only if $n = 2(m-1)$ and $m = 2,3,5,9$.*

Proof. Assume that $f \in H_{2p,m}(Z) = H_{2p,m}(A^m)$, and $\{f_e\}_{e \in S^{n-1}}$ are all nondegenerate. Then f has the following form:

$$f(x, y) = f(z) = [q_X(x) - q_Y(y)]\varepsilon + \sum_{i,j=1}^{p} x_i y_j b_{ij},$$

with $b_{ij} \in (R\varepsilon)^{\perp}$, $\varepsilon \in Z^m$, and $X \approx Y \approx R^p$. Each of the p^2 monomials $\pm x_\alpha y_\beta$, $1 \leqslant \alpha, \beta \leqslant p$, appears once in $f_i(z)$, for some i, $2 \leqslant i \leqslant m$ (since

$b_{ij} \in Z^m$ and $|b_{ij}| = 1$ for all i, j [9, p. 267]. Each coordinate map $f_i(z)$ contains at most p monomials, and $p \leqslant m - 1$. By the theorem of Hurwitz, if $p = m - 1$ then $m = 2,3,5,9$. If $p < m - 1$, there is a j, $2 \leqslant j \leqslant m$, such that $f_j(z)$ contains l monomials, with $l < p$. Clearly $f_j(z)$ is a degenerate form. Now $f_j(z)$ contains the monomial $\pm x_\alpha x_\beta$ for some α, β. Let $e \in S^{n-1}$, $e = (e_1, \ldots, e_n)$, with

$$e_i = \begin{cases} 1/\sqrt{2} & \text{if} \quad i = \alpha \text{ or } i = p + \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$f_i(e) = \begin{cases} \pm 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases}$$

and $f_e(z) = \pm f_j(z)$ is a degenerate form.

Assume that $f \in H_{n,m}(\Lambda)$. There is a $\tau \in O(V)$ such that ${}^\tau f \in H_{n,m}(Z)$. By the above argument, if $p < m - 1$, then ${}^\tau f_e$ is a degenerate form for some $e \in S^{n-1}$. But

$${}^\tau f_e(z) = \langle \tau(f(z)), \tau(f(e)) \rangle_V = \langle f(z), f(e) \rangle_V = f_e(z).$$

Let $f \in H_{2(m-1),m}(\Lambda)$ with $m = 2,3,5,9$. A calculation shows that if $g: S^{2m-3} \to S^{m-1}$ is the classical Hopf fibration, then $\{g_e\}_{e \in S^{2m-3}}$ are all nondegenerate. There is a $\tau \in O(U)$ such that ${}^\tau f \in H_{2(m-1),m}(Z)$. Therefore, there is a $\psi \in O(U)$, $\phi \in O(U)$ such that ${}^\tau f = {}^\phi g^\psi$ [9, p. 267, Theorem 2] and ${}^\phi g_e^\psi(z) = g_e^\psi(z) = \langle g^\psi(z), g^\psi(e) \rangle = {}^t z\, {}^t\psi G_{\psi e} \psi z = {}^t z F_e z = f_e(z) = {}^\tau f_e(z)$. Therefore, ${}^t\psi G_{\psi e} \psi = F_e$, and the determinant

$$|{}^t\psi G_{\psi e} \psi| = |G_{\psi e}| = |F_e| \neq 0 \qquad \text{for all} \quad e \in S^{n-1}. \qquad \blacksquare$$

REFERENCES

1 H. Hopf, Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche, *Math. Ann.* 104:637–665 (1931).

2 ———, Über die Abbildungen von Sphären auf Sphären neidrigerer Dimension, *Fund. Math.* 25:427–440 (1935).

3 A. Hurwitz, Über die Komposition der quadratischen Formen, *Math. Ann.* 88:1–25 (1923).

4 D. Husemoller, *Fibre bundles*, Springer, Berlin, 1966.
5 T. Ono, A note on spherical quadratic maps over Z, in *Algebraic Number Theory*, International Symposium, Kyoto, 1976 (S. Iyanaga, Ed.), Japan Society for the Promotion of Science, Tokyo, 1977.
6 ______, *Variations on a Theme of Euler* (in Japanese, untranslated), Jikkyo, Tokyo, 1980.
7 J. Radon, Lineare Scharen orthogonaler Matrizen, *Abh. Math. Sem. Univ. Hamburg* 1:1–14 (1922).
8 N. Steenrod, *The Topology of Fibre Bundles*, Princeton U.P., Princeton, N.J., 1951.
9 J. Turisco, Quadratic mappings of spheres, *Linear Algebra Appl.* 23:261–274 (1979).
10 R. Wood, Polynomial maps from spheres to spheres, *Invent. Math.* 5:163–168 (1968).

Received 13 January 1984; revised 16 March 1984