New inequalities of the Kantorovich type for bounded linear operators in Hilbert spaces

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Abstract

Some new inequalities of the Kantorovich type are established. They hold for larger classes of operators and subsets of complex numbers than considered before in the literature and provide refinements of the classical results in the case when the involved operator satisfies the usual conditions. Several new reverse inequalities for the numerical radius of a bounded linear operator are obtained as well.

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1. Introduction

Let \((H, \langle \cdot, \cdot \rangle)\) be a Hilbert space over the real or complex number field \(K\), \(B(H)\) the \(C^*\)-algebra of all bounded linear operators defined on \(H\) and \(A \in B(H)\). If \(A\) is invertible, then we can define the Kantorovich functional as

\[
K(A; x) := \langle Ax, x \rangle \langle A^{-1}x, x \rangle
\]

for any \(x \in H\), \(\|x\| = 1\).

As pointed out by Greub and Rheinboldt in their seminal paper [22], if \(M > m > 0\) and for the selfadjoint operator \(A\) we have

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\[ MI \geq A \geq mI \] (1.2)
in the partial operator order of \( B(H) \), where \( I \) is the identity operator, then the Kantorovich operator inequality holds true
\[ 1 \leq K(A; x) \leq \frac{(M + m)^2}{4mM} \] (1.3)
for any \( x \in H, \|x\| = 1 \).

An equivalent additive form of this result is incorporated in
\[ 0 \leq K(A; x) - 1 \leq \frac{(M - m)^2}{4mM} \] (1.4)
for any \( x \in H, \|x\| = 1 \).

For results related to the Kantorovich operator inequality we recommend the classical works of Strang [41], Diaz and Metcalf [2], Householder [24], Mond [29], and Mond and Shisha [32]. Other results have been obtained by Mond and Pečarić [30,31], Fujii et al. [11,12], Spain [38], Nakamoto and Nakamura [33], Furuta [15,16], Tsukada and Takahasi [42] and more recently by Yamazaki [45], Furuta and Giga [17], Fujii and Nakamura [13,14] and others.

Due to the important applications of the original Kantorovich inequality for matrices [25] in statistics [26,40,27,36,43,39,46,35,44,28] and numerical analysis [19,20,37,1,18], any new inequality of this type will have a flow of consequences into other areas.

Motivated by interests in both pure and applied mathematics outlined above, we establish in this paper some new inequalities of Kantorovich type. They are shown to hold for larger classes of operators and subsets of complex numbers than considered before in the literature and provide refinements of the classical result in the case when the involved operator \( A \) satisfies the usual condition (1.2). As natural tools in deriving the new results, the recent Grüss type inequalities for vectors in inner products obtained by the author in [3–8] are utilised. In the process, several new reverse inequalities for the numerical radius of a bounded linear operator are derived as well.

2. Some Grüss type inequalities

The following lemmas, that are of interest in their own right, collect some Grüss type inequalities for vectors in inner product spaces obtained earlier by the author

**Lemma 1.** Let \((H, \langle \cdot, \cdot \rangle)\) be an inner product space over the real or complex number field \( \mathbb{K} \), \( u, v, e \in H, \|e\| = 1 \), and \( \alpha, \beta, \gamma, \delta \in \mathbb{K} \) such that
\[ \text{Re} \langle \beta e - u, u - \alpha e \rangle \geq 0, \quad \text{Re} \langle \delta e - v, v - \gamma e \rangle \geq 0 \] (2.1)
or equivalently,
\[ \left\| u - \frac{\alpha + \beta}{2} e \right\| \leq \frac{1}{2} |\beta - \alpha|, \quad \left\| v - \frac{\gamma + \delta}{2} e \right\| \leq \frac{1}{2} |\delta - \gamma|. \] (2.2)

Then
\[ |\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \]
\[ \leq \frac{1}{4} |\beta - \alpha||\delta - \gamma| - \left\{ \left[ \text{Re} \langle \beta e - u, u - \alpha e \rangle \text{Re} \langle \delta e - v, v - \gamma e \rangle \right]^{\frac{1}{2}}, \quad \left\| \langle u, e \rangle - \frac{\alpha + \beta}{2} \right\| \left\| \langle v, e \rangle - \frac{\gamma + \delta}{2} \right\| \right\}. \] (2.3)
The first inequality has been obtained in [4] (see also [10, p. 44]) while the second result was established in [5] (see also [10, p. 90]). They provide refinements of the earlier result from [3] where only the first part of the bound, i.e., \(|\beta - \alpha||\delta - \gamma|\) has been given. Notice that, as pointed out in [5], the upper bounds for the Grüss functional incorporated in (2.3) cannot be compared in general, meaning that one is better than the other depending on appropriate choices of the vectors and scalars involved.

Another result of this type is the following one:

**Lemma 2.** With the assumptions in Lemma 1 and if \(\Re(\beta \bar{\alpha}) > 0, \Re(\delta \bar{\gamma}) > 0\) then

\[
|\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \leq \frac{1}{4} \frac{|\beta - \alpha||\delta - \gamma|}{|\Re(\beta \bar{\alpha})\Re(\delta \bar{\gamma})|^2} |\langle u, e \rangle \langle e, v \rangle|,
\]

(2.4)

The first inequality has been established in [6] (see [10, p. 62]) while the second one can be obtained in a canonical manner from the reverse of the Schwarz inequality given in [7]. The details are omitted.

Finally, another inequality of Grüss type that has been obtained in [8] (see also [10, p. 65]) can be stated as

**Lemma 3.** With the assumptions in Lemma 1 and if \(\beta \neq -\alpha, \delta \neq -\gamma\) then

\[
|\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \leq \frac{1}{4} \frac{|\beta - \alpha||\delta - \gamma|}{|\beta + \alpha||\delta + \gamma|}\left(|\|u\| + |\langle u, e \rangle|)(\|v\| + |\langle v, e \rangle|)\right)^{\frac{1}{2}}.
\]

(2.5)

3. Operator inequalities of Grüss type

For the complex numbers \(\alpha, \beta\) and the bounded linear operator \(A\) we define the following transform

\[C_{\alpha, \beta}(A) := (A^* - \bar{\alpha} I)(\beta I - A),\]

(3.1)

where by \(A^*\) we denote the adjoint of \(A\).

We list some properties of the transform \(C_{\alpha, \beta}(\cdot)\) that are useful in the following:

(i) For any \(\alpha, \beta \in \mathbb{C}\) and \(A \in B(H)\) we have

\[C_{\alpha, \beta}(I) = (1 - \bar{\alpha})(\beta I - 1), \quad C_{\alpha, \alpha}(A) = -(\alpha I - A)^*(\alpha I - A),\]

(3.2)

\[C_{\alpha, \beta}(\gamma A) = |\gamma|^2 C_{\frac{\alpha}{|\gamma|}, \frac{\beta}{|\gamma|}}(A) \quad \text{for each } \gamma \in \mathbb{C} \setminus \{0\},\]

(3.3)

\[[C_{\alpha, \beta}(A)]^* = C_{\beta, \alpha}(A)\]

(3.4)

and

\[C_{\bar{\beta}, \bar{\alpha}}(A^*) - C_{\alpha, \beta}(A) = A^* A - A A^*.\]

(3.5)
(ii) The operator $A \in B(H)$ is normal if and only if $C_{\bar{\beta},\bar{\alpha}}(A^*) = C_{\alpha,\beta}(A)$ for each $\alpha, \beta \in \mathbb{C}$.

(iii) If $A \in B(H)$ is invertible and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, then
\[
(A^{-1})^*C_{\alpha,\beta}(A)A^{-1} = \bar{\alpha}\beta C_{\frac{\alpha}{\bar{\alpha}},\frac{\beta}{\bar{\beta}}}(A^{-1}).
\]

(3.6)

We recall that a bounded linear operator $T$ on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is called accretive if $\text{Re}\langle Ty, y \rangle \geq 0$ for any $y \in H$.

The following simple characterization result is useful in the following:

**Lemma 4.** For $\alpha, \beta \in \mathbb{C}$ and $A \in B(H)$ the following statements are equivalent:

(i) The transform $C_{\alpha,\beta}(A)$ is accretive;

(ii) The transform $C_{\bar{\alpha},\bar{\beta}}(A^*)$ is accretive;

(iii) We have the norm inequality
\[
\left\| A - \frac{\alpha + \beta}{2} I \right\| \leq \frac{1}{2} |\beta - \alpha|.
\]

(3.7)

**Proof.** The proof of the equivalence “(i) $\iff$ (iii)” is obvious by the equality
\[
\text{Re}\langle (A^* - \bar{\alpha}I)(\beta I - A)x, x \rangle = \frac{1}{4} |\beta - \alpha|^2 - \left\| \left( A - \frac{\alpha + \beta}{2} I \right) x \right\|^2
\]
which holds for any $\alpha, \beta \in \mathbb{C}$, $A \in B(H)$ and $x \in H$, $\|x\| = 1$. □

**Remark 1.** In order to give examples of operators $A \in B(H)$ and numbers $\alpha, \beta \in \mathbb{C}$ such that the transform $C_{\alpha,\beta}(A)$ is accretive, it suffices to select a bounded linear operator $T$ and the complex numbers $z, w$ with the property that $\|T - zI\| \leq |w|$ and, by choosing $A = T$, $\alpha = \frac{1}{2}(z + w)$ and $\beta = \frac{1}{2}(z - w)$ we observe that $A$ satisfies (3.7), i.e., $C_{\alpha,\beta}(A)$ is accretive.

For two bounded linear operators $A, B \in B(H)$ and the vector $x \in H$, $\|x\| = 1$ define the functional

\[
G(A, B; x) := \langle Ax, Bx \rangle - \langle Ax, x \rangle \langle x, Bx \rangle.
\]

The following result concerning operator inequalities of Grüss type may be stated:

**Theorem 1.** Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ be such that the transforms $C_{\alpha,\beta}(A), C_{\gamma,\delta}(B)$ are accretive, then
\[
|G(A, B; x)| \leq \frac{1}{4} |\beta - \alpha| |\delta - \gamma|
\]
\[
- \left\| \left[ \text{Re}\langle C_{\alpha,\beta}(A)x, x \rangle \text{Re}\langle C_{\gamma,\delta}(B)x, x \rangle \right] \right\| \left\| \left( A - \frac{\alpha + \beta}{2} I \right) x, x \right\| \left\| \left( B - \frac{\gamma + \delta}{2} I \right) x, x \right\|
\]
\[
\leq \frac{1}{4} |\beta - \alpha| |\delta - \gamma|
\]

(3.9)

for any $x \in H$, $\|x\| = 1$. 

If $\Re(\beta \bar{\alpha}) > 0$, $\Re(\delta \bar{\gamma}) > 0$ then

$$|G(A, B; x)| \leq \begin{cases} 
\frac{1}{4} \frac{|\beta - \sigma||\delta - \gamma|}{|\Re(\beta \bar{\alpha})|^{\frac{1}{2}}} |\langle Ax, x \rangle \langle Bx, x \rangle|, \\
\left| \left( |\alpha + \beta| - 2|\Re(\beta \bar{\alpha})|^{\frac{1}{2}} \right) \left( |\delta \bar{\gamma}| - 2|\Re(\delta \bar{\gamma})|^{\frac{1}{2}} \right) \right|^\frac{1}{2} \\
\times \left| \left( \langle Ax, x \rangle \langle Bx, x \rangle \right) \right|^{\frac{1}{2}}
\end{cases}$$

(3.10)

for any $x \in H$, $\|x\| = 1$.

If $\beta \neq -\alpha$, $\delta \neq -\gamma$ then

$$|G(A, B; x)| \leq \frac{1}{4} \frac{|\beta - \sigma||\delta - \gamma|}{|\beta + \alpha||\delta + \gamma|} \left| \left( \|Ax\| + |\langle Ax, x \rangle| \right) \left( \|Bx\| + |\langle Bx, x \rangle| \right) \right|^{\frac{1}{2}}$$

(3.11)

for any $x \in H$, $\|x\| = 1$.

The proof follows by Lemmas 1–3 on choosing $u = Ax$, $v = Bx$ and $e = x$, $x \in H$, $\|x\| = 1$.

**Remark 2.** In order to give examples of operators $A \in B(H)$ and complex numbers $\alpha, \beta$ for which $C_{\alpha,\beta}(A)$ is accretive and $\Re(\beta \bar{\alpha}) > 0$ it is enough to select in Remark 1 $z, w \in C$ with $|z| > |w| > 0$. This follows from the fact that for $\alpha = \frac{1}{2}(z + w)$ and $\beta = \frac{1}{2}(z - w)$ we have $\Re(\beta \bar{\alpha}) = \frac{1}{4}(|z|^2 - |w|^2)$.

**Remark 3.** We observe that

$$G(A, B^*; x) = \langle B Ax, x \rangle - \langle Ax, x \rangle \langle Bx, x \rangle, \ x \in H, \ \|x\| = 1$$

and since, by Lemma 4 the transform $C_{\alpha,\beta}(A)$ is accretive if and only if $C_{\alpha,\beta}(A^*)$ is accretive, hence in the inequalities (3.9)–(3.11) we can replace $G(A, B; x)$ by $G(A, B^*; x)$ to obtain other Grüss type inequalities that will be used in the sequel.

In some applications, the case $B = A$ in both quantities $G(A, B; x)$ and $G(A, B^*; x)$ may be of interest. For the sake of simplicity, we denote

$$G_1(A; x) := G(A, A; x) = \|Ax\|^2 - |\langle Ax, x \rangle|^2 \geq 0$$

and

$$G_2(A; x) := G(A, A^*; x) = \langle A^2 x, x \rangle - |\langle Ax, x \rangle|^2$$

for $x \in H, \|x\| = 1$. For these quantities, related to the Schwarz’s inequality, we can state the following result which is of interest:

**Corollary 1.** Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{C}$ be such that the transform $C_{\alpha,\beta}(A)$ is accretive, then

$$G_1(A; x) \leq \frac{1}{4} |\beta - \alpha|^2 - \frac{\Re(C_{\alpha,\beta}(A)x, x)}{\left| \left( \left( A - \frac{\alpha + \beta}{2} I \right) x, x \right) \right|^2} \left( \leq \frac{1}{4} |\beta - \alpha|^2 \right)$$

(3.12)

for any $x \in H, \|x\| = 1$. 
If $\text{Re}(\beta\bar{\alpha}) > 0$ then

$$G_1(A; x) \leq \begin{cases} \frac{|\beta - \alpha|^2}{4 \text{Re}(\beta\bar{\alpha})} |(Ax, x)|^2, \\ |\alpha + \beta| - 2[\text{Re}(\beta\bar{\alpha})]^{\frac{1}{2}} |(Ax, x)| \end{cases}$$

for any $x \in H, \|x\| = 1$.

If $\beta \neq -\alpha$ then

$$G_1(A; x) \leq \frac{1}{4} |\beta - \alpha|^2 \left( \|Ax\| + |(Ax, x)| \right)$$

for any $x \in H, \|x\| = 1$.

A similar result holds for $G_2(A; x)$. The details are omitted.

4. Reverse inequalities for the numerical range

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The numerical range of an operator $A$ is the subset of the complex numbers $\mathbb{C}$ given by [21, p. 1] (see also [23]):

$$W(A) = \{ \langle Ax, x \rangle, \ x \in H, \ \|x\| = 1 \}.$$ 

The numerical radius $w(A)$ of an operator $A$ on $H$ is given by [21, p. 8]

$$w(A) = \sup \{ |\lambda|, \lambda \in W(A) \} = \sup \{ |\langle Ax, x \rangle|, \ \|x\| = 1 \}. \quad (4.1)$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$. This norm is equivalent with the operator norm. In fact, the following more precise result holds [21, p. 9]:

**Theorem 2** (Equivalent norm). For any $A \in B(H)$ one has

$$w(A) \leq \|A\| \leq 2w(A). \quad (4.2)$$

The following reverses of the first inequality in (4.2), i.e., upper bounds under appropriate conditions for the bounded linear operator $A$ for the nonnegative difference $\|A\|^2 - w^2(A)$ can be obtained.

**Theorem 3.** Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{C}$ be such that the transform $C_{\alpha, \beta}(A)$ is accretive, then

$$(0 \leq) \ \|A\|^2 - w^2(A) \leq \frac{1}{4} |\beta - \alpha|^2 - \left\{ \frac{\vartheta_1(C_{\alpha, \beta}(A))}{w_1^2 \left( A - \frac{\alpha + \beta}{2}I \right)} \right\} \left( \leq \frac{1}{4} |\beta - \alpha|^2 \right). \quad (4.3)$$

where, for a given operator $B$ we have denoted $\vartheta_1(B) := \inf \|x\|=1 \text{Re}(Bx, x)$ and $w_1(B) := \inf \|x\|=1 |\langle Bx, x \rangle|$.

If $\text{Re}(\beta\bar{\alpha}) > 0$ then

$$(0 \leq) \ \|A\|^2 - w^2(A) \leq \begin{cases} \frac{1}{4} |\beta - \alpha|^2 \ w^2(A), \\ |\alpha + \beta| - 2[\text{Re}(\beta\bar{\alpha})]^{\frac{1}{2}} w(A). \end{cases} \quad (4.4)$$

If $\beta \neq -\alpha$ then

$$(0 \leq) \ \|A\|^2 - w^2(A) \leq \frac{1}{4} \frac{|\beta - \alpha|^2}{|\beta + \alpha|} (\|A\| + w(A)). \quad (4.5)$$
Proof. We give a short proof for the first inequality. The other results follow in a similar manner.

Utilising the inequality (3.12) we can write that
\[
\|Ax\|_2^2 \leq |\langle Ax, x \rangle|^2 + \frac{1}{4} |\beta - \alpha|^2 - \text{Re}\{C_{\alpha,\beta}(A)x, x\}
\]  
(4.6)
for any \(x \in H, \|x\| = 1\). Taking the supremum over \(x \in H, \|x\| = 1\) in (4.6) we deduce the first inequality in (4.3). \(\square\)

Remark 4. An equivalent and perhaps more useful version of (4.5) is the inequality
\[
w(A) \leq \|A\| \leq \frac{1}{4} \cdot \frac{|\beta - \alpha|^2}{|\beta + \alpha|} + w(A),
\]
provided that \(\alpha\) and \(\beta\) satisfy the corresponding conditions mentioned in Theorem 3. Similar statements can be made for the other versions of this inequality presented below.

Corollary 2. If \(A \in B(H)\) and \(M > m > 0\) are such that the transform \(C_{m,M}(A) = (A^* - mI)(MI - A)\) is accretive, then
\[
0 \leq \|A\|^2 - w^2(A) \leq \begin{cases} 
\frac{1}{4} (M - m)^2 - \vartheta_i(C_{m,M}(A)), \\
\frac{1}{4} (M - m)^2 - w^2_i(A - \frac{m+M}{2}I), \\
\frac{1}{4} (M - m)^2 - w^2(A), \\
\left(\sqrt{M} - \sqrt{m}\right)^2 w(A), \\
\frac{1}{4} \left(\frac{M - m}{m+M}\right)^2 \|A\| + w(A). 
\end{cases}
\]  
(4.7)

Remark 5. The inequalities in (4.4) and their consequences for positive \(M\) and \(m\) were obtained previously in [9].

The following result is well known in the literature (see for instance [34]):
\[
w(A^n) \leq w^n(A),
\]
for each positive integer \(n\) and any operator \(A \in B(H)\).

The following reverse inequalities for \(n = 2\), can be stated:

Theorem 4. Let \(A \in B(H)\) and \(\alpha, \beta \in \mathbb{K}\) be so that the transform \(C_{\alpha,\beta}(A)\) is accretive, then
\[
0 \leq w^2(A) - w^2(A^2) \leq \frac{1}{4} |\beta - \alpha|^2 - \left\{ \frac{\vartheta_i(C_{\alpha,\beta}(A))}{w^2_i(A - \frac{\alpha+\beta}{2}I)} \left( \leq \frac{1}{4} |\beta - \alpha|^2 \right) \right\}
\]  
(4.8)
If \(\text{Re}(\beta \bar{\alpha}) > 0\) then
\[
0 \leq w^2(A) - w^2(A^2) \leq \begin{cases} 
\frac{1}{4} \frac{|\beta-\alpha|^2}{\text{Re}(\beta \bar{\alpha})} w^2(A), \\
(|\alpha + \beta| - 2 \left[ \text{Re}(\beta \bar{\alpha}) \right]^\frac{1}{2}) w(A). 
\end{cases}
\]  
(4.9)
If \(\beta \neq -\alpha\) then
\[
0 \leq w^2(A) - w^2(A^2) \leq \frac{1}{4} \frac{|\beta - \alpha|^2}{|\beta + \alpha|} (\|A\| + w(A)).
\]  
(4.10)

Proof. We give a short proof for the first inequality only. The other inequalities can be proved in a similar manner.
Utilising the inequality (3.9) for $B = A^*$, $\gamma = \bar{\alpha}$ and $\delta = \bar{\beta}$ we can write that
\[
|(Ax, x)|^2 - |(A^2x, x)| \leq |(Ax, x) - (Ax, x)|^2 \\
\leq \frac{1}{4} |\beta - \alpha|^2 - [\text{Re}(C_{\alpha, \beta}(A)x, x)\text{Re}(C_{\bar{\alpha}, \bar{\beta}}(A^*x, x))]^{1/2}
\]
for any $x \in H$, $\|x\| = 1$, which implies that
\[
|(Ax, x)|^2 \leq |(A^2x, x)| + \frac{1}{4} |\beta - \alpha|^2 - [\text{Re}(C_{\alpha, \beta}(A)x, x)\text{Re}(C_{\bar{\alpha}, \bar{\beta}}(A^*x, x))]^{1/2}
\]
(4.11) for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$ in (4.11) we deduce the desired inequality in (4.8).

**Remark 6.** If $A \in B(H)$ and $M > m > 0$ are such that the transform $C_{m, M}(A) = (A^* - mI)/(M - A)$ is accretive, then all the inequalities in (4.7) hold true with the left side replaced by the nonnegative quantity $w^2(A) - w(A^2)$.

### 5. New inequalities of Kantorovich type

The following result comprising some inequalities for the Kantorovich functional can be stated:

**Theorem 5.** Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be such that the transform $C_{\alpha, \beta}(A)$ is accretive. If $\text{Re}(\bar{\beta}\bar{\alpha}) > 0$ and the operator $-i\text{Im}(\bar{\beta}\bar{\alpha})C_{\alpha, \beta}(A)$ is accretive, then

\[
|K(A; x) - 1| \\
\leq \begin{cases} 
\frac{1}{4} |\beta - \alpha|^2 - \left[\text{Re}(C_{\alpha, \beta}(A)x, x)\text{Re}\left(\frac{1}{2\beta}\left(A^{-1}\right)x, x\right)\right]^2, \\
\frac{1}{4} |\beta - \alpha|^2 - \left|\left(A - \frac{\alpha + \beta}{2}\right)x, x\right|\left|\left(A^{-1} - \frac{\alpha + \beta}{2\beta\bar{\alpha}}\right)x, x\right|, \\
\frac{1}{4} |\beta - \alpha|^2 \left|\frac{1}{2}\frac{|\beta + \alpha - 2\text{Re}(\bar{\beta}\bar{\alpha})|^2}{|\beta\bar{\alpha}|^2}|K(A; x)| \right|^2, \\
\frac{1}{4} |\beta - \alpha|^2 \left|\frac{1}{2\beta\bar{\alpha}}\left((\|Ax\| + |Ax, x|)(\|A^{-1}x\| + |Ax^{-1}x, x|)\right)\right|^2
\end{cases}
\]

(5.1)

for any $x \in H$, $\|x\| = 1$.

**Proof.** Utilising the identity (3.6), we have for each $x \in H$, $\|x\| = 1$ that
\[
\text{Re}\left(\frac{1}{2\beta}\left(A^{-1}\right)x, x\right) \\
= \frac{1}{|\beta\bar{\alpha}|^2} \text{Re}\left[\bar{\beta}\bar{\alpha}\left((A^{-1})^*C_{\alpha, \beta}(A)A^{-1}x, x\right)\right] \\
= \frac{1}{|\beta\bar{\alpha}|^2} \left[\text{Re}(\bar{\beta}\bar{\alpha}) \cdot \text{Re}\left((A^{-1})^*C_{\alpha, \beta}(A)A^{-1}x, x\right) + \text{Im}(\bar{\beta}\bar{\alpha}) \cdot \text{Im}\left((A^{-1})^*C_{\alpha, \beta}(A)A^{-1}x, x\right)\right]
\]
\[
\begin{align*}
&= \frac{1}{|\beta \alpha|^2} \left[ \text{Re}(\beta \bar{\alpha}) \cdot \text{Re} \left( (A^{-1})^* C_{\alpha, \beta}(A) A^{-1} x, x \right) \\
&\quad + \text{Re} \left( (A^{-1})^* (-i \text{Im}(\beta \bar{\alpha}) C_{\alpha, \beta}(A)) A^{-1} x, x \right) \right] \geq 0,
\end{align*}
\]
showing that the operator \( C_{\frac{1}{\alpha^*}, \frac{1}{\beta^*}}(A^{-1}) \) is also accretive.

Now, on applying Theorem 1 for the difference \( \langle B A x, x \rangle - \langle A x, x \rangle \langle B x, x \rangle \) and for the choices \( B = A^{-1} \) and \( \delta = 1/\beta, \gamma = 1/\alpha \), we get the desired inequality (5.1). The details are omitted. □

**Remark 7.** A sufficient simple condition for the second assumption to hold in the above theorem is that \( \beta \bar{\alpha} \) is a positive real number.

**Remark 8.** The third and the fourth inequalities in (5.1) can be written in the following equivalent forms that perhaps are more useful

\[
\left| K^{-1}(A; x) - 1 \right| \leq \frac{1}{4} \frac{|\beta - \alpha|^2}{\text{Re}(\beta \bar{\alpha})}
\]

and

\[
\left| K^{1/2}(A; x) - K^{-1/2}(A; x) \right| \leq \frac{|\beta + \alpha| - 2[\text{Re}(\beta \bar{\alpha})]^{1/2}}{|\beta \alpha|^{1/2}},
\]

provided that \( \alpha \) and \( \beta \) satisfy the assumptions in Theorem 5. Similar comments apply for the other related results listed below.

However, for practical applications the following even more particular case is of interest:

**Corollary 3.** Let \( A \in B(H) \) and \( M > m > 0 \) are such that the transform \( C_{m, M}(A) = (A^* - mI)(M I - A) \) is accretive. Then

\[
\left| K(A; x) - 1 \right| \leq \begin{cases} 
\frac{1}{4} \frac{(M - m)^2}{m M} & - \left[ \text{Re} \left( C_{m, M}(A)x, x \right) \text{Re} \left( C_{\frac{1}{m}, \frac{1}{M}}((A^*)^{-1})x, x \right) \right]^{1/2}, \\
\frac{1}{4} \frac{(M - m)^2}{m M} & - \left[ |(A - m I)$
\]

\[
\left| (\langle A I \rangle x, x \rangle) + |(A^{-1} - \frac{m + M}{2m M} I)x, x \rangle \right| \right]^{1/2}, \\
\frac{(\sqrt{M} - \sqrt{m})^2}{\sqrt{M} M} |K(A; x)|, \\
\frac{1}{4} \frac{(M - m)^2}{\sqrt{M} M} \left[ (\|x x\| + |(A x, x) |) (\|A^* x\|^{-1} x\| + |(A^{-1} x, x)|) \right]^{1/2},
\end{cases}
\]

for any \( x \in H, \|x\| = 1 \).

Finally, on returning to the original assumptions, we can state the following results which provide refinements for the additive version of the operator Kantorovich inequality (1.4) as well as other similar results that apparently are new:

**Corollary 4.** Let \( A \) be a selfadjoint operator on \( H \) and \( M > m > 0 \) such that \( MI \geq A \geq m I \) in the partial operator order of \( B(H) \). Then
\[0 \leq K(A; x) - 1 \leq \left\{ \begin{array}{l}
\frac{1}{4} \left( \frac{(M-m)^2}{mM} \right) - \left[ \text{Re} \left( \langle C_{m,M}(A)x, x \rangle \right) \right]^2, \\
\frac{1}{4} \left( \frac{(M-m)^2}{mM} \right) - \left| \langle (A - \frac{m+M}{2} I)x, x \rangle \right| \left| \langle (A^{-1} - \frac{m+M}{2mM} I)x, x \rangle \right|, \\
\frac{1}{4} \left( \frac{(M-m)^2}{\sqrt{mM}} \right) \left( \|Ax\| + \langle Ax, x \rangle \right) \left( \|A^{-1}x\| + \langle A^{-1}x, x \rangle \right). \end{array} \right. \] (5.3)

for any \( x \in H, \|x\| = 1. \)

The proof is obvious by Corollary 4 on noticing the fact that \( MI \geq A \geq mI \) for a selfadjoint operator \( A \) implies that \( C_{m,M}(A) = (A^* - mI)(MI - A) \) is accretive. The reverse is not true in general.

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