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The Geometry of Frequency Squares

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TO PROFESSOR HANFRIED LENZ ON THE OCCASION OF HIS 85TH BIRTHDAY

This paper establishes a correspondence between mutually orthogonal frequency squares (MOFS) and nets satisfying an extra property ("framed nets"). In particular, we provide a new proof for the bound on the maximal size of a set of MOFS and obtain a geometric characterisation of the case of equality: necessary and sufficient conditions for the existence of a complete set of MOFS are given in terms of the existence of a certain type of PBIBD based on the L_2-association scheme. We also discuss examples obtained from classical affine geometry and recursive construction methods for (complete) sets of MOFS. © 2001 Academic Press

1. INTRODUCTION

The concept of a frequency square is a generalisation of that of a latin square. In the case of latin squares there is a concept of orthogonality. It is well known that the maximum number of mutually orthogonal latin squares of order n is bounded by $n-1$. Sets of latin squares realising this bound are called complete. Bose (see, e.g., [11]) established that the existence of a complete set is equivalent to that of an affine plane.

There is a natural extension of the concept of orthogonality to frequency squares. Laywine and Mullen (see [10, 11] for details and further references)

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investigated this situation with the aim of extending Bose's theorem to frequency squares. They established links between affine 2-designs and complete sets of mutually orthogonal frequency squares (MOFS). However, they do not give equivalence. Indeed, Laywine [9] proved that there are affine 2-designs which do not arise from complete sets of MOFS in the way corresponding to the links mentioned.

The problem seems to stem from the fact that the theorem of Hedayat *et al.* [5] on the maximum size of a set of MOFS does not explicitly give any information about the optimal case that may possibly be used to construct a corresponding design to the affine plane of the latin square case.

In this paper, we provide a definitive answer to the problem of finding a geometric interpretation of MOFS by showing that they are equivalent to a certain special type of nets which we decided to call *framed nets*. We use this result to give a new proof for the bound on the maximum size of a set of MOFS by a counting argument which permits us to show that the existence of a complete set (that is one realising the bound) of MOFS is equivalent to the existence of a certain type of PBIBD. In the case of latin squares, this PBIBD is an affine plane with two parallel classes deleted. In consequence, we finally obtain the proper generalisation of Bose's theorem which had proved elusive for such a long time. We also discuss examples obtained from classical affine geometry and provide recursive construction methods for (completely) framed nets, in this way unifying all known construction methods for complete sets of MOFS.

2. BASIC DEFINITIONS

A *type* $F(n;\mu)$ *frequency square* over a set S of order m ($\geqslant 2$) is an $n\times n$ matrix over S such that each element of S appears exactly μ times in every row and column, so $n=\mu m$. Thus latin squares of order n are type $F(n;1)$ frequency squares.

Two frequency squares M, N of type $F(n;\mu)$ are *orthogonal* if they are over the same set S and every element of S^2 occurs exactly μ^2 times among the pairs (M_{ij}, N_{ij}), $1\leqslant i, j\leqslant n$. This generalizes the concept of orthogonality for latin squares.

A theorem of Hedayat *et al.* [5] (or see [4, 6, 11]) states that the order of a set of MOFS of type $F(n;\mu)$ is at most $(n-1)^2/(m-1)$; when this bound is reached, the set of MOFS is said to be *complete*.

A *2-class partially balanced incomplete block design* (2-class PBIBD) D with parameters $v, k, r, n_1, n_2, \lambda_1, \lambda_2$ satisfies (a) there are v points, with k on each block and r blocks on each point; (b) any two distinct points are contained in λ_1 or λ_2 blocks and called first or second associates accordingly; (c) the number of ith associates of each point is n_i ($i=1,2$); (d) for each $j, k \in \{1, 2\}$ and for each pair of points, the number of points which are jth associates of the first point and kth associates of the second point depends only on the associate class of the pair of points. Thus, D is a 1-(v, k, r) design.

A PBIBD is said to be based on the L_2-association scheme, and sometimes called an L_2-design (see, e.g., Raghavarao [17]) if $v = n^2$ and the n^2 points can be put in an $n \times n$ square array so that distinct points are first associates if, and only if, they are in the same row or column. We also refer to Bailey [1] for a concise (and nice) introduction to PBIBSs.

A PBIBD with $\lambda_1 = \lambda_2 = \lambda$ is a balanced incomplete block design (BIBD) or, simply, a 2-(v, k, λ) design.

A design D is *resolvable* if its blocks can be partitioned into subsets called *parallel classes*, each of which partitions the point set of the design. If the resolution of D is such that any two blocks from different parallel classes (i.e., non-parallel blocks) meet in a constant number of points μ, then D is said to be *affine*. An $(m, r; \mu)$-net N (see [2]) is an affine 1-$(\mu m^2, \mu m, r)$ design. The number of blocks in a parallel class of N is m, any two non-parallel blocks meet in μ points and there are r parallel classes.

The dual of a net is also known as a transversal design or semi-regular group divisible design.

It is well known (see, e.g., [2]) that $r \leqslant (\mu m^2 - 1)/(m-1)$ for a $(m, r; \mu)$-net N, with equality if and only if N is a 2-design. In this case N is a 2-$(\mu m^2, \mu m, \lambda)$ design with $\lambda = (\mu m - 1)/(m-1)$ and is known as a *complete net*. Since the parameters of the complete net N are determined by μ and m, we can refer to N as a *complete* (m, μ)-*net*. This is denoted by $AD(\mu m, m)$ in [11] and by $A_\mu(m)$ in [2].

If the dual design N^* of an $(m, r; \mu)$-net N is resolvable, it follows that $r \leqslant \mu m$; furthermore, N^* is a net if, and only if, $r = \mu m$ (see [2, II.8.18 and II.8.21]). We say that N is a *symmetric net* in the latter case, and call the block classes of N^* *point classes* of N. The parallel classes of N will be called *block classes* of N if there is any danger of confusion. Since the parameters of a symmetric net N are determined by μ and m, we can refer to N as an (m, μ)-*symmetric net*.

Note that the complete $(m, 1)$-nets are precisely the affine planes of order m. The $(m, 1)$-symmetric nets are precisely the designs obtained by deleting one parallel class from an affine plane of order m.

3. NETS AND MOFS

The maximum size of a set of MOFS was obtained by Hedayat *et al.* (see [4, 6, 11]). In [6, Theorem 8.23] the existence of a set of MOFS is shown to imply that of an orthogonal array. Orthogonal arrays are closely linked to nets (see, e.g., [2]). In Theorem 3.2 we obtain the above results in terms of nets but we also give a converse. That is, the existence of a certain type of net implies that of a set of MOFS.

In Theorem 3.5 we show that the maximum cardinality for a set of MOFS is achieved if, and only if, the net is a PBIBD based on the L_2-association scheme.

DEFINITION 3.1. Let D be any design with n^2 points and μn points on each block. A *frame* $[X:Y]$ of D consists of two partitions $X=\{X_1, X_2, ..., X_n\}$ and $Y=\{Y_1, Y_2, ..., Y_n\}$ of the points of D into subsets of order n, such that for all i, j:

(a) $|X_i \cap Y_j| = 1$;

(b) $|X_i \cap B| = |Y_j \cap B| = \mu$ for all blocks B.

D is a *framed design* if it has a frame.

Clearly, any subdesign of a framed design obtained by removing blocks will also be framed.

THEOREM 3.2. *There exists a set of r MOFS of type $F(n;\mu)$ if, and only if, there exists a framed $(m, r; \mu^2)$-net, where $n=\mu m$.*

Proof. Let $F^{(1)}, F^{(2)}, ..., F^{(r)}$ be MOFS of type $F(n;\mu)$ defined over $S=\{1, 2, ..., m\}$. Define a design D whose points are the ordered pairs (i, j), $1 \leqslant i, j \leqslant n$, and whose blocks are the point-sets $B_t^{(u)} = \{(i, j) \mid F_{ij}^{(u)} = t\}$ for $1 \leqslant u \leqslant r$, $t \in S$.

Since a given $t \in S$ appears μ times in each row of a square $F^{(u)}$, $B_t^{(u)}$ has $n\mu$ points. Clearly, $\{B_t^{(u)} \mid t \in S\}$ partitions the point set of D, for any u, $1 \leqslant u \leqslant r$. Furthermore, $B_t^{(u)}$ and $B_y^{(x)}$ do not meet if $u=x$ and $t \neq y$. If $u \neq x$, $B_t^{(u)}$ and $B_y^{(x)}$ meet in μ^2 points, using the orthogonality of $F^{(u)}$ and $F^{(x)}$. It follows that D is an $(m, r; \mu^2)$-net. Define $X_i = \{(i, j) \mid 1 \leqslant j \leqslant m\}$, $Y_j = \{(i, j) \mid 1 \leqslant i \leqslant m\}$, and $X=\{X_1, X_2, ..., X_n\}$, $Y=\{Y_1, Y_2, ..., Y_n\}$. Then $[X:Y]$ is easily verified to be a frame.

Conversely, suppose D is an $(m, r; \mu^2)$-net with a frame $[X:Y]$, where $X=\{X_1, X_2, ..., X_n\}$ and $Y=\{Y_1, Y_2, ..., Y_n\}$, with $n=m\mu$. Then each point of D is uniquely expressible as an intersection $X_i \cap Y_j$. For a general parallel class $K=\{K_1, K_2, ..., K_m\}$, define a frequency square F by $F_{ij}=t$ if, and only if, $X_1 \cap Y_j$ is on K_t. It is straightforward to verify that F is a frequency square using the fact that $[X:Y]$ is a frame.

Let F and F' be the frequency squares corresponding to different parallel classes K and K'. Then, for a given pair x, y $(1 \leqslant x, y \leqslant m)$, the number of ordered pairs (i, j) with $F_{ij}=x$ and $F'_{ij}=y$ is $|K_x \cap K'_y| = \mu^2$. Hence, F and F' are orthogonal. Thus, the r parallel classes of D give a set of r MOFS of type $F(n;\mu)$. ∎

Consider the case $\mu = 1$. Given an $(m, r; 1)$-net Δ, with $r \geqslant 2$, choose two distinct parallel classes X and Y. Then the subnet of Δ obtained by deleting the blocks of X and Y from Δ is a framed net with frame $[X : Y]$. Conversely, by adjoining the frame to a framed $(m, r; 1)$-net we get an $(m, r+2; 1)$-net. This proves the following theorem.

THEOREM 3.3. *An $(m, r; 1)$-net is framed if, and only if, it can be extended by 2 parallel classes; that is, it is a subnet of some $(m, s; 1)$-net, where $s \geqslant r+2$.*

By a theorem of Shrikhande (see, e.g., [6, p. 170]) any $(m, m-1; 1)$-net with $m \neq 4$ is an affine plane of order m with two parallel classes deleted; so the net is framed and is a PBIBD with $\lambda_1 = 0$ and $\lambda_2 = 1$. The exceptional net for $m = 4$ is unique. Its points are the elements of the group $G = H \times H$, where G is the cyclic group of order 4, and its blocks are all subsets of the form $U_i g$ with $i = 1, 2, 3$ and $g \in G$, where the U_i are the three subgroups $H \times \{1\}$, $\{1\} \times H$ and $\{(h, h): h \in H\}$. This net is therefore a non-framed $(4, 3; 1)$-net; in fact, it is a PBIBD based on the pseudo L_2-association scheme with $\lambda_1 = 0$ and $\lambda_2 = 1$. To sum up, we have the following corollary to Theorem 3.3.

COROLLARY 3.4. *Any framed $(m, m-1; 1)$-net is embeddable in an affine plane of order m. There is a unique non-framed $(m, m-1; 1)$-net and this net has $m = 4$.*

In the next theorem we give a new proof for the known upper bound on the size of a complete set of MOFS by counting arguments applied to the *corresponding net*, as obtained in Theorem 3.2. More importantly, we will be able to characterise the case of equality.

THEOREM 3.5. *Suppose there exists a framed $(m, r; \mu^2)$-net or, equivalently, a set of r MOFS of type $F(n; \mu)$, where $n = \mu m$ and $m > 1$. Then $r \leqslant (n-1)^2/(m-1)$ with equality if, and only if, the net is a PBIBD based on the L_2-association scheme. In this case, $\lambda_1 = (n-1)(\mu-1)/(m-1)$ and $\lambda_2 = (\mu n - 2\mu + 1)/(m-1)$.*

Proof. Consider the framed $(m, r; \mu^2)$-net Δ. Let Δ have frame $[X : Y]$, where $X = \{X_1, X_2, ..., X_n\}$ and $Y = \{Y_1, Y_2, ..., Y_n\}$. Any point of Δ is uniquely expressible in the form $X_i \cap Y_j$. Choose a fixed point P. We can assume $P = X_1 \cap Y_1$. Let N_1 be the set of points $X_i \cap Y_j \neq P$ with either $i = 1$ or $j = 1$.

Let λ_Q be the number of blocks containing P and Q, for any point Q of Δ. Then, since $[X : Y]$ is a frame, it follows that $|B \cap N_1| = 2\mu - 2$ for any block B containing P. So, counting pairs (Q, B), where B is a block on P

and $Q \in B \cap N_1$, we get $\sum_{Q \in N_1} \lambda_Q = 2r(\mu-1)$ and $\sum_{Q \notin N_1} \lambda_Q = r(\mu n - 2\mu + 1)$.

Next, counting ordered triples (Q, B, B'), where B and B' are distinct blocks on P and $P \neq Q \in B \cap B'$, we get $\sum_Q \lambda_Q(\lambda_Q - 1) = r(r-1)(\mu^2-1)$. In this summation, we have used the fact that distinct blocks of Δ meet in 0 or μ^2 points and we sum over all points Q different from P. It follows that $\sum_Q \lambda_Q^2 = r(\mu n - 1) + r(r-1)(\mu^2-1)$.

Let $\bar{\lambda}_1 = r(\mu-1)/(n-1)$ and $\bar{\lambda}_2 = r(\mu n - 2\mu + 1)/(n-1)^2$.

Next, we compute $V = \sum_{Q \in N_1} (\bar{\lambda}_1 - \lambda_Q)^2 + \sum_{Q \notin N_1} (\bar{\lambda}_2 - \lambda_Q)^2$. Simplifying, with the help of the above equations and the definitions of $\bar{\lambda}_1$ and $\bar{\lambda}_2$, we obtain $V = r(n-\mu)(\mu - r(n-\mu)/(n-1)^2)$.

Since $r > 0$ and $n = \mu m > \mu$, we get $r \leqslant \mu(n-1)^2/(n-\mu) = (n-1)^2/(m-1)$, with equality if, and only if, Δ is a PBIBD based on the L_2-association scheme, with $\lambda_1 = \bar{\lambda}_1$ and $\lambda_2 = \bar{\lambda}_2$. In this case, the rows and columns of the association scheme are the subsets X_i and Y_j of the frame $[X : Y]$ of Δ. ∎

Definition 3.6. In view of the above theorem we shall refer to the net Δ corresponding to a complete set of MOFS as a *complete-framed net*.

Note that Δ is a PBIBD based on the L_2-association scheme. Δ is not a complete net in the usual sense (see Section 2).

Consider the case $\mu = 1$ of Theorem 3.5. Then $n = m$ and the MOFS are MOLS. The theorem asserts $r \leqslant m-1$ with equality if, and only if, the associated framed net is a PBIBD based on the L_2-association scheme. Adjoining the frame subsets as lines gives an affine plane of order m. Thus, Theorem 3.5 is equivalent to Bose's theorem for the case $\mu = 1$.

A theorem of Raghavarao is the key to characterising the framed nets which are also PBIBDs based on the L_2-association scheme.

Theorem 3.7. *Let D be a PBIBD based on the L_2-association scheme and suppose D is an $(n/\mu, r; \mu^2)$-net.*

Then D is a complete-framed net if, and only if, $\lambda_1 = r(\mu-1)/(n-1)$ and $\lambda_2 = r(\mu n - 2\mu + 1)/(n-1)^2$. In this case, $r = (n-1)^2/(m-1)$.

Proof. If λ_1 and λ_2 have the values given, it is easily verified that the relation $r + (n-2)\lambda_1 = (n-1)\lambda_2$ holds. By a theorem of Raghavarao [17, Theorem 8.9.3], it follows that D is framed by the rows and columns of the association scheme. The value of r and the converse follow from Theorem 3.5. ∎

The next result is in Hedayat *et al.* [6, Theorem 8.23] in the language of orthogonal arrays and frequency squares.

Theorem 3.8. *If there exists a framed $(m, r; \mu^2)$-net or, equivalently, a set of r MOFS of type $F(\mu m; \mu)$, and there exists an $(m, s; \mu/m)$-net, then there exists an $(m, r+2s; \mu^2)$-net.*

Proof. If the frame is $[X:Y]$, define an $(m, s; \mu^2)$-net on the μm subsets of X (as point set), and similarly define an $(m, s; \mu^2)$-net on the μm subsets of Y. Then a routine verification shows that these two nets may be used to extend the $(m, r; \mu^2)$-net to an $(m, r+2s; \mu^2)$-net. ∎

Remark 3.9. In the above theorem, $[X:Y]$ is not a frame in the extended net since a new block does not meet each subset X_i in X in μ points but either contains X_i or is disjoint from X_i.

4. THE CLASSICAL CASE

The construction of a complete set of MOFS obtained from classical affine geometry has appeared in various forms. It is not immediately obvious what the corresponding complete-framed net is (see, e.g., [15]). We give a direct construction for this complete-framed $(q, (q^{2d}-1)/(q-1); q^{2d-2})$-net for any prime power q and positive integer d.

Denote by $V = V(2d, q)$ the $2d$-dimensional vector space over the field $GF(q)$ and by $AG(2d, q)$ the corresponding $2d$-dimensional affine geometry.

Let P and Q be any two d-dimensional subspaces of V which satisfy $P \cap Q = \{0\}$.

Let Δ be the net whose points are those of $AG(2d, q)$ and whose blocks are the affine hyperplanes U for which $|P \cap U| = |Q \cap U| = q^{d-1}$. Then Δ is a net since, if U is a block of Δ, so is every block in the parallel class of U.

The number of $(2d-1)$-dimensional subspaces containing a d-dimensional subspace S is the number of $(d-1)$-dimensional subspaces in the quotient space $V/S \simeq V(d, q)$; namely, $(q^d-1)/(q-1)$. Since $P+Q=V$, no hyperplane contains both P and Q. So the number of parallel classes of hyperplanes containing neither P nor Q is $(q^{2d}-1)/(q-1)-2((q^d-1)/(q-1) = (q^d-1)^2/(q-1)$. Therefore, Δ is an $(m, r; \mu)$-net with $m=q$, $\mu=q^{d-1}$ and $r=(\mu m-1)^2/(m-1)$. Furthermore, the cosets of P and Q (i.e., the d-dimensional affine subspaces parallel to P or Q in $AG(2d, q)$) are easily shown to form a frame for Δ, since P and Q each has $q^d = \mu m = n$ cosets and, moreover, $P+\mathbf{x}$ and $Q+\mathbf{y}$ meet in exactly one point in $AG(2d, q)$, for any $\mathbf{x}, \mathbf{y} \in V$.

The following algebraic presentation of the above construction is essentially the "permutation polynomial" construction of Mullen described in [11, Theorem 4.2].

First, note that the equation of any hyperplane U of $AG(2d, q)$ is uniquely expressible in the standard form $\mathbf{a}\cdot\mathbf{x} = a_1x_1 + a_2x_2 + \cdots + a_{2d}x_{2d} = c$, where $\mathbf{x} = (x_1, x_2, ..., x_{2d})$ is a general vector in V, the a_i and c are constants and, for some j with $1 \leqslant j \leqslant 2d$, we have $a_j = 1$ while $a_i = 0$ for any $i < j$. We get the parallel class of U by varying c.

Choose a basis for V so that $P = \{\mathbf{x} \mid x_i = 0, d+1 < i \leqslant 2d\}$ and $Q = \{\mathbf{x} \mid x_i = 0, 1 \leqslant i \leqslant d\}$. Let $\mathbf{a}$ be any vector satisfying $a_i \neq 0$ for some $i \in \{1, ..., d\}$ and also $a_j \neq 0$ for some $j \in \{d+1, ..., 2d\}$. A frequency square F, corresponding to the parallel class determined by $\mathbf{a}$, can be formed as follows: the rows and columns of F are labelled by the vectors in P and Q, respectively, and one sets $F_{\mathbf{xy}} = \mathbf{a} \cdot (\mathbf{x} + \mathbf{y})$.

By taking all parallel classes determined by vectors $\mathbf{a}$ satisfying the above restriction, we obtain a complete set of MOFS of type $F(q^d, q^{d-1})$. Observe that the parameters of Δ and the frequency squares just obtained have the properties that $m = n/\mu$ is a prime power and μ a power of m. We shall refer to complete-framed net parameters with these properties as *classical*. We will see that a complete-framed net endowed with a suitable automorphism group necessarily has such parameters.

Let D be a net. A *translation* of D is any automorphism α which fixes every parallel class and either α is the identity or α is fixed-point-free. D is a *translation net* if D admits a group G of translations which is point transitive. In this case, G is called a *translation group* for D. (One has to be a bit careful here, as the set of all translations of a net does not necessarily form a group, and as the same net may admit more than one translation group—though these phenomena cannot occur in the special situation we consider now.) Note that the exceptional net described after Theorem 3.3 provides an example of a translation net.

The next theorem shows that if a complete-framed net is a translation net, then the parameters, in general are classical.

THEOREM 4.1. *If the complete-framed net corresponding to a complete set of type $F(n; \mu)$ MOFS is a translation net and $m = n/\mu \neq 4$, then its parameters are classical.*

Proof. We consider separately the cases $\mu = 1$ and $\mu > 1$.

$\mu = 1$. Δ is then a translation plane of order n $(= m)$ with 2 parallel classes deleted. It is well known that $n = |G|$ is a prime power and G is elementary abelian (see, e.g., [7, Theorem 2.1]).

$\mu > 1$. By a result of Jungnickel [8, Theorem 1.7], m and μ are powers of some prime p and G is elementary abelian as $r = (n-1)^2/(m-1) \geqslant k = \mu n$. So, $m = p^i$ and $\mu = p^j$ for some i and j. Since $r = (n-1)^2/(m-1)$ is an integer, $m-1$ divides $(n-1)^2$ and elementary arguments show that i divides j. Hence μ is a power of m. ∎

We conclude this section with a remark. The construction of the classical examples of framed nets presented above admits obvious generalisations. For instance, we might use three mutually skew d-dimensional subspaces in a vector space $V(3d, q)$ to define nets framed by 3 partitions from which we may then obtain classical examples of mutually orthogonal frequency *cubes* by an analogue of Theorem 3.2. It seems clear that bounds corresponding to Theorem 3.5 could be obtained, and that the classical examples will turn out to be complete. We do not feel that it is worthwhile doing so now, but this observation should provide a good topic for a graduate student thesis.

5. GENERAL CONSTRUCTIONS

The two constructions we shall describe cover between them all known constructions for complete sets of MOFS. We adopt the following labelling practice. If R is a parallel class of a resolvable design, the blocks of R are labelled $R_1, R_2, \ldots$.

The following construction is motivated by a design construction technique in Mavron [13, Theorem 1].

THEOREM 5.1. *Suppose there exists an* $(m, r; \mu/m)$*-net* Γ *and an* $(m, s; \mu)$*-net* Δ *and the dual of* Δ *is resolvable. Then:*

(a) *There exists a framed* $(m, r(s-1); \mu^2)$*-net and therefore a set of* $r(s-1)$ *MOFS of type* $F(\mu m; \mu)$.

(b) *The set of MOFS in* (a) *is complete if and only if* Γ *is a complete net and* Δ *a symmetric net.*

Proof. Let Γ have point set K. Then $|K| = \mu m$. Let $\mathscr{P}$ be the set of point classes of Δ. Then $|\mathscr{P}| = \mu m$. Define a design Σ_0 as follows.

The points of Σ_0 are the elements of $K \times \mathscr{P}$. For any parallel class L of Γ and any block B of Δ we define a block $[L, B] = \{(P, C) \mid P \in L_t;\ C_t \in B;\ 1 \leqslant t \leqslant m\}$ of Σ_0. All blocks of Σ_0 are defined in this way.

It is straightforward to verify that Σ_0 is an $(m, rs; \mu^2)$-net, where two blocks $[L, B]$ and $[L', B']$ are parallel if, and only if, $L = L'$ and B is parallel to B'. To get our framed net we need to delete some parallel classes and label in a special way.

Choose any block class $E = \{E_1, E_2, \ldots, E_m\}$ of Δ. The labelling of E is arbitrary. Label the points of each point class Q of Δ so that $Q_i \in E_i$, $1 \leqslant i \leqslant m$. Delete the r parallel classes from Σ_0 that consist of blocks of the form $[L, E_i]$, $E_i \in E$. The remaining design Σ is an $(m, r(s-1); \mu^2)$-net.

We show that Σ is a framed net. Let $P \in K$. Define X_P to be the set of all points (P, C) of Σ, where $C \in \mathscr{P}$. Given $C \in \mathscr{P}$, define $Y_C = \{(P, C) \mid P \in K\}$. Let $X = \{X_P \mid P \in K\}$ and let $Y = \{Y_C \mid C \in \mathscr{P}\}$. We show that $[X : Y]$ is a frame.

Let $P \in K$ and let $[L, B]$ be a block of Σ. Then $[L, B] \cap X_P$ consists of all points (P, C) for which $C_t \in B$, where $P \in L_t$. The number of such points is $|B \cap E_t| = \mu$.

Now, let C be a point class of Δ. A point $(P, C) \in [L, B] \cap Y_C$ if, and only if, $P \in L_t$ where t is such that $C_t \in B$. The number of such points is therefore $|L_t| = \mu$.

It is now readily seen that $[X : Y]$ is a frame for Σ.

Σ is a complete-framed net if, and only if, $r(s-1) = (\mu m-1)^2/(m-1)$. From Section 2, we have $r \leqslant (\mu m-1)/(m-1)$ with equality if, and only if, Γ is complete. We also have $s \leqslant \mu m$ with equality if, and only if, Δ is a symmetric net. The proof is now easily completed. ▮

The existence of a complete $(m, \mu/m)$-net and a symmetric (m, μ)-net is equivalent to the existence of a complete (m, μ)-net with a parallel class of lines all of size m. This is proved in [12], but see also [16]. In [15], it is shown that all but one of the known existence results for complete sets of MOFS can be obtained using complete nets with parallel classes of lines. The exception will be dealt with in the next construction.

First we make some further comments on the foregoing construction. An important construction due to Street [18] is not mentioned in [3]. Essentially, Street shows that a complete set of MOFS exists if there exist a generalised Hadamard matrix and a complete net with suitable parameters. Any Hadamard matrix is of course a generalized Hadamard matrix and Hadamard matrices give the complete $(2, \mu)$-nets. Then Federer's theorem, that a complete set of type $F(n; n/2)$ MOFS exists if there is a Hadamard matrix of order n, which is quoted in [3], can be deduced from Street's result.

Moreover, from any generalized Hadamard matrix one can construct a symmetric net but not all symmetric nets arise in this way. In this way, one may deduce Street's result from Theorem 5.1. (See Mavron and Tonchev [16] for more details.)

The next construction covers the missing existence result alluded to earlier. The construction is essentially the net form of a generalisation of the construction in Mavron [14], which in turn generalises a construction of Laywine (see [11]). The construction we describe shows that complete sets of MOFS in certain cases may be generated recursively.

THEOREM 5.2. *Suppose there exists an* $(m, r; \mu)$*-net* Γ *and a* $(\mu m^2, s; v^2)$*-net* Δ*. Then there exists an* $(m, rs; \mu^2 m^2 v^2)$*-net* Σ*. If* Δ *is framed then so is* Σ*. In this case* Σ *is complete-framed if, and only if,* Δ *is complete-framed and* Γ *is a complete net.*

Proof. Let Γ have point set J. Then $|J| = \mu m^2$. Let Δ have point set K. Then $|K| = \mu^2 m^4 v^2$. We may assume that the μm^2 blocks of any parallel class L of Δ are labelled L_t, $t \in J$. Define Σ as follows.

The point set of Σ is that of Δ; namely K. Given a parallel class L of Δ and block B of Γ a block $[L, B]$ of Σ is defined by $[L, B] = \{P \mid P \in L_t;\ t \in B\}$. It is straightforward to verify that Σ is an $(m, rs; \mu^2 m^2 v^2)$-net. Two blocks $[L, B]$ and $[L', B']$ are parallel in Σ if, and only if, $L = L'$ and B is parallel to B' in Γ.

It is easy to verify that any frame of Δ is a frame of Σ. For, if a subset X of K meets every block of Δ in v points, then X meets every block of Σ in $v\mu m$ points.

Suppose Δ is framed. From Section 2, $r \leqslant (\mu m^2 - 1)/(m-1)$ with equality if, and only if, Γ is complete. Since Σ is complete-framed if, and only if, $rs = (\mu m^2 v - 1)^2/(m-1)$, then, using Theorem 3.5, we see that Σ is complete-framed if, and only if, $s = (\mu m^2 v - 1)^2/(\mu m^2 - 1)$, or, equivalently, Δ is complete-framed. ∎

The following corollary is immediate using Theorem 3.2.

COROLLARY 5.3. *If there exists an $(m, r; \mu)$-net and a set of s MOFS of type $F(\mu v m^2; v)$, then there exists a set of rs MOFS of type $F(\mu v m^2; \mu m v)$. The second set of MOFS is complete if, and only if, the first set is complete and the net is complete.*

The above corollary was essentially proved in Mavron [14]. It extended an earlier result of Laywine (see [3, 11]) for the case in which $v = 1$ and μ is a power of m.

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