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Lyapunov Theory and Perturbation of Stable and Asymptotically Stable Systems

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1. INTRODUCTION

In this paper we study the vector ordinary differential equation

$$\dot{x} = f(t, x) \tag{E}$$

for which the identically zero function is a solution, i.e., $f(t, 0) = 0$ for all time t. We denote this special solution simply by 0. Now suppose one knows that all the solutions of (E) which start near 0 remain near 0 for all future time, or even that they approach 0 as time increases. If the differential equation (E) is subjected to certain small perturbations, the above property concerning the solutions near 0 may or may not remain true. A more precise formulation of this problem is as follows: if 0 is asymptotically stable for (E), and if the function $p(t)$ is small in some sense, give conditions on f so that 0 is (eventually) asymptotically stable for the perturbed equation

$$\dot{x} = f(t, x) + p(t). \tag{P}$$

In particular, an example is known [1, Theorem C] in which p $(= \epsilon \exp(-t^2))$ tends to 0 faster than exponentials and f is a real-valued, uniformly continuous, and locally Lipschitz function, and all solutions of (E) approach zero exponentially and monotonically as $t \to \infty$; yet many solutions of (P) starting near

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$x = 0$ fail to approach zero as $t \to \infty$ no matter how small ϵ is. More precisely, if x is a solution such that $x(t_0) \geqslant 0$ for some t_0 then

$$\limsup_{t\to\infty} x(t) > 0.$$

A great deal of work has been done in an attempt to provide positive answers to this problem. Historically, there have been two approaches. One approach is to set conditions of f, such as being uniformly Lipschitz, and find out what kind of perturbations $p(t)$ preserve stability (e.g., [1; 2; 3, Chap. 13]). The second approach is to set the kind of perturbations $p(t)$ that will be allowed, such as $\int_0^\infty |p(t)|\, dt < \infty$, and find out which differential equations (E) will have their asymptotic stability preserved by all such $p(t)$ [4–7, 10].

A monumental paper along the lines of the second approach was published in 1959 by Vrkoč [4]. In this paper be defined the concepts of integral stability and integral asymptotic stability. A definition similar to integral stability had been given by Okamura (cf. [8, p. 89]). Vrkoč's main results are that (i) integral stability of (E) (i.e., solutions of (P) remain small whenever $\int_0^\infty |p(t)|\, dt$ is small; precise statements appear in Section 4) is equivalent to the existence of a certain kind of Lyapunov function for (E), and (ii) integral asymptotic stability of (E) is equivalent to the existence of another kind of Liapunov function for (E). "Integral asymptotic stability" means "integral stability plus integral attraction", i.e., solutions of (P) starting in some neighborhood of 0 become small after time T provided $\int_0^\infty |p(t)|\, dt$ is small.

The first objective of our paper is to give substantially simpler proofs of Vrkoč's results based on methods used in [8], thereby relating Vrkoč's results to those of Okamura [9] and Yoshizawa [8]. This simpler proof allows us to extend Vrkoč's result on integral-asymptotic stability in two directions. First we show that (E) is integral-asymptotically stable if and only if (E) is integrally attractive (Theorem 3). That is, the assumption of integral stability is redundant. Second we show that every integral-asymptotically stable system behaves nicely not only for perturbations integrable on $[0, \infty)$ but also for the larger class B_{IB} of *interval bounded* functions $p(t)$, i.e.,

$$\| p \|_{B_{IB}} = \sup_{\alpha \geqslant 0} \int_t^{t+1} | p(s)|\, ds < \infty.$$

We prove specifically that an asymptotically stable system remains "attractive" for the class of interval bounded perturbations p if and only if the system is integral-asymptotically stable (Theorem 4).

Throughout this paper the perturbation $p(t)$ could have been replaced by a perturbation $g(t, x)$ which satisfies $| g(t, x)| \leqslant | p(t)|$ for all x in a neighborhood of 0. Since this would have introduced no new ideas, we chose to present the notationally simpler case.

There is a large literature on applications of Lyapunov functions to the study of perturbations. Indeed, the study of perturbations is one of the motivations for studying Lyapunov theory. Vrkoč's results show that a superior theory can be derived if the perturbations are introduced during the development of the Lyapunov-type results so that necessary and sufficient conditions can be given for (E) to be perturbable.

2. Notations and Assumptions

For $c > 0$ write S_c for $\{x \in R^n: |x| < c\}$, letting $|\cdot|$ denote some norm in R^n. We assume the following conditions throughout this paper.

(H1) For some $c > 0$, $f: [0, \infty) \times S_c \to R^n$.

(H2) $f(\cdot, x)$ is measurable for each x.

(H3) $f(t, \cdot)$ is continuous for each t.

(H4) $|f|$ is bounded on every compact subset of $[0, \infty) \times S_c$.

(H5) $f(t, 0) \equiv 0$.

(H6) $p: [0, \infty) \to R^n$ is measurable.

Remark. (H4) can be replaced by the following more general assumption (notationally more difficult to work with):

(H4*) For each compact $S \subset S_c$, there is a locally integrable function $m_S(t)$ such that $|f(t, x)| \leqslant m_S(t)$, for all $t \in [0, \infty)$, $x \in S$. The results in the paper still hold true.

We will denote a solution of (E) through (t_0, x_0) by $\phi_E(t; t_0, x_0)$, and similarly that of (P) by $\phi_P(t; t_0, x_0)$.

Let $0 < a < c$. For each $x \in S_a$ and each $t \in (0, \infty)$, $A_a(t, x)$ will denote the set of absolutely continuous functions $\Psi: [0, \infty) \to R^n$ satisfying:

$$\Psi(0) = 0, \qquad \Psi(t) = x, \qquad \text{and} \qquad \sup_{s \in [0,t]} |\Psi(s)| < a.$$

Let $V: [0, \infty) \times S_a \to R$ be a function. By "$\dot{V}(t, x)$ taken along solutions of (E)", we mean

$$\dot{V}(t, x) = \limsup_{h \to 0+} (1/h)\{V(t + h, x_E(t + h)) - V(t, x)\} \tag{2.1}$$

where $x_E(\cdot)$ is a solution of (E) satisfying $x_E(t) = x$. We say that V is *Lipschitzean with respect to x* with constant L if $|V(t, x_1) - V(t, x_2)| \leqslant$

$L| x_1 - x_2 |$ for all $t \geqslant 0$ and all x_1 and x_2 near 0. If V is Lipschitzean with respect to x, then for $\dot{V}(t, x)$ taken along solutions of (E), we have

$$\dot{V}(t, x) = \limsup_{h \to 0+}(1/h)\{V(t + h, x + hf(t, x)) - V(t, x)\}, \text{ a.e. in } t$$

(see [3, pp. 4–5; 10–11] for example).

We sometimes write "$\dot{V}_{(E)}(t, x)$" to mean "$\dot{V}(t, x)$ is taken along solutions of (E)."

3. Uniqueness of Solutions and Lyapunov Functions

The main technique of this paper is the study of certain natural "Lyapunov functions". First we define a measure of how much an absolutely continuous function differs from being a solution. This measure is only defined on $[0, t]$ if $\phi(s)$ and $f(s, \phi(s))$ are defined for all $s \in [0, t]$. We define

$$E(\phi, [0, t]) = \int_0^t | \dot{\phi}(s) - f(s, \phi(s))| \, ds.$$

Letting $p(s) = \dot{\phi}(t) - f(s, \phi(s))$, we see ϕ satisfies Eq. (P), so E is also a measure of the size of the perturbation $\int | p | \, dt$ required for ϕ to be a solution on $[0, t]$. Notice $E = 0$ if an only if ϕ is a solution. We now define a function $V_0(t, x)$ which measures the minimum "energy" or minimum perturbation required for an absolutely continuous ϕ to start at 0 at time 0 and be at x at time t, restricting $| \phi(s)| \leqslant a$. Define

$$V_0(t, x) = \inf_{\phi \in A_a(t,x)} E(\phi, [0, t]). \tag{3.1}$$

This function was first used by Okamura [4] in his investigation of uniqueness of solutions. His definition was equivalent to the one here. Our definition is due to Yoshizawa. The significance of V_0 is that $V_0(t_1, x_1) = 0$ for $t_1 > 0$ if and only if there is a solution $x(t)$ in $\{| x | \leqslant a\}$ such that $x(0) = 0$ and $x(t_1) = x_1$. The function V_0 will be used in our investigation of integral stability.

We are interested in functions $V: [0, \infty) \times S_a \to R$ which have the following properties for some $a \in (0, c]$.

$$V \text{ is continuous on } [0, \infty) \times S_a . \tag{3.2}$$

$$0 \leqslant V(t, x) \leqslant | x | \qquad \text{for all} \quad (t, x) \in [0, \infty) \times S_a . \tag{3.3}$$

$$| V(t, x) - V(t, y)| \leqslant | x - y | \qquad \text{if} \quad (t, x), (t, y) \in [0, \infty) \times s_a . \tag{3.4}$$

$$\dot{V}(t, x) \leqslant 0 \text{ along solutions of (E) for all } (t, x). \tag{3.5}$$

$$V(t, 0) = 0, \text{ and } 0 < V(t, x) \text{ for } x \neq 0 \text{ for all } t \geqslant 0. \tag{3.6}$$

$$\dot{V}(t, x) \leqslant -V(t, x) \text{ along solutions of (E) for all } (t, x). \tag{3.7}$$

We will say $x \equiv 0$ is *unique in the future* for (E) if $x \equiv 0$ is a solution and whenever $x(t)$ satisfies (E) and $x(t_0) = 0$ for $t_0 \geqslant 0$, we have $x(t) \equiv 0$ for $t \geqslant t_0$. The following uniqueness theorem was proved by Okamura using V_0 to prove that such a function V always exists (see [8]). This result can be strengthened in a sense as will be seen in Lemma 2.

The following Lemma is the basis of Okamura's results on necessary and sufficient conditions for uniqueness in the future, and will be applied to the study of integral stability (see [8], p. 8).

LEMMA 1. *The function V_0 satisfies* (3.2), (3.3), (3.4), *and* (3.5). *Furthermore* (3.6) *is satisfied if $x \equiv 0$ is unique in the future.*

Okamura proved that the solution $x \equiv 0$ of (E) is unique in the future if and only if there exists a function $V\colon [0, \infty) \times S_a \to [0, \infty)$ (for some a) such that V satisfies conditions (3.2) through (3.6). It is immediate that if such a function V exists, then the uniqueness of 0 in the future follows. The more difficult converse follows from Lemma 1.

We shall investigate integral-asymptotic stability by means of the function

$$U(t, x) = \begin{cases} \inf\limits_{\Psi \in A_a(t,x)} \displaystyle\int_0^t e^{-(t-u)} \,|\, \dot{\Psi}(u) - f(u, \Psi(u))| \, du, & t > 0, \\ |\, x \,|, \quad t = 0. \end{cases} \tag{3.8}$$

LEMMA 2. *The function U in* (3.8) *satisfies* (3.2), (3.3), (3.4), *and* (3.7), *and if the solution $x \equiv 0$ of* (E) *is unique in the future, U satisfies* (3.6).

Another definition of U which would satisfy Lemma 2 would be $U(t, x) = e^{-t} V_0(t, x)$, but this choice of U would not be useful later. The advantage of U in (3.8) is that if (E) is integrally attracting, U is "positive definite".

It may be seen that for $t > 0$ and $x \in S_a$ if $U(t, x) = 0$, then there is a solution ϕ of (E) such that $\phi(0) = 0$ and either $\phi(t) = x$ or ϕ leaves S_a before time t, so future uniqueness of $x \equiv 0$ implies (3.6). The proof of this result is almost identical to the "Ascoli" argument in [8, p. 6] and so it is omitted here. Write $M(T) = \sup\{|\, f(t, y)| : 0 \leqslant t \leqslant T \text{ and } y \in S_a\}$.

LEMMA 3. *For any $t \geqslant s > 0$ and $x, y \in S_a$*

$$|\, U(s, x) - U(t, y)| \leqslant |\, x - y \,| + |\, s - t \,| M(t) + (1 - e^{s-t}) a. \tag{3.9}$$

Proof of Lemma 3. Considering $| U(s, x) - U(s, y)| + | U(s, y) - U(t, y)|$ we see it is sufficient to prove two facts:

$$| U(s, x) - U(s, y)| \leqslant | x - y |, \tag{3.10}$$

and

$$| U(s, y) - U(t, y)| \leqslant | s - t | M(t) + (1 - e^{s-t})a. \tag{3.11}$$

For $\Psi \in A_a(s, x)$ and $h \in (0, s)$, let $\Psi_h \in A_a(s, y)$ be a function such that $\Psi_h = \Psi$ on $[0, s - h]$ and such that the graph of Ψ_h on $[s - h, y]$ is a straight line between $(s - h, \Psi(s - h))$ and (s, y). Then for all $h > 0$

$$\begin{aligned} U(s, y) &\leqslant \int_0^s e^{-s+u} | \dot{\Psi}_h(u) - f(u, \Psi_h(u))| \, du \\ &\leqslant \int_0^{s-h} e^{-s+u} | \dot{\Psi}(u) - f(u, \Psi(u))| \, du + e^h \int_{s-h}^s | \dot{\Psi}_h(u)| \, du \\ &\quad + \int_{s-h}^s | f(s, \Psi_h(s))| \, ds \\ &\leqslant \int_0^s e^{-s+u} | \dot{\Psi}(u) - f(u, \Psi(u))| \, du + e^h | y - \Psi(s - h)| \\ &\quad + hM(t). \end{aligned}$$

Since this is true for all $h > 0$, we may take the infimum (as $h \to 0$). Also since this is true for all $\Psi \in A_a(s, x)$, we may replace the integral by $U(s, x)$, yielding

$$U(s, y) \leqslant U(s, x) + | y - x |.$$

This inequality is symmetric in x and y so (3.10) is proved.

Note that $U(t, 0) = 0$ by definition of U (letting $\Psi \equiv 0$ in (3.8)). Hence,

$$0 \leqslant U(s, y) \leqslant | y |. \tag{3.12}$$

Let Ψ be in $A_a(t, y)$. Then (writing h for $t - s \geqslant 0$),

$$\begin{aligned} &\int_0^t e^{-(t-u)} | \dot{\Psi}(u) - f(u, \Psi(u))| \, du \\ &\quad \geqslant e^{-t+s} \left[\int_0^s e^{-(s-u)} | \dot{\Psi}(u) - f(u, \Psi(u))| \, du + \int_s^t | \dot{\Psi}(u)| \, du \right] \\ &\qquad - \int_s^t | f(u, \psi(u))| \, du \\ &\quad \geqslant e^{-t+s}[U(s, \Psi(s)) + | y - \Psi(s)|] - M(t)h \\ &\quad \geqslant e^{-t+s} U(s, y) - M(t)h \end{aligned}$$

from (3.10), replacing x in (3.10) by $\psi(s)$. Since this is true for all $\Psi \in A_a(t, y)$,

$$U(t, y) \geqslant U(s, y) - M(t)h - (1 - e^{s-t})a \tag{3.13}$$

since from (3.12) $(e^{-t+s} - 1)\, U(s, y) \geqslant (e^{-t+s} - 1)|\, y \,| \geqslant -(1 - e^{s-t})a$. For any $\Psi \in A_a(s, y)$ define $\Psi^* \in A_a(t, y)$ by $\Psi^* \equiv \Psi$ on $[0, s]$ and $\Psi^* \equiv y$ on (s, ∞). Then for any $\psi \in A_a(s, y)$

$$\begin{aligned} U(t, y) &\leqslant \int_0^t e^{-(t-u)}\, |\, \dot{\Psi}^*(u) - f(u, \Psi^*(u))|\, du \\ &= e^{-t+s} \int_0^s e^{u-s}\, |\, \dot{\Psi}(u) - f(u, \psi(u))|\, du + \int_s^t |\, f(u, y)|\, du. \end{aligned}$$

Since this inequality is satisfied for all $\Psi \in A_a(s, y)$, and since $e^{-t+s} \leqslant 1$ and $e^{u-s} \leqslant 1$,

$$U(t, y) \leqslant U(s, y) + (t - s)\, M(t).$$

This inequality with (3.13) yields (3.11). ∎

Proof of Lemma 2. Lemma 3 shows U satisfies (3.3) and (3.4) and continuity at (t, x) if $t > 0$. To see that U is continuous at $(0, x)$ and thereby prove (3.2) is satisfied, it suffices to prove

$$U(t, x) \geqslant e^{-t}|\, x \,| - tM(t).$$

For $\Psi \in A_a(t, x)$,

$$\begin{aligned} &\int_0^t e^{-t+u}\, |\, \dot{\Psi}(u) - f(u, \Psi(u))|\, du \\ &\qquad \geqslant e^{-t} \int_0^t |\, \dot{\Psi}(u)|\, du - \int_0^t |\, f(u, \Psi(u))|\, du \\ &\qquad \geqslant e^{-t} \left| \int_0^t \dot{\Psi}(u)\, du \right| - tM(t) = e^{-t}\, |\, x \,| - tM(t) \end{aligned}$$

which proves our claim.

To prove (3.7) let ϕ_1 be a solution of (E) with $\phi_1(t) = x$. For $\Psi \in A_a(t, x)$ define $\Psi_h \in A_a(t + h, \phi_1(t + h))$ equal to Ψ on $[0, t]$ and equal to ϕ_1 on $[t, t + h]$. Then

$$\begin{aligned} U(t + h, \phi_1(t + h)) &\leqslant \int_0^{t+h} e^{-(t+h-u)}\, |\, \dot{\Psi}(u) - f(u, \Psi_h(u))|\, du \\ &= e^{-h} \int_0^t e^{-(t-u)}\, |\, \dot{\Psi}(u) - f(u, \Psi(u))|\, du \\ &\qquad \text{for all } \Psi \in A_a(t, x) \end{aligned}$$

which, implies $U(t + h, \phi_1(t + h)) \leqslant e^{-h} U(t, \phi_1(t))$. Computing $\dot{U}$ (with respect to h at $h = 0$) gives (3.7), completing the proof of the lemma. ∎

4. Asymptotic Stability Under Perturbation

Let B be a normed vector space of measurable functions p: $[0, \infty) \to R^n$. The norm of p will be denoted $\| p \|_B$.

Definition 1. We say $x \equiv 0$ is *stable under B perturbations for* (E) if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$, such that

$$| x_0 | < \delta \quad \text{and} \quad \| p \|_B < \delta \quad \text{imply} \quad | \phi_P(t, t_0, x_0)| < \epsilon$$

for all $| x_0 | < \delta$ and $t \geqslant t_0 \geqslant 0$. When $B = L^1[0, \infty)$, we sometimes say $x \equiv 0$ *is integrally stable.* It is clear that $x \equiv 0$ is a solution when $x \equiv 0$ is stable under B perturbations.

Definition 2. A function V: $[0, \infty) \times S_a \to R$ is called *positive definite* if there exists a continuous, nondecreasing, real-valued function $b(\cdot)$, and $b(r) = 0$ if and only if $r = 0$, such that

$$b(| x |) \leqslant V(t, x) \quad \text{for all} \quad (t, x) \in [0, \infty) \times S_a . \tag{4.1}$$

Theorems 1 and 2 were given by Vrkoč [4], and Theorem 3 is apparently new.

Theorem 1. *The following conditions are equivalent.*

(IS): $x \equiv 0$ *is integrally stable for* (E).

(V_{IS}): *For some* $a \in (0, c)$ *there exists a continuous function*

$$V: [0, \infty) \times S_a \to R$$

having the following four properties:

$$V(t, x) \text{ is positive definite,} \tag{4.2}$$

$$V(t, x) \leqslant | x | \quad \text{for all} \quad (t, x) \in [0, \infty) \times S_a , \tag{4.3}$$

$$| V(t, x) - V(t, y)| \leqslant | x - y |, \quad \text{for all} \quad (t, x), (t, y) \in [0, \infty) \times S_a , \tag{4.4}$$

$$\dot{V}(t, x) \leqslant 0 \text{ taken along solutions of (E).} \tag{4.5}$$

Definition 3. We say $x \equiv 0$ *is attracting under B perturbations* for (E) if there exists $\delta_0 > 0$ and for each $\epsilon > 0$ there exists $T = T(\epsilon) \geqslant 0$ and $\eta = \eta(\epsilon) > 0$ such that

$$| x_0 | < \delta_0 \quad \text{and} \quad \| p \|_B < \eta \quad \text{imply} \quad | \phi_P(t, t_0, x_0)| < \epsilon$$

for all $t \geqslant t_0 + T$ and $t_0 \geqslant 0$. When $B = L_1[0, \infty)$, we sometimes say $x \equiv 0$ *is integrally attracting for* (E).

DEFINITION 4. We say $x \equiv 0$ is *asymptotically stable under B perturbations for* (E) if it is stable under B perturbations and attracting under B perturbations. When $B = L_1[0, \infty)$ we sometimes say $x \equiv 0$ *is integral-asymptotically stable for* (E).

For a function $U\colon [0, \infty) \times S_a \to R$ we will write $\dot{U}_{(\mathrm{E})}$ or $\dot{U}_{(\mathrm{P})}$ for the "trajectory derivative" (see (2.1)) taken along solutions of (E) and (P), respectively.

THEOREM 2. *The following conditions are equivalent.*

(IAS): $x \equiv 0$ *is integral-asymptotically stable for* (E).

(U_{IAS}): *For some* $a \in (0, c)$ *there exists a continuous function* $U\colon [0, \infty) \times S_a \to S_a$ *satisfying* (4.2), (4.3), (4.4), *and*

$$\dot{U}_{(\mathrm{E})}(t, x) \leqslant -U(t, x). \tag{4.6}$$

THEOREM 3. *The above conditions* (IAS) *and* (U_{IAS}) *are equivalent to*

($IA+$): $x \equiv 0$ *is a solution and is unique in the future and is integrally attracting for* (E).

Notice that integral stability of 0 implies $x \equiv 0$ is a unique-in-the-future solution of (E), which explains why the uniqueness hypothesis of Theorem 3 is not needed in Theorems 1 and 2.

EXAMPLE. For the following scalar equation 0 is integrally attracting but 0 is not even a solution:

$$\dot{x} = -x + (t+1)^{-1}.$$

Integral asymptotically stable systems can in fact be perturbed by a larger class of functions and still maintain their stability and attraction.

DEFINITION 6. A function $p\colon [0, \infty) \to R^n$ is said to be *interval bounded* if it is measurable and

$$\sup_{t \geqslant 0} \int_t^{t+1} |p(s)|\, ds < \infty.$$

We will denote the space of interval bounded functions by B_{IB} with norm

$$\| p \|_{B_{IB}} = \sup_{t \geqslant 0} \int_t^{t+1} |p(s)|\, ds.$$

Notice that a measurable function $p(\cdot)$ is interval bounded if for example $|p(t)|$ is bounded or if for some $\gamma \geqslant 1$

$$\int_0^\infty |p(s)|^\gamma \, ds < \infty.$$

Since $L_1 \subset B_{IB}$, if $x \equiv 0$ is asymptotically stable under B_{IB}, then $x \equiv 0$ is also integral-asymptotically stable. It is perhaps surprising that the converse is true.

THEOREM 4. *The solution $x \equiv 0$ is integral asymptotically stable (IAS) if and only if the following condition is satisfied:*

(AS_{IB}): *$x \equiv 0$ is asymptotically stable under perturbations in the space of interval bounded functions.*

5. PROOFS THAT (IS) IMPLIES (V_{IS}) AND (IAS) IMPLIES (U_{IAS})

To prove that there exist functions satisfying (V_{IS}) and (U_{IAS}) we need only prove that the functions V_0 (in (3.1)) and U (in (3.8)) are positive definite. Lemmas 1 and 2 have shown that, respectively, V_0 and U satisfy all the other conditions of (V_{IS}) and (U_{IAS}). Actually we omit the proof in the case of V_0 since the proof is quite similar to the one we now present showing ($IA+$) implies U is positive definite.

Proof that ($IA+$) implies U is positive definite

Suppose that $x \equiv 0$ is unique in the future and is integrally attracting for (E). Let δ_0 correspond to the δ_0 in Definition 3. Letting $a = \delta_0$, we only have to prove that U: $[0, \infty) \times S_a \to R^n$ is positive definite. We assume the contrary and will arrive at a contradiction. Since U is not positive definite there exists an $\epsilon \in (0, \delta_0)$ and sequences $\{t_n\}$ and $\{x_n\}$ with $t_n \to \infty$ and $\epsilon \leqslant x_n < \delta_0$ such that

$$U(t_n, x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Let $T(\epsilon)$ and $\eta(\epsilon)$ be numbers corresponding to those in Definition 4.

Let N be sufficiently large that $t_N > T + 1$ and

$$U(t_N, x_N) < \eta \exp[-(T+1)]$$

and let $\Psi_N \in A_{\delta_0}(t_N, x_N)$ be chosen such that

$$\int^{t_N} e^{-(t_N - s)} \, |\dot{\Psi}_N(s) - f(s, \Psi_N(s))| \, ds < e^{-(T+1)}\eta.$$

Then (writing $t_0 = t_N - (T+1)$)

$$\int_{t_0}^{t_N} e^{-(t_N - s)} \, | \dot{\Psi}_N(s) - f(s, \Psi_N(s))| \, ds < e^{-(T+1)}\eta$$

$$\Rightarrow e^{-(T+1)} \int_{t_0}^{t_N} | \dot{\Psi}_N(s) - f(s, \Psi_N(s))| \, ds < e^{-(T+1)}\eta$$

$$\Rightarrow \int_{t_0}^{t_N} | \dot{\Psi}_N(s) - f(s, \Psi_N(s))| \, ds < \eta.$$

Define

$$p(t) = \begin{cases} \dot{\Psi}_N(t) - f(t, \Psi_N(t)), & \text{for} \quad t \in [0, t_N] \\ 0, \quad \text{for} \quad t \in (t_N, \infty). \end{cases}$$

Then $\phi_P(t, t_0, \Psi_N(t_0)) = \Psi(t)$ for $0 \leqslant t \leqslant t_N$ and $| \phi_P(t_0, t_0, \Psi_N(t_0))| = | \Psi_N(t_0)| < \delta_0$, and

$$\int_{t_N - T - 1}^{t_N} | p(t)| \, dt \leqslant \int_0^\infty | p(t)| \, dt < \eta. \tag{4.7}$$

However $| \phi_P(t_N, t_0, \Psi_N(t_0)| = | x_N | \geqslant \epsilon$ since $\Psi_N(\cdot)$ is a solution of (P) through $(t_0, \Psi_N(t_0))$, which contradicts the definition of integral attraction since $t_N > t_0 + T$. ∎

6. Proof of Theorem 4: $(IAS) \Rightarrow (AS_{IB})$

LEMMA 5 [8, pp. 4, 5]. *Let $a \in (0, c)$ and let U: $[0, \infty) \times S_a \to [0, \infty)$ be a continuous function satisfying* (3.4). *Then*

$$\dot{U}_{(P)}(t, x) \leqslant \dot{U}_{(E)}(t, x) + | p(t)|. \tag{6.1}$$

LEMMA 6. *Assume*

$$\int_t^{t+1} | p(s)| \, ds \leqslant \delta \qquad \textit{for all} \quad t \geqslant 0. \tag{6.2}$$

Then for every $t_2 > t_1$

$$\int_{t_1}^{t_2} | p(s)| \, ds \leqslant (t_2 - t_1 + 1)\delta. \tag{6.3}$$

This result is clear and the proof is omitted.

Assume (IAS) is true for the remainder of this section. Then from Theorem 2, (U_{IAS}) is true. From Theorem 2, there is a continuous function $U\colon [0, \infty) \times S_a \to R$ satisfying (4.3), (4.4), and (for some δ as in Def. 2)

$$b(|x|) \leqslant U(t, x) \leqslant |x|, \qquad \text{for all} \quad (t, x) \in [0, \infty) \times S_a \tag{6.4}$$

$$\dot{U}_{(\mathrm{E})}(t, x) \leqslant -U(t, x) \tag{6.5}$$

where $b(\cdot)$ is a continuous, nondecreasing, real-valued function and $b(r) = 0$ if and only if $r = 0$. With these hypotheses we prove two lemmas whose proof will complete the proof of Theorem 4.

LEMMA 7. *$x \equiv 0$ is stable under B_{IB} perturbation for* (E).

Proof. Suppose not. Then there exists $\epsilon > 0$ such that for any $\delta > 0$ there exists an absolutely diminishing p with $\|p\|_{\mathrm{dim}} < \delta$ and x_0 with $|x_0| < \delta$ and $t_0 \geqslant 0$ and a solution $\phi_P = \phi_P(\cdot, t_0, x_0)$ such that for some $t_2 > t_0$

$$|\phi_P(t_2, t_0, x_0)| \geqslant \epsilon.$$

We may assume δ is sufficiently small that

$$\begin{aligned} \delta < b(b(\epsilon/2)) < b(\epsilon/2) < b(\epsilon) \\ b(\epsilon/2) + \delta < b(\epsilon). \end{aligned} \tag{6.6}$$

Since $|x_0| < \delta < b(\epsilon/2) < \epsilon$ (from (6.4)), there is some t_1 (t_0, t_2) such that

$$|\phi_p(t_1)| = b(\epsilon/2), \qquad |\phi_p(t)| > b(\epsilon/2) \qquad \text{for} \quad t \in (t_1, t_2).$$

From Lemma 5 and Eq. (6.5)

$$\dot{U}_{(\mathrm{P})}(t, x) \leqslant \dot{U}_{(\mathrm{E})}(t, x) + |p(t)| \leqslant -U(t, x) + p(t).$$

Using (6.4) we have for $t \in [t_1, t_2]$

$$\begin{aligned} \dot{U}_{(\mathrm{P})}(t, \phi_P(t)) &\leqslant -U(t, \phi_P(t)) + |p(t)| \leqslant -b(|\phi_P(t)| + |p(t)| \\ &\leqslant -b(b(\epsilon/2)) + |p(t)| < -\delta + |p(t)|. \end{aligned}$$

Now integrating from t_1 to t_2 and using Lemma 6

$$U(t_2, \phi_P(t_2)) - U(t_1, \phi_P(t_1)) \leqslant -(t_2 - t_1)\delta + \int_{t_1}^{t_2} |p(t)|\, dt < \delta.$$

But this means that

$$\begin{aligned} b(\epsilon) = b(|\phi_P(t_2)|) &\leqslant U(t_2, \phi_P(t_2)) \\ &< U(t_1, \phi_P(t_1)) + \delta \\ &\leqslant |\phi_P(t_1)| + \delta < b(\epsilon/2) + \delta \end{aligned}$$

which contradicts (6.6), proving the lemma. ∎

LEMMA 8. *$x \equiv 0$ is attracting for* (E) *under interval bounded perturbations for* (E).

Proof. Let $\delta = \delta(\epsilon)$ and $\eta = \eta(\epsilon)$ be from Lemma 7, i.e.,

$$\| p \|_{IB} < \eta \qquad \text{implies} \qquad |\phi_P(t; t_0, x_0)| < \epsilon$$

for all $|x_0| < \delta$ and $t \geqslant t_0 \geqslant 0$. Choose $\delta_0 = \delta(c)$. Let $\epsilon > 0$ be given. We claim that

$$\eta(\epsilon) = \min\{\eta(\epsilon), (1/2)\, b(\delta(\epsilon))\}$$

and

$$T(\epsilon) = (c + (1/2)\, b(\delta(\epsilon)))/(1/2)\, b(\delta(\epsilon))$$

are the required estimates in Definition 3 for our system. In view of Lemma 7, we only need to show that there exists $t^* \in [t_0, T(\epsilon) + t_0]$, for any $t_0 \geqslant 0$, such that

$$|\phi_P(t^*; t_0, x_0)| < \delta(\epsilon), \qquad \text{where} \quad |x_0| < \delta_0 = \delta(c).$$

Since $\eta(\epsilon) \leqslant \eta(\epsilon)$, we have from Lemma 7,

$$|\phi_P(t; t_0, x_0)| < \epsilon \qquad \text{for all} \quad t \geqslant t_0 + T \geqslant t^*.$$

Now suppose there does not exist such t^*, then

$$\delta(\epsilon) \leqslant |\phi_P(t)| \leqslant c, \qquad \text{for all } t \in [t_0, t_0 + T(\epsilon)]$$

where $\phi_P(t) = \phi_P(t; t_0, x_0)$. As in the proof of Lemma 7, we obtain

$$\begin{aligned} 0 &< b(\delta(\epsilon)) \\ &\leqslant U(t_0 + T, \phi_P(t_0 + T)) \\ &\leqslant U(t_0, \phi_P(t_0)) - [b(\delta) - g(t_0)]\, T + g(t_0) \\ &\leqslant c - [b(\delta) - \eta]\, T + \eta \\ &\leqslant c - [b(\delta) - (1/2)\, b(\delta)]\, T + (1/2)\, b(\delta) \\ &= c - c = 0. \end{aligned}$$

This is a contradiction. ∎

7. An Application

Lemma 9 (Massera [11]). *Let $f(t, x)$ be Lipschitzean in x and assume* 0 *is uniform asymptotically stable for* (E). *Then for some $a \in (0, c)$, there is a function $V: (0, \infty) \times S_a \to [0, \infty)$ such that V has an infinitesimal upper bound, $\dot{V}$ (taken along solutions of* (E)) *is negative definite, and V is Lipschitzean in x.*

Corollary. *Let f be Lipschitzean in x and assume* 0 *is uniform-asymptotically stable for* (E). *Then* 0 *is integral-asymptotically stable.*

This Corollary follows immediately from Lemma 9 and Theorem 3. It was proved in [1] using quite different methods.

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