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Potential optimality of Pareto optima

Vladislav V. Podinovski *

National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russia

Abstract

In this paper the notion of potential optimality without an assumption that a value function exists is used to investigate multicriterial optimization problems. Our results show that the notions of potential optimality and strong Pareto optimality (weak Pareto optimality, properly Pareto optimality) are equivalent for special forms of objective functions which are increasing with respect to strong Pareto relation (weak Pareto relation).

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1. Introduction

The notion of potential optimality was originally introduced in the context of an additive value or utility function with unknown parameters and finite number of alternatives. See, for example, [1, 2]. White [3] proposed a definition of potential optimality using the set of value functions. Podinovski [4] introduced the notion of potential optimality without an assumption of value function existence. In this paper the author investigates the potential optimality for Pareto optima in multicriterial decision making problems.

2. Basic definitions and some results for the potential optimality

For convenience, the basic definitions and selected results from [4] are presented below. Let the preferences of the decision maker (DM) be described by the *non-strict preference relation* R on the set A of objects: aRb if the DM considers object a to be not less preferable than object b. Objects a and b are *comparable* if aRb or bRa

* Corresponding author. Tel.: +7-495-621-1342; fax: +7-495-621-1342.
E-mail address: podinovski@mail.ru.

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is true. Relation R is *complete* if any two elements of A are comparable. If relation R is not complete it is called *partial*. The non-strict preference relation R induces the relations of (*strict*) *preference relation* P and *indifference relation* I on A: aPb holds if aRb is true but bRa is not true; aIb holds if both aRb and bRa are true. The relation R is assumed to be a quasi-order, i.e. R is a reflexive and transitive.

Definition 1. An object a^* is *optimal* (*with respect to R*), if a^*Ra holds for any $a \in A$.

Definition 2. An object a^0 is *non-dominated* (*with respect to P*), if there exists no object $a \in A$ such that aPa^0. Otherwise object a^0 is *dominated* (*with respect to P*).

A quasi-order R' is a (*consistent*) *extension* of a quasi-order R if the following is true: $R' \supset R$; $I' \supseteq I$; $P' \supseteq P$ (note that the first embedding implies the second). According to the theorem of Szpilrajn [5], any partial quasi-order R on A can be extended to a complete quasi-order R'. Let $\aleph$ be the class (set) of all complete quasi-orders on A that extend R, and $\Re$ be some class (non-empty subset of $\aleph$) of such complete quasi-orders.

Definition 3. An object a^* is *potentially optimal for the class* $\Re$ if there exists a complete quasi-order $R \in \Re$ with respect to which a^* is optimal.

Definition 4. An object a^* is *potentially optimal*, if it potentially optimal for the class $\aleph$.

Theorem 1. *The following statements hold:*

1.1. *Each potentially optimal for the class* $\Re$ *object is non-dominated.*
1.2. *Any non-dominated object is potentially optimal.*
1.3. *Any object is non-dominated iff (if and only if) it is potentially optimal.*

3. Basic definitions and some results from the theory of multicriteria optimization

Let X be a set of alternatives (strategies, variants, plans,...) and let $f = (f_1,\ldots, f_m)$ be a vector criterion. Each criterion f_i is a function defined on X and taking its values from the range $Z_i \subseteq \mathrm{Re} = (-\infty, +\infty)$; its larger values are preferred to smaller ones. The vector $y = f(x)$ is called a *criterial estimate* of alternative x, and the set $Y = f(X)$ is called *a set of feasible criterial estimates*. The set $Z = Z_1 \times \ldots \times Z_m$ is *a set of* (all) *criterial estimates*. It is located in the m-dimentional arithmetical space Re^m called *the criterial space*.

The fundamental role in multicriterial optimization problems belong to the (*strong*) *Pareto relation* $R^{\geq}$. It is defined on the set Z as follows:

$$z' R^{\geq} z'' \Leftrightarrow z_i' \geq z_i'',\ i = 1,\ldots,m .$$

If all non-strict inequalities are satisfied as equalities, then $z' I^{\geq} z''$ (and therefore $I^{\geq}$ is an equality relation for vectors from Z), otherwise $z' P^{\geq} z''$.

Definition 5. A point $y^0 \in Y$ that is non-dominated in Y with respect to $P^{\geq}$ is (*strongly*) *Pareto optimal*, or (*strongly*) *efficient*, or *a* (*strong*) *Pareto optimum.*

Weak Pareto relation $P^{>}$ on the set Z is defined as follows:

$$z' R^{>} z'' \Leftrightarrow z_i' > z_i'',\ i = 1,\ldots,m .$$

Let $R^{>}$ be a union of $P^{>}$ and the equality relation for vectors from Z.

Definition 6. A point $y^0 \in Y$ that is non-dominated in Y with respect to $P^{>}$ is *weakly Pareto optimal*, or *Slater optimal*, or *a weak Pareto optimum.*

Definition 7. A Pareto optimal point $y^0 \in Y$ is *properly Pareto optimal*, or *properly efficient*, or *Geoffrion optimal*, if there exists a positive number $\theta(y^0)$ such that for any point $y \in Y$ the following statement is true: if $y_i > y_i^0$ then there exists $j \in M = \{1,\ldots, m\}$ such that $y_j < y_j^0$ and

$$\frac{y_i - y_i^0}{y_j^0 - y_j} \le \theta(y^0) .$$

Let $Y^{\gtrsim}$, $Y^{\gtrsim *}$ and $Y^{\succ}$ denote the sets of Pareto optimal points, properly Pareto optimal points and weakly Pareto optimal points respectively. Note that $Y^{\gtrsim *} \subseteq Y^{\gtrsim} \subseteq Y^{\succ}$ and these embeddings are generally strict. If the set Y is finite then each Pareto optimal point is properly Pareto optimal: $Y^{\gtrsim *} = Y^{\gtrsim}$.

Theorem 2. [6] *A point $y^0 \in Y$ is weakly Pareto optimal iff*

$$\max_{y \in Y} \min_{i \in M} (y_i - y_i^0) = 0 . \tag{1}$$

Let ε be a positive number such that $\varepsilon < 1/m$. Using vectors $\mu^j(\varepsilon)$:

$$\mu_i^j(\varepsilon) = \begin{cases} \varepsilon, \, i \in M, i \ne j, \\ 1-(m-1)\varepsilon, \, i = j, \end{cases}$$

define Λ_ε a polyhedral cone in Re^m:

$$\Lambda_\varepsilon = \{u \in \mathrm{Re}^m \, \Big| \, \sum_{i=1}^{m} \mu_i^j(\varepsilon) u_i \ge 0, \, j = 1,\ldots,m\} .$$

Let $0_{(m)} = (0, \ldots, 0)$ denote the origin of Re^m and

$$Y - y^0 = \{u \in \mathrm{Re}^m \,|\, u = y - y^0, y \in Y\}.$$

Theorem 3. [6] *A point $y^0 \in Y$ is properly Pareto optimal iff there exists a positive number $\varepsilon < 1/m$ such that*

$$(Y - y^0) \cap \Lambda_\varepsilon = 0_{(m)}. \tag{2}$$

The set Y is called *effectively convex* if its *Edgeworth-Pareto hull* $Y_* = Y - \mathrm{Re}^m_{\geq}$, where $\mathrm{Re}^m_{\geq} = \{u \in \mathrm{Re}^m \,|\, u_1 \ge 0,\ldots,u_m \ge 0\}$, is convex. Note that if the set X is convex and all criteria f_i are concave functions then Y is effectively convex [7].

Theorem 4. *Let the set Y be effectively convex. The following statements hold*:

4.1. [7, 8] *A point $y^0 \in Y$ is weakly Pareto optimal iff there exist non-negative numbers $c_1, \ldots, c_m$ not all equal to zero such that*

$$\sum_{i=1}^{m} c_i y_i^0 = \max_{y \in Y} \sum_{i=1}^{m} c_i y_i . \tag{3}$$

4.2. [9] *A point $y^0 \in Y$ is properly Pareto optimal iff there exist positive numbers $c_1, \ldots, c_m$ such that (3) is true.*

Let Y be polyhedral (see, e.g., [10]) and therefore effectively convex. (If X is polyhedral and all criteria f_i are linear functions then Y is polyhedral.) In this case the set of Pareto optimal points and the set of properly Pareto optimal points are equal (see, e.g., [11]). Note that if the criterial estimate $y = f(x)$ of the alternative x is Pareto optimal (weakly Pareto optimal, properly Pareto optimal) then the alternative x is called *Pareto optimal* (*weakly Pareto optimal*, *properly Pareto optimal* respectively).

4. Potential optimality for Pareto optima

Let $R_Y^{\geq}$ and $R_Y^{>}$ be restrictions of the relations $R^{\geq}$ and $R^{>}$ to Y respectively. The relations $R_Y^{\geq}$ and $R_Y^{>}$ are partial quasi-orders and can be extended to a complete quasi-order. Theorem 1 shows that the following statement is true.

Theorem 5. *A point $y^0 \in Y$ is potentially optimal for the class of all complete quasi-orders on Y that extend $R_Y^{\geq}$ (respectively, $R_Y^{>}$) iff it is Pareto optimal (respectively, weakly Pareto optimal).*

But for some classes of quasi-orders, the Pareto optimal point may not be potentially optimal (see, e.g., [12]).

A function $\varphi(z)$ is said to be increasing with respect to $P^{\geq}$ (increasing with respect to $P^{>}$) on the set Z if

$$z' P^{\geq} z'' \Rightarrow \varphi(z') > \varphi(z'') \text{ (respectively, } z' P^{>} z'' \Rightarrow \varphi(z') > \varphi(z'')).$$

Each such function induces on Z a complete quasi-order R^{φ}:

$$z' R^{\varphi} z'' \Leftrightarrow \varphi(z') \geq \varphi(z''),$$

and the set of maximum points of this function on Y is equal to the set of points that are optimal with respect to R_Y^{φ}. Let $\mathfrak{R}^{\geq}$ ($\mathfrak{R}^{>}$) be a class (set) of complete quasi-orders that are induced on Y by all functions that are continuous and increasing on Z (therefore, on Y) with respect to $P^{\geq}$ (increasing with respect to $P^{>}$). Many multicriterial optimization methods use such (objective) functions. Therefore question of potential optimality for the classes $\mathfrak{R}^{\geq}$ and $\mathfrak{R}^{>}$ could be interesting for application. Note that $\mathfrak{R}^{\geq} \subset \mathfrak{R}^{>}$.

Theorem 6. *A point $y^0 \in Y$ is weakly Pareto optimal iff it is potentially optimal for the class $\mathfrak{R}^{>}$.*

Proof of Theorem 6. If a point $y^0 \in Y$ is potentially optimal for the class $\mathfrak{R}^{>}$, then, according to Theorem 1, y^0 is weakly Pareto optimal. Now assume that $y^0 \in Y$ is weakly Pareto optimal. The complete quasi-order R^{φ} induces on Z by the function

$$\varphi_{\Pi}(z \mid b) = \min_{i \in M}(z_i - b_i), \tag{4}$$

where $b \in Z$, extends $R^{>}$ on Z. According to Theorem 2, y^0 is a maximum point of $\varphi_{\Pi}(z \mid y^0)$ on Y. Hence, y^0 is optimal with respect to R_Y^{φ}. Therefore, y^0 is potentially optimal for the class $\mathfrak{R}^{>}$.

Using the function (4) with $b \in \mathrm{Re}^m$, define the following lexicographic quasi-order $R^{lex}(b)$ on Re^m:

$$z' R^{lex} z'' \Leftrightarrow [\varphi_{\Pi}(z' \mid b) > \varphi_{\Pi}(z'' \mid b)] \vee [\varphi_{\Pi}(z' \mid b) = \varphi_{\Pi}(z'' \mid b) \wedge \sum_{i=1}^{m} z_i' \geq \sum_{i=1}^{m} z_i''].$$

This quasi-order is complete and extends $R^{\geq}$ on Z.

Theorem 7. *A Pareto optimal point $y^0 \in Y$ is optimal with respect to $R_Y^{lex}(y^0)$.*

Proof of Theorem 7. Let a point $y^0 \in Y$ be Pareto optimal. Since $R^{>} \subset R^{\geq}$, this point is weakly Pareto optimal. Let y be any point from Y. Assume that $yP_Y^{lex}(y^0)y^0$ is true. Theorem 2 shows that inequality $\varphi_\Pi(y|y^0) > \varphi_\Pi(y^0|y^0)$ is not true. Therefore, $\varphi_\Pi(y|y^0) = \varphi_\Pi(y^0|y^0) = 0$ holds and

$$\sum_{i=1}^{m} y_i > \sum_{i=1}^{m} y_i^0 . \tag{5}$$

The equality $\varphi_\Pi(y|y^0) = 0$ implies that $\min_{i\in M}(y_i - y_i^0) = 0$, hence

$$y_i \geq y_i^0, \quad i = 1,\ldots,m . \tag{6}$$

According to (5), at least one of inequalities in (6) is strict. This contradicts the assumption that y^0 is Pareto optimal. Hence, $y_0 R_Y^{lex}(y^0)y$ is true. Therefore, y^0 is optimal with respect to $R_Y^{lex}(y^0)$.

Theorem 8. *A Pareto optimal point may not be potential optimal for the class* $\Re^{\geq}$.

The following example shows that Theorem 8 is true.

Example 1. Let $m = 2$, $Z = \mathrm{Re}^2$, $Y = [0, 1) \times [0, 1) \cup \{(1, 0)\}$. The point $y^0 = (1, 0)$ is Pareto optimal. Let φ be any continuous function, increasing with respect to $P^{\geq}$ on Re^2. Suppose that y^0 is the maximum point of φ on Y. Since $z^* P^{\geq} y^0$, where $z^* = (1, 1)$, the inequality $\varphi(z^*) > \varphi(y^0)$ holds. The function φ is continuous, consequently there is a point $y^* \in Y$ for which the inequality $\varphi(z^*) - \varphi(y^*) < \varphi(z^*) - \varphi(y^0)$ is true. Therefore, $\varphi(y^*) > \varphi(y^0)$. But this is impossible, because by assumption y^0 is the maximum point of $\Re^{\geq}$ on Y. Since φ is an arbitrary continuous function, increasing with respect to $P^{\geq}$ on Re^2, the point y^0 is not potential optimal for the class $\Re^{\geq}$.

Theorem 9. *A properly Pareto optimal point is potential optimal for the class* $\Re^{\geq}$.

Proof of Theorem 9. Let a point $y^0 \in Y$ be properly Pareto optimal. According to Theorem 3, there exists a positive number $\varepsilon < 1/m$ such that (2) is true. Let us consider the function

$$\varphi(z|\mu(\varepsilon), b) = \min_{j\in M} \sum_{i=1}^{m} \mu_i^j(\varepsilon)(z_i - b_i) . \tag{7}$$

where $\varepsilon < 1/m$ and $b \in \mathrm{Re}^m$. It is continuous and increasing with respect to $P^{\geq}$ on Re^m. Its hypersurface defining by the equation $\varphi(z|\mu(\varepsilon), 0_{(m)}) = 0$ coincides with the frontier of the cone Λ_ε. It is easy to see that y^0 is the maximum point of the function (7) on Y. Hence y^0 is optimal with respect to the complete quasi-order R_Y^{φ} that is induced on Y by this function. Therefore, y^0 is potential optimal for the class $\Re^{\geq}$.

Note that a point $y^0 \in Y$ that is potential optimal for the class $\Re^{\geq}$ may not be properly Pareto optimal.

Example 2. Let $m = 2$, $Z = \mathrm{Re}_+^2$, $Y = \{z \in \mathrm{Re}_+^2 \mid z_1^2 + z_2^2 \leq 1\}$. The point $y^0 = (1, 0)$ is Pareto optimal but is not properly Pareto optimal. The function $\varphi(z) = 4z_1^2 + z_2^2$ is continuous and increasing with respect to $P^{\geq}$ on Z. It is easy to see that y^0 is the maximum point of φ on Y. Therefore, y^0 is optimal for the class $\Re^{\geq}$.

Let $\Re_Y^{L\geq}$ ($\Re_Y^{L>}$) be a class of complete quasi-orders induced on Y by all functions

$$\varphi(y|c) = \sum_{i=1}^{m} c_i y_i ,$$

where all $c_i > 0$ (respectively, all $c_i \geq 0$ and at least one of them is positive). Each such function is continuous and increasing with respect to $P^{\geq}$ (respectively, with respect to $P^{>}$) on Re^m. Note $\mathfrak{R}_Y^{L>} \subset \mathfrak{R}_Y^{L\geq}$.

The following statements are corollaries of Theorem 4.

Theorem 10. *Let the set Y be effectively convex. The following statements hold*:

10.1. *A point $y^0 \in Y$ is properly Pareto optimal iff it is potentially optimal for the class* $\mathfrak{R}_Y^{L>}$.

10.2. *A point $y^0 \in Y$ is weakly Pareto optimal iff it is potentially optimal for the class* $\mathfrak{R}_Y^{L\geq}$.

10.3. *Let the set Y be polyhedral. A point $y^0 \in Y$ is Pareto optimal iff it is potentially optimal for the class* $\mathfrak{R}_Y^{L>}$.

5. Conclusion

The notion of potential optimality is one of the basic notions in the theory of multicriterial optimization. This paper reveals the interdependence of (strong, week, properly) Pareto optimality and potential optimality in multicriterial optimization problems. Also, it demonstrates that the methods, developed for construction of Pareto optimal sets of alternatives (particularly, for convex problems), can be employed to form a set of potentially optimal alternatives.

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