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Reflection subgroups of finite complex reflection groups

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ABSTRACT

All simple extensions of the reflection subgroups of a finite complex reflection group G are determined up to conjugacy. As a consequence, it is proved that if the rank of G is n and if G can be generated by n reflections, then for every set R of n reflections which generate G, every subset of R generates a parabolic subgroup of G.

1. Introduction

Let V be a complex vector space of dimension n. A *reflection* is a linear transformation of V of finite order whose space of fixed points is a hyperplane [3, Ch. V, §2]. A *complex reflection group* on V is a group generated by reflections. The finite complex reflection groups were determined by Shephard and Todd [15] in 1954; other proofs of the classification can be found in [4,5,12].

Every finite subgroup of $\mathrm{GL}(V)$ preserves a positive definite hermitian form $(-,-)$ on V. Therefore, a finite complex reflection group G is a *unitary* reflection group; that is, G is a group of unitary transformations with respect to a positive definite hermitian form. From now on by *reflection group* we mean a finite unitary reflection group. If r is a reflection, a *root* of r is an eigenvector corresponding to the unique eigenvalue not equal to 1.

A *reflection subgroup* of G is a subgroup generated by reflections. A *parabolic subgroup* is the pointwise stabiliser in G of a subset of V. By a fundamental theorem of Steinberg [16] a parabolic subgroup is a reflection subgroup.

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Definition 1.1. If H is a reflection subgroup of G, a *simple extension* of H is a subgroup K such that $K = \langle H, r\rangle$ for some reflection $r \notin H$. The simple extension K is a *minimal extension* of H if for all reflection subgroups L such that $H \subsetneqq L \subseteq K$ we have $L = K$.

It will be evident from Theorem 3.12 below that not every simple extension is minimal.

The set of one-dimensional subspaces spanned by the roots of the reflections in G is a *line system*. Simple extensions of line systems correspond to simple extensions of reflection subgroups and they were investigated in Chapters 7 and 8 of [12]. But not all possible extensions were determined—only those needed for a proof of the Shephard and Todd classification theorem.

In this paper all conjugacy classes of reflection subgroups of a reflection group and all pairs (H, K) where K is a simple extension of H are determined. As a consequence, if G is a reflection group of rank n and if R is a set of n reflections which generate G, then every subset of R generates a parabolic subgroup. (I thank Professor Gus Lehrer for drawing this problem to my attention.) For the primitive reflection groups the proof depends on calculations carried out using the computer algebra system Magma [2] and tabulated in Section 6 (see Tables 1–19).

Prior work on this subject dealt with special cases. For example, the classification of the parabolic subgroups of an imprimitive reflection group can be derived from the work of Orlik and Solomon [13,14] on arrangements of hyperplanes. In [17] Wang and Shi describe all irreducible reflection subgroups of the imprimitive reflection groups in terms of graphs.

There is an extensive literature on the classification of the reflection subgroups of a Coxeter group beginning with the works of Borel and de Siebenthal [1] and Dynkin [10]. See [6] and [9] for recent results and a history of the finite case. Tables of conjugacy classes of the parabolic subgroups of finite Coxeter groups can be found in the book by Geck and Pfeiffer [11].

2. Notation and preliminaries

Suppose that G is a reflection group acting on V. The *support* of G is the subspace M of V spanned by the roots of the reflections in G. The *rank* of G is the dimension of its support. The orthogonal complement $M^{\perp}$ of M is the space V^G of fixed points of G.

For $v \in V$, let G_v denote the stabiliser of v in G and for a subset $X \subseteq V$, let $G(X)$ denote the pointwise stabiliser of X. If H is a parabolic subgroup of G, then $H = G(U)$ for some subspace U of V. Thus $H \subseteq G(V^H) \subseteq G(U) = H$ and so $H = G(V^H)$.

If H is a reflection subgroup of G the *parabolic closure* of H is the subgroup $G(V^H)$, which is the smallest parabolic subgroup containing H.

If Ω is a set, $\mathrm{Sym}(\Omega)$ is the group of all permutations of Ω; if n is a positive integer, $\mathrm{Sym}(n) = \mathrm{Sym}([n])$, where $[n] = \{1, 2, \dots, n\}$. If a and b are integers, the notation $a \mid b$ means that a divides b.

We write $\lambda \vdash n$ to denote that the sequence $\lambda = (n_1, n_2, \dots, n_d)$ is a *partition* of n; that is, $n_1, n_2, \dots, n_d$ are integers such that $n_1 \geqslant n_2 \geqslant \dots \geqslant n_d > 0$ and $n = n_1 + n_2 + \dots + n_d$.

The following well-known property of a cyclic group is used at several places throughout the paper.

Lemma 2.1. *If G is a cyclic group generated by an element x of order m and if $H_1 = \langle x^{n_1}\rangle$ and $H_2 = \langle x^{n_2}\rangle$, then $|H_1H_2| = \mathrm{lcm}(|H_1|, |H_2|)$ and $H_1H_2 = \langle x^{\gcd(m,n_1,n_2)}\rangle$.*

3. Imprimitive reflection groups

Definition 3.1. A group G acting on a vector space V is *imprimitive* if, for some $k > 1$, V is a direct sum of non-zero subspaces V_i $(1 \leqslant i \leqslant k)$ such that the action of G on V permutes the subspaces $V_1, V_2, \dots, V_k$ among themselves; otherwise G is *primitive*. The set $\Omega = \{V_1, V_2, \dots, V_k\}$ is called a *system of imprimitivity* for G. If G acts transitively on Ω we say that Ω is a *transitive* system of imprimitivity.

The imprimitive complex reflection groups $G(m, p, n)$ introduced by Shephard and Todd can be defined as follows. Let V be a complex vector space of dimension n with a positive definite hermitian form $(-,-)$. Let $\{e_i \mid i \in [n]\}$ be an orthonormal basis for V and let $\boldsymbol{\mu}_m$ be the group of mth roots of unity. Given a function $\theta : [n] \to \boldsymbol{\mu}_m$, the linear transformation which maps e_i to $\theta(i)e_i$ $(1 \leqslant i \leqslant n)$ will be denoted by $\hat{\theta}$.

If $p \mid m$, let $A(m, p, n)$ be the group of all linear transformations $\hat{\theta}$ such that $\prod_{i=1}^{n} \theta(i)^{m/p} = 1$. If $\pi \in \mathrm{Sym}(n)$, define the action of π on V by $\pi(e_i) = e_{\pi(i)}$.

The group $G(m, p, n)$ is the semidirect product of $A(m, p, n)$ by the symmetric group $\mathrm{Sym}(n)$ with the action on V given above. In particular, $G(m, 1, n)$ is the wreath product $\boldsymbol{\mu}_m \wr \mathrm{Sym}(n)$ and $G(m, p, n)$ is a normal subgroup of $G(m, 1, n)$ of index p.

If $n > 1$, the group $G(m, p, n)$ is imprimitive and $\{V_1, V_2, \ldots, V_n\}$ is a transitive system of imprimitivity, where $V_i = \mathbb{C}e_i$. The group $G(m, p, 1) = G(m/p, 1, 1)$ is cyclic of order m/p and therefore we shall require $p = 1$ whenever $n = 1$.

Shephard and Todd [15] proved that every irreducible imprimitive complex reflection group is isomorphic to $G(m, p, n)$ for some m, p and n, where $n > 1$ and $p \mid m$. The group $G(1, 1, n) \simeq \mathrm{Sym}(n)$ is imprimitive in its action on V but for $n \geqslant 5$ its action on its support is primitive.

Definition 3.2. Suppose that $\{V_1, V_2, \ldots, V_n\}$ is a transitive system of imprimitivity for $G(m, p, n)$ and that H is a reflection subgroup of $G(m, p, n)$. The *penumbra* of H is the sum of the subspaces V_i such that $a \in V_i + r(V_i)$, where a is a root of a reflection $r \in H$.

The support M of H is contained in the penumbra because M is spanned by the roots of the reflections in H. The definition of penumbra depends on a choice of system of imprimitivity. However, except for $G(4, 2, 2)$, $G(2, 1, 2) \simeq G(4, 4, 2)$, $G(3, 3, 3)$ and $G(2, 2, 4)$, the group $G(m, p, n)$ has a unique transitive system of imprimitivity (see [12, Theorem 2.16]).

Lemma 3.3. *Let H be a reflection subgroup of $G(m, p, n)$ and suppose that Γ is an orbit of H on a transitive system of imprimitivity $\Omega = \{V_1, V_2, \ldots, V_n\}$ for $G(m, p, n)$ such that the subspaces in $\Omega \setminus \Gamma$ are fixed pointwise by H. Let M and P be the support and penumbra of H and choose the notation so that $\Gamma = \{V_1, V_2, \ldots, V_d\}$. Then $P = V_1 \oplus V_2 \oplus \cdots \oplus V_d$ and the group of permutations induced by H on Γ is* $\mathrm{Sym}(\Gamma)$. *Furthermore, either*

(i) *$P = M$ and $H \simeq G(m', p', d)$, where p' divides m', m' divides m and m'/p' divides m/p; or*
(ii) *$P = V_i \oplus M$ for all $i \in [d]$ and $H \simeq G(1, 1, d) \simeq \mathrm{Sym}(d)$. In this case let $e_1, e_2, \ldots, e_n$ be the orthonormal basis for V such that $V_i = \mathbb{C}e_i$. Then $P \cap M^{\perp}$ is spanned by a vector $u = u_1 + u_2 + \cdots + u_d$, where $u_i = \theta_i e_i$, $\theta_i \in \boldsymbol{\mu}_m$ for all $i \in [d]$ and where $H \simeq \mathrm{Sym}(d)$ is the group of all permutations of $\{u_1, u_2, \ldots, u_d\}$.*

Proof. If $d = 1$, H is cyclic and we have case (i) with $H \simeq G(m', 1, 1)$, where $m' = |H|$. Thus we may suppose that $d > 1$ and hence the set $\{V_1, V_2, \ldots, V_d\}$ is a system of imprimitivity for H.

Then for all $V_i \in \Gamma$, there exists a reflection $r \in H$ with root a such that $V_i \neq r(V_i)$. By [12, Lemma 2.7] the order of r is 2, $a \in V_i + r(V_i)$ and r fixes every element of $\Omega \setminus \{V_i, r(V_i)\}$ pointwise. Therefore $P = \bigoplus_{i=1}^{d} V_i$ and, by [12, Lemma 2.13], H acts on Γ as $\mathrm{Sym}(\Gamma)$.

Suppose that M_1 is a proper H-invariant subspace of P and that $a \in M_1$ is the root of a reflection $r \in H$. If $V_i \subseteq M_1$ for some $i \in [d]$, the H-orbit of V_i would be contained in M_1 and so $P = M_1$, contrary to assumption. Thus there exists $i \in [d]$ such that $V_i \neq r(V_i)$ and $a \in (V_i + r(V_i)) \cap M_1$. Hence $V_i \oplus M_1 = r(V_i) \oplus M_1$ and since $\mathrm{Sym}(\Gamma)$ is at least doubly transitive it follows that $V_i \oplus M_1 = V_j \oplus M_1 = P$ for all $j \in [d]$.

(i) Suppose that $P = M$. If H is reducible, then $M = M_1 \perp M_2$ for some proper H-invariant subspaces M_1 and M_2. If a is a root of a reflection of H, then by [12, Corollary 1.23] $a \in M_1 \cup M_2$. As shown above, if $a \in M_1$, then $V_1 \oplus M_1 = P = M$ and it follows that $\dim M_2 = 1$. Since M_1 cannot contain roots of all the reflections of H the same argument shows that $\dim M_1 = 1$. Therefore $H \simeq G(2, 2, 2)$.

If H is irreducible, it follows from [12, Theorem 2.14] that $H \simeq G(m', p', d)$ for some $m' > 1$ and some divisor p' of m'. In all cases m' is the order of the cyclic group of products rs where $r, s \in H$ are reflections interchanging V_1 and V_2; thus $m' \mid m$. If $p' \neq m'$, the group H contains a reflection of order m'/p' whose root belongs to V_1 and therefore $m'/p' \mid m/p$.

(ii) If $P \neq M$, then M itself is a proper H-invariant subspace of P and it follows that $P = V_i \oplus M$ for all $i \in [d]$. Therefore $\dim(P \cap M^{\perp}) = 1$ and if u is a basis vector of $P \cap M^{\perp}$ we may write $u = u_1 + u_2 + \cdots + u_d$, where $u_i \in V_i$. For $h \in H$ we have $h(u) = \sum_{i=1}^{d} \xi_i u_{\pi(i)}$ for some ξ_i and some permutation $\pi \in \mathrm{Sym}(d)$. But $u = h(u)$ and therefore $\xi_i = 1$ for all i. Thus $H \simeq \mathrm{Sym}(d)$, as claimed.

For $i \in [d]$ we have $u_i = \lambda_i e_i$ for some $\lambda_i \neq 0$. If $h \in H$ is the transposition which interchanges u_1 and u_i then $h(e_1) = \theta_i e_i$ for some $\theta_i \in \boldsymbol{\mu}_m$ and thus $\lambda_i = \theta_i \lambda_1$. Replacing u by $\lambda_1^{-1} u$ completes the proof. □

If $d > 1$, the distinction between cases (i) and (ii) of the previous lemma is almost, but not quite, the distinction between imprimitive and primitive reflection subgroups of $G(m, p, n)$. The exceptions are the groups Sym(3) and Sym(4) which occur in case (ii). These groups are imprimitive on their support and also occur in case (i) as $G(3, 3, 2)$ and $G(2, 2, 3)$.

Corollary 3.4. *The reflection subgroups which are complements to the normal subgroup $A(m, p, n)$ in $G = G(m, p, n)$ are the stabilisers G_v, where $v = e_1 + \theta_2 e_2 + \cdots + \theta_n e_n$ for some $\theta_2, \ldots, \theta_n \in \boldsymbol{\mu}_m$. If $e = e_1 + e_2 + \cdots + e_n$, the stabilisers G_v and G_e are conjugate in $G(m, p, n)$ if and only if $\theta_2 \cdots \theta_n \in \boldsymbol{\mu}_k$, where $k = m/\gcd(n, p)$.*

3.1. Reflection subgroups

Suppose $\Omega = \{V_1, V_2, \ldots, V_n\}$ is a transitive system of imprimitivity for G, where $V_i = \mathbb{C}e_i$ and where $\Lambda = \{e_i \mid i \in [n]\}$ is an orthonormal basis of the space V on which G acts.

Definition 3.5. Call (m', p', n') a *feasible triple* for $G(m, p, n)$ if m', p' and n' are positive integers such that $n' \leqslant n$, p' divides m', m' divides m, and m'/p' divides m/p.

Define a total order on feasible triples by writing $(m_1, p_1, n_1) \geqslant (m_2, p_2, n_2)$ if

(a) $n_1 > n_2$; or
(b) $n_1 = n_2$ and $m_1 > m_2$; or
(c) $n_1 = n_2$, $m_1 = m_2$ and $p_1 \geqslant p_2$.

It follows from Lemma 3.3 that (m', p', n') is feasible if and only if $G(m', p', n')$ is a reflection subgroup of $G(m, p, n)$. The triples which occur in case (i) of the lemma correspond to the subgroups whose support equals their penumbra; triples in case (ii) have $m' = p' = 1$ and correspond to the symmetric groups $\mathrm{Sym}(n')$. We shall say that a feasible triple (m', p', n') is *thick* if $m' > 1$ and that it is *thin* if $m' = p' = 1$.

Definition 3.6. An *augmented partition* for $G(m, p, n)$ is a decreasing sequence $\Delta = [\tau_1, \tau_2, \ldots, \tau_d]$ of feasible triples $\tau_i = (m_i, p_i, n_i)$ such that $\lambda = (n_1, n_2, \ldots, n_d) \vdash n$.

Let $k_0 = 0$ and for $1 \leqslant i \leqslant d$ let $k_i = n_1 + n_2 + \cdots + n_i$, then set $\Lambda_i = \{e_j \mid k_{i-1} < j \leqslant k_i\}$. We say that $(\Lambda_1, \Lambda_2, \ldots, \Lambda_d)$ is the *standard partition* of Λ associated with Δ.

The *standard* reflection subgroup of type Δ is

$$G_\Delta = \prod_{i=1}^{d} G(m_i, p_i, n_i) \tag{3.1}$$

where $G(m_i, p_i, n_i)$ acts on the subspace of V with basis Λ_i. The factors in (3.1) will be called *thick* or *thin* whenever the corresponding triple is thick or thin. The thin factors are the groups $G(1, 1, n_i) \simeq \mathrm{Sym}(\Lambda_i)$ and furthermore if $n_i = 1$, this factor is trivial and may be omitted from (3.1).

Given $\alpha \in \boldsymbol{\mu}_m$, define $\theta : [n] \to \boldsymbol{\mu}_m$ by $\theta(1) = \alpha$ and $\theta(i) = 1$ for $i > 1$, then put

$$G_\Delta^\alpha = \hat{\theta} G_\Delta \hat{\theta}^{-1}. \tag{3.2}$$

Theorem 3.7. *If H is a reflection subgroup of $G(m,p,n)$, then there is an augmented partition Δ and an element $\alpha \in \boldsymbol{\mu}_m$ such that H is conjugate to G_Δ^α.*

Proof. Let $\Omega = \{V_1, V_2, \ldots, V_n\}$ be a transitive system of imprimitivity for $G(m,p,n)$ and let $\Omega_1, \Omega_2, \ldots, \Omega_d$ be the orbits of H on Ω.

For $i \in [d]$ let P_i be the (direct) sum of the subspaces in Ω_i, let H_i be the subgroup of H generated by the reflections whose roots belong to P_i and let M_i be the support of H_i. For $i \neq j$, the elements of H_i fix every vector in P_j and $H = H_1 \times H_2 \times \cdots \times H_d$. (If no reflections of H belong to P_i, then H_i is the trivial subgroup and M_i is the zero subspace.)

Since H_i is transitive on Ω_i, if $H_i \neq 1$, the penumbra of H_i is P_i. For $i \in [d]$ we set $n_i = |\Omega_i|$. It follows from Lemma 3.3 that for all i, either

(i) $P_i = M_i$ and $H_i \simeq G(m_i, p_i, n_i)$ for some feasible triple (m_i, p_i, n_i), or
(ii) $P_i \neq M_i$ and $H_i \simeq G(1,1,n_i) \simeq \mathrm{Sym}(\Omega_i)$.

By conjugating H by an element of $\mathrm{Sym}(n)$ we may suppose that $(n_1, n_2, \ldots, n_d) \vdash n$ and that $\Delta = [(m_1, p_1, n_1), \ldots, (m_d, p_d, n_d)]$ is an augmented partition for $G(m,p,n)$.

The group $G(m,p,n)$ is normalised by $A(m,1,n)$ and so, by Lemma 3.3, there exists $\hat\theta \in A(m,1,n)$ such that $\hat\theta H \hat\theta^{-1} = G_\Delta$. Modify the definition of θ by choosing $\theta(1)$ so that $\hat\theta \in A(m,p,n)$. This choice of θ shows that H is conjugate in $G(m,p,n)$ to G_Δ^α for some $\alpha \in \boldsymbol{\mu}_m$. □

Corollary 3.8. *If H is a reflection subgroup of the symmetric group $\mathrm{Sym}(n)$, then H is conjugate to $\prod_{i=1}^d \mathrm{Sym}(n_i)$, where $(n_1, \ldots, n_d) \vdash n$. In particular, every reflection subgroup is parabolic.*

Proof. We have $\mathrm{Sym}(n) \simeq G(1,1,n)$ and it follows from the theorem that H is conjugate to G_Δ, where the feasible triples of Δ have the form $(1,1,n_i)$ for $i = 1, 2, \ldots, d$. □

Theorem 3.9. *Suppose that $\Delta = [(m_1, p_1, n_1), \ldots, (m_d, p_d, n_d)]$ is an augmented partition for $G(m,p,n)$. Then for $\alpha, \beta \in \boldsymbol{\mu}_m$, the groups G_Δ^α and G_Δ^β are conjugate in $G(m,p,n)$ if and only if $\alpha\beta^{-1} \in \boldsymbol{\mu}_k$ where*

$$k = m/\gcd(p, n_1, n_2, \ldots, n_d, m/m_1, m/m_2, \ldots, m/m_d).$$

In particular, if $m_i = m$ for some $i \leqslant d$, there is a single conjugacy class of reflection subgroups of type Δ.

Proof. Without loss of generality we may suppose that $\beta = 1$. As in Definition 3.6, for $i \in [d]$, let $k_i = n_1 + n_2 + \cdots + n_i$ and set $k_0 = 0$.

Suppose that $\hat\theta \in A(m,p,n)$ conjugates G_Δ^α to G_Δ. For $i \in [d]$ let $\xi_i = \theta(k_i)$. Then there are elements γ_j such that for $k_{i-1} < j \leqslant k_i$ we have $\gamma_j \in \boldsymbol{\mu}_{m_i}$ and

(i) $\theta(j) = \gamma_j \xi_i$ for $k_{i-1} < j \leqslant k_i$ and $i > 1$;
(ii) $\theta(j) = \alpha^{-1}\gamma_j \xi_1$ for $1 \leqslant j \leqslant k_1$.

Therefore, for $i \in [d]$, there are elements $\delta_i \in \boldsymbol{\mu}_{m_i}$ such that the order of $\alpha^{-1}(\xi_1^{n_1}\delta_1)(\xi_2^{n_2}\delta_2)\cdots(\xi_d^{n_d}\delta_d)$ divides m/p. That is, $\alpha \in \boldsymbol{\mu}_k$, where $k = m/\gcd(p, n_1, n_2, \ldots, n_d, m/m_1, m/m_2, \ldots, m/m_d)$.

Conversely, if $\alpha \in \boldsymbol{\mu}_k$, there exist $\delta_i \in \boldsymbol{\mu}_{m_i}$ and $\xi_i \in \boldsymbol{\mu}_m$ such that the order of $\alpha^{-1}(\xi_1^{n_1}\delta_1)(\xi_2^{n_2}\delta_2)\cdots(\xi_d^{n_d}\delta_d)$ divides m/p. Using these elements and the fact that for all i the group $A(m_i, 1, n_i)$ normalises $G(m_i, p_i, n_i)$ we may construct an element $\hat\theta \in A(m,p,n)$ which conjugates G_Δ^α to G_Δ. □

Example 3.10. The group $G = G(2,2,n)$ is the Weyl group of type D_n and if $\Delta = [(1,1,n)]$, we have $G_\Delta^1 \simeq G_\Delta^{-1} \simeq \mathrm{Sym}(n)$. According to Theorem 3.9, the reflection subgroups G_Δ^1 and G_Δ^{-1} are conjugate if and only if n is odd. For all n there is a single conjugacy class of reflection subgroups $G_{[(2,2,n-1),(1,1,1)]}$ of type D_{n-1}. When $n = 4$ we have $G(2,2,3) \simeq \mathrm{Sym}(4)$ and the three conjugacy classes of reflection subgroups isomorphic to Sym(4) are fused by the triality automorphism of D_4. Similarly, there are three conjugacy classes of subgroups isomorphic to $G(2,2,2) \simeq \mathrm{Sym}(2) \times \mathrm{Sym}(2)$.

3.2. Parabolic subgroups

Theorem 3.11. *Retaining the notation of the previous section, if H is a parabolic subgroup of $G(m,p,n)$ then H is conjugate to either*

(i) *a reflection subgroup G_Δ, where Δ has exactly one thick feasible triple (m,p,n_0) for some n_0, or*
(ii) *a reflection subgroup G_Δ^α for some $\alpha \in \boldsymbol{\mu}_m$, where all the feasible triples of Δ are thin.*

Proof. Let H be a parabolic subgroup of $G(m,p,n)$. By Steinberg's theorem H is a reflection subgroup and, up to conjugacy in $G(m,1,n)$, we may suppose that $H = G_\Delta$. Let $(\Lambda_1, \Lambda_2, \dots, \Lambda_d)$ be the standard partition of $\{e_1, e_2, \dots, e_n\}$ associated with Δ and let T be the set of indices i such that $(m_i, p_i, n_i) \in \Delta$ is thin.

For $i \in T$, the elements $E_i = \sum_{e \in \Lambda_i} e$ form a basis for V^H. Therefore, if $v = \sum_{i \in T} iE_i$, the parabolic closure of H is G_v. If Δ contains at least one thick triple, then $G_v = G_\Gamma$, where Γ is the augmented partition consisting of the thin triples of Δ and a single thick triple (m,p,n_0), where n_0 is the sum of the n_i such that (m_i, p_i, n_i) is thick. In this case it follows from Theorem 3.9 that G_Γ is conjugate to G_Γ^α for all $\alpha \in \boldsymbol{\mu}_m$.

If there are no thick triples in Δ, then $G_v = G_\Delta$. □

The well-known fact that the partially ordered set of parabolic subgroups of $G(m,1,n)$ is isomorphic to the Dowling lattice $Q(\boldsymbol{\mu}_m)$ introduced in [7,8] is a consequence of this theorem and Lemma 3.3.

The conjugacy classes of parabolic subgroups of the Coxeter groups of types A_n, B_n and D_n are described in Propositions 2.3.8, 2.3.10 and 2.3.13 of [11]. These are the groups $\mathrm{Sym}(n+1) \simeq G(1,1,n+1)$, $G(2,1,n)$ and $G(2,2,n)$ in the notation of Shephard and Todd.

3.3. Simple extensions

The following theorem describes, up to conjugacy, all simple extensions (see Definition 1.1) of the reflection subgroups of $G(m,p,n)$.

Theorem 3.12. *Suppose that H is the reflection subgroup G_Δ of $G(m,p,n)$, as defined in (3.1), where $\Delta = [(m_1,p_1,n_1), \dots, (m_d,p_d,n_d)]$. Let $(\Lambda_1, \Lambda_2, \dots, \Lambda_d)$ be the standard partition of $\Lambda = \{e_1, e_2, \dots, e_n\}$ associated with Δ. If $r \notin H$ is a reflection with root a, then $\langle H, r\rangle = gG_\Gamma g^{-1}$ for some Γ and some element g in the centraliser of H in $A(m,1,n)$. The augmented partition Γ is obtained from Δ in one of the following ways.*

(i) *$a \in V_i$ for some i and the order of r is k, where $k \mid m/p$. In this case we may take $g = 1$. If $e_i \in \Lambda_j$, the triple (m_j, p_j, n_j) is replaced by (m', p', n_j), where*

$$m' = \mathrm{lcm}(m_j, k) \quad \textit{and} \quad p' = \frac{\gcd(m_j, kp_j)}{\gcd(m_j, k)}.$$

In particular, if (m_j, p_j, n_j) is thin, the factor $\mathrm{Sym}(n_j)$ is replaced by $G(k,1,n_j)$.

(ii) *$a \in V_i + V_j$ for some $i \neq j$, the order of r is 2 and $r(e_i) = \xi e_j$ for some $\xi \in \boldsymbol{\mu}_k$.*
  (a) *If $e_i, e_j \in \Lambda_h$, the triple (m_h, p_h, n_h) is replaced by (m', p', n_h), where $m' = \mathrm{lcm}(m_h, k)$ and $p' = m'p_h/m_h$. In this case we may take $g = 1$.*

(b) *If $e_i \in \Lambda_h$ and $e_j \in \Lambda_\ell$ where $h \neq \ell$, the triples (m_h, p_h, n_h) and (m_ℓ, p_ℓ, n_ℓ) in Δ are replaced by a single triple (m', p', n'), where $m' = \operatorname{lcm}(m_h, m_\ell)$, $p' = m'/\operatorname{lcm}(m_h/p_h, m_\ell/p_\ell)$ and $n' = n_h + n_\ell$.*

Proof. In the group $G(m, p, n)$, if $n > 1$, the integer m is characterised as the order of the cyclic subgroup whose elements are rs, where r and s are reflections which interchange V_1 and V_2. Similarly, m/p is the order of the cyclic group generated by the reflections whose roots lie in V_1.

The theorem is the result of explicit calculations using the observation of the previous paragraph coupled with Lemma 2.1. In cases (i) and (ii)(a) this is straightforward.

In case (ii)(b), let $\hat{\theta}$ be the element of $A(m, 1, n)$ which fixes each element of Λ_s for $s \neq j$ and which multiplies each element of Λ_j by ξ. Then $\hat{\theta}$ centralises H and conjugates r to the reflection which interchanges e_i and e_j. Thus $n' = n_h + n_\ell$ and the values of m' and p' follow from Lemma 2.1. □

4. Simple extensions and parabolic subgroups

As an application of the results of the previous section, we have the following characterisation of parabolic subgroups.

Theorem 4.1. *Suppose that H is a reflection subgroup of the finite unitary reflection group G and that K is a simple extension of H. If K is parabolic and the rank of K is greater than the rank of H, then H is parabolic.*

Proof. We may suppose that G is irreducible and that $K = \langle H, r\rangle$, where r is a reflection.

If $G = G(m, p, n)$ and if H is not parabolic, then not all factors of H are thin. If H has more than one thick factor, these factors must be contained in a single thick factor of K. But then, from Theorem 3.12(ii)(b), $\operatorname{rank}(K) = \operatorname{rank}(H)$, which is a contradiction.

Thus we may suppose that H has a single thick factor $G(m', p', n')$ with support M and either $m' \neq m$ or $p' \neq p$. Let a be a root of r. Since $\operatorname{rank}(K) > \operatorname{rank}(H)$ we have $a \notin M$ and since K is parabolic, $a \notin M^\perp$. Thus for some $V_j \subseteq M$ we have $V_j \neq r(V_j)$ and hence r is a reflection of order 2. It follows from Theorems 3.11 and 3.12 that K cannot be a parabolic subgroup. This is a contradiction and therefore we may suppose that G is primitive.

We have used the computer algebra system Magma [2] to obtain a case-by-case description of the conjugacy classes of simple extensions of all reflection subgroups of the 15 primitive reflection groups G_k $(23 \leqslant k \leqslant 37)$ and the results can be found in Section 6. An inspection of the tables shows that if H is a non-parabolic reflection subgroup, either H has no parabolic simple extension or else the only parabolic simple extension of H is its parabolic closure.

If the rank of G is 2, then G is one of the groups G_k for $4 \leqslant k \leqslant 22$. If H is a non-parabolic subgroup of rank 1, then H is generated by the square of a reflection of order 4. Thus from [12, Table D.1], G is G_8, G_9, G_{10} or G_{11}. The only rank 2 parabolic subgroup is G itself and it can be seen from [12, §6.3] that G is not a simple extension of H. This completes the proof. □

Theorem 4.2. *If G is a finite reflection group of rank n and if $R = \{r_1, r_2, \dots, r_n\}$ is a set of n reflections which generate G, then for any subset S of R, the subgroup generated by S is a parabolic subgroup of G.*

Proof. For $1 \leqslant i \leqslant n$ let $H_i = \langle r_1, r_2, \dots, r_i\rangle$, let M_i be the support of H_i and let $d_i = \dim M_i$. Then $1 = d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n = n$. But $d_{i+1} \leqslant d_i + 1$ and therefore $d_i = i$ for all i.

If there is a subset of R which generates a non-parabolic subgroup, we may reorder the r_i if necessary and find a subgroup H_k which is not parabolic but such that H_{k+1} is parabolic. This contradiction to Theorem 4.1 completes the proof. □

5. Reading the tables

The tables in Section 6 were constructed with the assistance of approximately 300 lines of Magma code. The code and accompanying documentation is available at http://www.maths.usyd.edu.au/u/don/details.html#programs.

The conjugacy classes of reflection subgroups and their simple extensions are calculated by an iterative process which begins with the conjugacy classes of subgroups of rank one.

Cohen [4] introduced a notation for primitive complex reflection groups of rank at least 3 which extends the standard Killing–Cartan notation for Coxeter groups. In this notation the complex reflection groups which are not Coxeter groups are labelled $J_3^{(4)}$, $J_3^{(5)}$, K_5, K_6, L_3, L_4, M_3, N_4 and EN_4. In the tables which follow the labels for the conjugacy classes of reflection subgroups use this notation except that, as in [12], O_4 is used instead of EN_4. The captions on the tables use both the Cohen and the Shephard and Todd naming schemes.

A reflection subgroup which is the direct product of irreducible reflection groups of types $T_1, T_2, \ldots, T_k$ will be labelled $T_1 + T_2 + \cdots + T_k$ and if $T_i = T$ for all i we denote the group by kT.

For the imprimitive reflection subgroups which occur in the tables we use the notation introduced in [12, section 7.5] rather than the Shephard and Todd notation $G(m, p, n)$. That is, $B_n^{(2p)}$ denotes the group $G(2p, p, n)$ and $D_n^{(p)}$ denotes the group $G(p, p, n)$. For consistency with the Killing–Cartan names we write B_n instead of $B_n^{(2)}$ and D_n instead of $D_n^{(2)}$. Similarly A_{n-1} denotes the symmetric group $\mathrm{Sym}(n) \simeq G(1, 1, n)$. However, we use $D_2^{(m)}$ rather than $I_2(m)$ to denote the dihedral group of order $2m$.

For small values of the parameters there are isomorphisms between the groups: $A_2 \simeq D_2^{(3)}$, $A_3 \simeq D_3$ and $B_2 \simeq D_2^{(4)}$. The tables use the first named symbol for these groups. The cyclic groups of order 2 and 3 are denoted by A_1 and L_1 respectively, and L_2 denotes the Shephard and Todd group G_4.

If there is more than one conjugacy class of reflection subgroups of type T we label the conjugacy classes $T.1$, $T.2$, and so on. There is no significance to the order in which these indices occur.

For those (conjugacy classes of) reflection subgroups H whose parabolic closure is a simple extension of H we place the parabolic closure first in the list of simple extensions and use a bold font.

The conjugacy classes of parabolic subgroups are labelled with the symbol $\wp$.

6. The tables

Tables of conjugacy classes of the reflection subgroups of the Coxeter groups of types E_6, E_7, E_8, F_4, H_3 and H_4 can also be found in [6].

The data in Table 11 below corrects an error in [12, Table D.4], where $D_3^{(3)} + A_2$ is incorrectly listed as a subsystem of K_5.

Table 1
Reflection subgroup classes of $G_{23} = H_3$.

	Class	Simple extensions
$\wp$	A_1	$D_2^{(5)}$, A_2, $2A_1$
$\wp$	$2A_1$	H_3, $3A_1$
$\wp$	A_2	H_3
$\wp$	$D_2^{(5)}$	H_3
	$3A_1$	$\mathbf{H}_3$

Table 2
Reflection subgroup classes of $G_{24} = J_3^{(4)}$.

	Class	Simple extensions
$\wp$	A_1	B_2, A_2, $2A_1.1$, $2A_1.2$
	$2A_1.1$	$\mathbf{B}_2$, $B_3.1$, $A_1 + B_2$, $A_3.1$, $3A_1.1$
	$2A_1.2$	$\mathbf{B}_2$, $B_3.2$, $A_1 + B_2$, $A_3.2$, $3A_1.2$
$\wp$	A_2	$J_3^{(4)}$, $B_3.1$, $B_3.2$, $A_3.1$, $A_3.2$
$\wp$	B_2	$J_3^{(4)}$, $B_3.1$, $B_3.2$, $A_1 + B_2$
	$3A_1.1$	$B_3.1$, $A_1 + B_2$
	$3A_1.2$	$B_3.2$, $A_1 + B_2$
	$A_1 + B_2$	$\mathbf{J}_3^{(4)}$, $B_3.1$, $B_3.2$
	$A_3.1$	$\mathbf{J}_3^{(4)}$, $B_3.1$
	$A_3.2$	$\mathbf{J}_3^{(4)}$, $B_3.2$
	$B_3.1$	$\mathbf{J}_3^{(4)}$
	$B_3.2$	$\mathbf{J}_3^{(4)}$

Table 3
Reflection subgroup classes of $G_{25} = L_3$.

	Class	Simple extensions
$\wp$	L_1	$L_2,\ 2L_1$
$\wp$	$2L_1$	$L_3,\ 3L_1$
$\wp$	L_2	L_3
	$3L_1$	$\mathbf{L}_3$

Table 4
Reflection subgroup classes of $G_{26} = M_3$.

	Class	Simple extensions
$\wp$	L_1	$L_2,\ 2L_1,\ B_2^{(3)},\ A_1 + L_1$
$\wp$	A_1	$B_2^{(3)},\ A_1 + L_1,\ A_2$
	$2L_1$	$\mathbf{B}_2^{(3)},\ L_3,\ B_2^{(3)} + L_1,\ 3L_1$
	A_2	$\mathbf{B}_2^{(3)},\ B_3^{(3)},\ D_3^{(3)},\ A_2 + L_1$
$\wp$	L_2	$M_3,\ L_3,\ A_1 + L_2$
$\wp$	$B_2^{(3)}$	$M_3,\ B_3^{(3)},\ B_2^{(3)} + L_1$
$\wp$	$A_1 + L_1$	$M_3,\ B_3^{(3)},\ B_2^{(3)} + L_1,\ A_1 + L_2,\ A_2 + L_1$
	$D_3^{(3)}$	$B_3^{(3)}$
	$3L_1$	$L_3,\ B_2^{(3)} + L_1$
	$B_2^{(3)} + L_1$	$\mathbf{M}_3,\ B_3^{(3)}$
	$A_2 + L_1$	$\mathbf{M}_3,\ B_3^{(3)},\ B_2^{(3)} + L_1$
	$B_3^{(3)}$	$\mathbf{M}_3$
	L_3	$\mathbf{M}_3$
	$A_1 + L_2$	$\mathbf{M}_3$

Table 5
Reflection subgroup classes of $G_{27} = J_3^{(5)}$.

	Class	Simple extensions
$\wp$	A_1	$B_2,\ D_2^{(5)},\ A_2.1,\ A_2.2,\ 2A_1.1,\ 2A_1.2$
	$2A_1.1$	$\mathbf{B}_2,\ H_3.1,\ B_3.1,\ A_1 + B_2,\ A_3.1,\ 3A_1.1$
	$2A_1.2$	$\mathbf{B}_2,\ H_3.2,\ B_3.2,\ A_1 + B_2,\ A_3.2,\ 3A_1.2$
$\wp$	$A_2.1$	$J_3^{(5)},\ H_3.2,\ B_3.1,\ D_3^{(3)},\ A_3.1$
$\wp$	$A_2.2$	$J_3^{(5)},\ H_3.1,\ B_3.2,\ D_3^{(3)},\ A_3.2$
$\wp$	$D_2^{(5)}$	$J_3^{(5)},\ H_3.1,\ H_3.2$
$\wp$	B_2	$J_3^{(5)},\ B_3.1,\ B_3.2,\ A_1 + B_2$
	$3A_1.1$	$H_3.1,\ B_3.1,\ A_1 + B_2$
	$3A_1.2$	$H_3.2,\ B_3.2,\ A_1 + B_2$
	$A_3.1$	$\mathbf{J}_3^{(5)},\ B_3.1$
	$A_3.2$	$\mathbf{J}_3^{(5)},\ B_3.2$
	$A_1 + B_2$	$\mathbf{J}_3^{(5)},\ B_3.1,\ B_3.2$
	$H_3.1$	$\mathbf{J}_3^{(5)}$
	$H_3.2$	$\mathbf{J}_3^{(5)}$
	$B_3.1$	$\mathbf{J}_3^{(5)}$
	$B_3.2$	$\mathbf{J}_3^{(5)}$
	$D_3^{(3)}$	$\mathbf{J}_3^{(5)}$

Table 6
Reflection subgroup classes of $G_{28} = F_4$.

	Class	Simple extensions
$\wp$	$A_1.1$	$B_2,\ A_2.1,\ 2A_1.1,\ 2A_1.3$
$\wp$	$A_1.2$	$B_2,\ A_2.2,\ 2A_1.2,\ 2A_1.3$
	$2A_1.1$	$\mathbf{B}_2,\ (A_1+B_2).2,\ A_3.1,\ 3A_1.1,\ 3A_1.2$
	$2A_1.2$	$\mathbf{B}_2,\ (A_1+B_2).1,\ A_3.2,\ 3A_1.3,\ 3A_1.4$
$\wp$	$2A_1.3$	$B_3.1,\ B_3.2,\ (A_1+B_2).2,\ (A_1+B_2).1,$ $(A_1+A_2).1,\ (A_1+A_2).2,\ 3A_1.2,\ 3A_1.3$
$\wp$	$A_2.1$	$B_3.1,\ A_3.1,\ (A_1+A_2).1$
$\wp$	$A_2.2$	$B_3.2,\ A_3.2,\ (A_1+A_2).2$
$\wp$	B_2	$B_3.1,\ B_3.2,\ (A_1+B_2).1,\ (A_1+B_2).2$
	$A_3.1$	$\mathbf{B}_3.1,\ B_4.1,\ D_4.1,\ (A_1+A_3).1$
	$A_3.2$	$\mathbf{B}_3.2,\ B_4.2,\ D_4.2,\ (A_1+A_3).2$
	$3A_1.1$	$(\mathbf{A}_1+\mathbf{B}_2).2,\ (2A_1+B_2).1,\ D_4.1,\ 4A_1.1$
	$3A_1.2$	$\mathbf{B}_3.1,\ (A_1+B_3).2,\ (A_1+B_2).1,$ $(2A_1+B_2).1,\ (A_1+A_3).1,\ 4A_1.2$
	$3A_1.3$	$\mathbf{B}_3.2,\ (A_1+B_3).1,\ (A_1+B_2).2,$ $(2A_1+B_2).2,\ (A_1+A_3).2,\ 4A_1.2$
	$3A_1.4$	$(\mathbf{A}_1+\mathbf{B}_2).1,\ (2A_1+B_2).2,\ D_4.2,\ 4A_1.3$
	$(A_1+B_2).1$	$\mathbf{B}_3.1,\ B_4.2,\ 2B_2,\ (A_1+B_3).1,\ (2A_1+B_2).2$
	$(A_1+B_2).2$	$\mathbf{B}_3.2,\ B_4.1,\ 2B_2,\ (A_1+B_3).2,\ (2A_1+B_2).1$
$\wp$	$(A_1+A_2).1$	$F_4,\ B_4.1,\ (A_1+B_3).1,\ 2A_2,\ (A_1+A_3).1$
$\wp$	$(A_1+A_2).2$	$F_4,\ B_4.2,\ (A_1+B_3).2,\ 2A_2,\ (A_1+A_3).2$
$\wp$	$B_3.1$	$F_4,\ B_4.1,\ (A_1+B_3).1$
$\wp$	$B_3.2$	$F_4,\ B_4.2,\ (A_1+B_3).2$
	$4A_1.1$	$D_4.1,\ (2A_1+B_2).1$
	$4A_1.2$	$(2A_1+B_2).1,\ (2A_1+B_2).2,$ $(A_1+B_3).1,\ (A_1+B_3).2$
	$4A_1.3$	$D_4.2,\ (2A_1+B_2).2$
	$D_4.1$	$B_4.1$
	$D_4.2$	$B_4.2$
	$(2A_1+B_2).1$	$2B_2,\ B_4.1,\ (A_1+B_3).2$
	$(2A_1+B_2).2$	$2B_2,\ B_4.2,\ (A_1+B_3).1$
	$2B_2$	$B_4.1,\ B_4.2$
	$(A_1+A_3).1$	$\mathbf{F}_4,\ B_4.1,\ (A_1+B_3).1$
	$(A_1+A_3).2$	$\mathbf{F}_4,\ B_4.2,\ (A_1+B_3).2$
	$(A_1+B_3).1$	$\mathbf{F}_4,\ B_4.1$
	$(A_1+B_3).2$	$\mathbf{F}_4,\ B_4.2$
	$B_4.1$	$\mathbf{F}_4$
	$B_4.2$	$\mathbf{F}_4$
	$2A_2$	$\mathbf{F}_4$

Table 7
Reflection subgroup classes of $G_{29} = N_4$.

	Class	Simple extensions
$\wp$	A_1	$B_2,\ A_2,\ 2A_1.1,\ 2A_1.2$
	$2A_1.1$	$\mathbf{B}_2,\ A_1+B_2,\ A_3.1,\ A_3.4,\ 3A_1.1,\ 3A_1.2$
$\wp$	$2A_1.2$	$B_3,\ A_1+B_2,\ A_1+A_2,\ 3A_1.2,\ A_3.2,\ A_3.3$
$\wp$	A_2	$B_3,\ D_3^{(4)},\ A_1+A_2,\ A_3.1,\ A_3.2,\ A_3.3,\ A_3.4$
$\wp$	B_2	$B_3,\ D_3^{(4)},\ A_1+B_2$
	$3A_1.1$	$A_1+B_2,\ 2A_1+B_2,\ D_4.1,\ 4A_1.1$
	$3A_1.2$	$\mathbf{B}_3,\ A_1+B_2,\ A_1+B_3,\ 2A_1+B_2,\ D_4.2,\ 4A_1.2,\ A_1+A_3$
	A_1+B_2	$\mathbf{B}_3,\ 2B_2,\ B_4,\ A_1+B_3,\ 2A_1+B_2,\ D_4^{(4)}$
	$A_3.1$	$\mathbf{B}_3,\ D_4.1,\ D_4^{(4)},\ A_1+A_3$
	$A_3.4$	$\mathbf{D}_3^{(4)},\ B_4,\ D_4.1,\ D_4.2,\ D_4^{(4)}$
$\wp$	$A_3.2$	$N_4,\ D_4.2,\ D_4^{(4)},\ A_4.1$
$\wp$	$A_3.3$	$N_4,\ D_4.2,\ D_4^{(4)},\ A_4.2$
$\wp$	A_1+A_2	$N_4,\ B_4,\ A_1+B_3,\ A_1+A_3,\ A_4.1,\ A_4.2$
$\wp$	B_3	$N_4,\ B_4,\ A_1+B_3$
$\wp$	$D_3^{(4)}$	$N_4,\ D_4^{(4)}$
	$4A_1.1$	$D_4.1,\ 2A_1+B_2$
	$4A_1.2$	$D_4.2,\ 2A_1+B_2,\ A_1+B_3$
	$2A_1+B_2$	$B_4,\ A_1+B_3,\ D_4^{(4)},\ 2B_2$
	$2B_2$	$B_4,\ D_4^{(4)}$
	$D_4.1$	$B_4,\ D_4^{(4)}$
	$D_4.2$	$\mathbf{N}_4,\ D_4^{(4)}$
	A_1+A_3	$\mathbf{N}_4,\ A_1+B_3,\ B_4$
	A_1+B_3	$\mathbf{N}_4,\ B_4$
	$A_4.1$	$\mathbf{N}_4$
	$A_4.2$	$\mathbf{N}_4$
	B_4	$\mathbf{N}_4$
	$D_4^{(4)}$	$\mathbf{N}_4$

Table 8
Reflection subgroup classes of $G_{30} = H_4$.

	Class	Simple extensions
$\wp$	A_1	$D_2^{(5)},\ A_2,\ 2A_1$
$\wp$	$2A_1$	$H_3,\ A_1+D_2^{(5)},\ A_1+A_2,\ A_3,\ 3A_1$
$\wp$	A_2	$H_3,\ A_1+A_2,\ A_3$
$\wp$	$D_2^{(5)}$	$H_3,\ A_1+D_2^{(5)}$
	$3A_1$	$\mathbf{H}_3,\ A_1+H_3,\ D_4,\ 4A_1$
$\wp$	A_3	$H_4,\ D_4,\ A_4$
$\wp$	A_1+A_2	$H_4,\ A_1+H_3,\ A_4,\ 2A_2$
$\wp$	$A_1+D_2^{(5)}$	$H_4,\ A_1+H_3,\ 2D_2^{(5)}$
$\wp$	H_3	$H_4,\ A_1+H_3$
	$4A_1$	$A_1+H_3,\ D_4$
	A_1+H_3	$\mathbf{H}_4$
	D_4	$\mathbf{H}_4$
	$2D_2^{(5)}$	$\mathbf{H}_4$
	A_4	$\mathbf{H}_4$
	$2A_2$	$\mathbf{H}_4$

Table 9
Reflection subgroup classes of $G_{31} = O_4$.

Class	Simple extensions
℘ A_1	$B_2,\ A_2,\ 2A_1.1,\ 2A_1.2$
$2A_1.2$	$B_2,\ (A_1+B_2).1,\ A_3.2,\ 3A_1.1,\ 3A_1.2$
B_2	$\mathbf{B}_2^{(4)},\ B_3,\ D_3^{(4)},\ (A_1+B_2).1,\ (A_1+B_2).2$
℘ $2A_1.1$	$B_3,\ (A_1+B_2).1,\ (A_1+B_2).2,\ A_1+A_2,\ A_3.1,\ 3A_1.1$
℘ A_2	$B_3,\ D_3^{(4)},\ A_1+A_2,\ A_3.1,\ A_3.2$
℘ $B_2^{(4)}$	$B_3^{(4)},\ A_1+B_2^{(4)}$
$A_3.2$	$B_3,\ D_3^{(4)},\ B_4.1,\ D_4^{(4)},\ D_4.1,\ D_4.2,\ A_1+A_3$
$3A_1.1$	$B_3,\ (A_1+B_2).1,\ (A_1+B_2).2,\ D_4.1,\ A_1+B_3,\ A_1+A_3,$ $(2A_1+B_2).1,\ (2A_1+B_2).2,\ 4A_1.1$
$3A_1.2$	$(A_1+B_2).1,\ D_4.2,\ (2A_1+B_2).1,\ 4A_1.2$
$(A_1+B_2).1$	$B_3,\ A_1+B_2^{(4)},\ B_4.1,\ 2B_2.1,\ 2B_2.2,\ D_4^{(4)},\ A_1+B_3,\ (2A_1+B_2).1$
$(A_1+B_2).2$	$\mathbf{B}_3^{(4)},\ A_1+B_2^{(4)},\ B_4.2,\ A_1+D_3^{(4)},\ 2B_2.2,\ (2A_1+B_2).2$
$A_1+B_2^{(4)}$	$\mathbf{B}_3^{(4)},\ B_4^{(4)},\ A_1+B_3^{(4)},\ B_2+B_2^{(4)},\ 2A_1+B_2^{(4)}$
B_3	$\mathbf{B}_3^{(4)},\ N_4,\ F_4,\ B_4.1,\ B_4.2,\ A_1+B_3$
$D_3^{(4)}$	$\mathbf{B}_3^{(4)},\ N_4,\ D_4^{(4)},\ A_1+D_3^{(4)}$
℘ $A_3.1$	$N_4,\ B_4.2,\ D_4^{(4)},\ D_4.1,\ A_4.1,\ A_4.2$
℘ A_1+A_2	$N_4,\ F_4,\ B_4.1,\ B_4.2,\ A_1+B_3,\ A_1+D_3^{(4)},\ A_1+A_3,\ 2A_2,\ A_4.1,\ A_4.2$
℘ $B_3^{(4)}$	$O_4,\ B_4^{(4)},\ A_1+B_3^{(4)}$
$4A_1.1$	$A_1+B_3,\ (2A_1+B_2).1,\ (2A_1+B_2).2,\ D_4.1$
$4A_1.2$	$D_4.2,\ (2A_1+B_2).1$
$(2A_1+B_2).1$	$B_4.1,\ D_4^{(4)},\ A_1+B_3,\ 2A_1+B_2^{(4)},\ 2B_2.1,\ 2B_2.2$
$(2A_1+B_2).2$	$A_1+B_3^{(4)},\ 2B_2.2,\ B_4.2,\ 2A_1+B_2^{(4)}$
A_1+A_3	$N_4,\ F_4,\ B_4.1,\ B_4.2,\ A_1+D_3^{(4)},\ A_1+B_3$
A_1+B_3	$N_4,\ F_4,\ A_1+B_3^{(4)},\ B_4.1,\ B_4.2$
$2B_2.1$	$B_4.1,\ D_4^{(4)},\ B_2+B_2^{(4)}$
$2B_2.2$	$B_4.2,\ B_4^{(4)},\ B_2+B_2^{(4)}$
$2A_1+B_2^{(4)}$	$B_4^{(4)},\ A_1+B_3^{(4)},\ B_2+B_2^{(4)}$
$B_2+B_2^{(4)}$	$B_4^{(4)},\ 2B_2^{(4)}$
$D_4.1$	$N_4,\ B_4.2,\ D_4^{(4)}$
$D_4.2$	$B_4.1,\ D_4^{(4)}$
$2B_2^{(4)}$	$B_4^{(4)}$
$D_4^{(4)}$	$N_4,\ B_4^{(4)}$
$B_4.1$	$N_4,\ B_4^{(4)},\ F_4$
$B_4.2$	$\mathbf{O}_4,\ B_4^{(4)}$
$A_1+D_3^{(4)}$	$\mathbf{O}_4,\ B_4^{(4)},\ A_1+B_3^{(4)}$
$A_1+B_3^{(4)}$	$\mathbf{O}_4,\ B_4^{(4)}$
$A_4.1$	$\mathbf{O}_4,\ N_4$
$A_4.2$	$\mathbf{O}_4,\ N_4$
$2A_2$	$\mathbf{O}_4,\ F_4$
$B_4^{(4)}$	$\mathbf{O}_4$
F_4	$\mathbf{O}_4$
N_4	$\mathbf{O}_4$

Table 10
Reflection subgroup classes of $G_{32} = L_4$.

	Class	Simple extensions
$\wp$	L_1	$L_2,\ 2L_1$
$\wp$	$2L_1$	$L_3,\ L_1+L_2,\ 3L_1$
$\wp$	L_2	$L_3,\ L_1+L_2$
	$3L_1$	$\mathbf{L}_3,\ L_1+L_3,\ 4L_1$
$\wp$	L_3	$L_4,\ L_1+L_3$
$\wp$	L_1+L_2	$L_4,\ L_1+L_3,\ 2L_2$
	$4L_1$	L_1+L_3
	L_1+L_3	$\mathbf{L}_4$
	$2L_2$	$\mathbf{L}_4$

Table 11
Reflection subgroup classes of $G_{33} = K_5$.

	Class	Simple extensions
$\wp$	A_1	$A_2,\ 2A_1$
$\wp$	$2A_1$	$A_1+A_2,\ A_3,\ 3A_1$
$\wp$	A_2	$D_3^{(3)},\ A_1+A_2,\ A_3$
$\wp$	A_1+A_2	$D_4^{(3)},\ A_1+A_3,\ A_4,\ 2A_2$
$\wp$	A_3	$D_4,\ D_4^{(3)},\ A_1+A_3,\ A_4$
$\wp$	$3A_1$	$D_4,\ A_1+A_3,\ 4A_1$
$\wp$	$D_3^{(3)}$	$D_4^{(3)}$
	$4A_1$	$\mathbf{D}_4,\ A_1+D_4,\ 5A_1$
	$2A_2$	$\mathbf{D}_4^{(3)},\ A_5$
$\wp$	A_1+A_3	$K_5,\ A_1+D_4,\ A_5$
$\wp$	D_4	$K_5,\ A_1+D_4$
$\wp$	A_4	$K_5,\ A_5$
$\wp$	$D_4^{(3)}$	K_5
	$5A_1$	A_1+D_4
	A_1+D_4	$\mathbf{K}_5$
	A_5	$\mathbf{K}_5$

Table 12
Reflection subgroup classes of $G_{34} = K_6$.

	Class	Simple extensions (ranks 1 to 4)
$\wp$	A_1	$A_2,\ 2A_1$
$\wp$	$2A_1$	$A_1+A_2,\ A_3,\ 3A_1$
$\wp$	A_2	$D_3^{(3)},\ A_1+A_2,\ A_3$
$\wp$	A_1+A_2	$D_4^{(3)},\ A_1+D_3^{(3)},\ A_1+A_3,\ 2A_1+A_2,\ A_4,\ 2A_2.1,\ 2A_2.2$
$\wp$	A_3	$A_4,\ D_4,\ D_4^{(3)},\ A_1+A_3$
$\wp$	$3A_1$	$D_4,\ A_1+A_3,\ 2A_1+A_2,\ 4A_1$
$\wp$	$D_3^{(3)}$	$D_4^{(3)},\ A_1+D_3^{(3)}$
	$4A_1$	$\mathbf{D}_4,\ A_1+D_4,\ 2A_1+A_3,\ 5A_1$
	$2A_2.2$	$\mathbf{D}_4^{(3)},\ A_2+D_3^{(3)},\ A_1+2A_2,\ A_5.2$
$\wp$	$2A_2.1$	$D_5^{(3)},\ A_2+D_3^{(3)},\ A_2+A_3,\ A_5.1,\ A_5.3$
$\wp$	A_1+A_3	$K_5,\ D_5,\ A_1+D_4^{(3)},\ A_1+D_4,\ A_2+A_3,\ A_1+A_4,\ 2A_1+A_3,\ A_5.1,\ A_5.2,\ A_5.3$
$\wp$	$2A_1+A_2$	$D_5,\ A_1+D_4^{(3)},\ A_2+A_3,\ A_1+A_4,\ A_1+2A_2,\ 2A_1+A_3$
$\wp$	$A_1+D_3^{(3)}$	$D_5^{(3)},\ A_2+D_3^{(3)},\ A_1+D_4^{(3)}$
$\wp$	D_4	$K_5,\ D_5,\ A_1+D_4$
$\wp$	A_4	$K_5,\ D_5,\ D_5^{(3)},\ A_1+A_4,\ A_5.1,\ A_5.2,\ A_5.3$
$\wp$	$D_4^{(3)}$	$K_5,\ D_5^{(3)},\ A_1+D_4^{(3)}$

Table 13
Reflection subgroup classes of $G_{34} = K_6$ (continued).

	Class	Simple extensions (ranks 5 and 6)
	$5A_1$	$A_1 + D_4,\ 2A_1 + D_4,\ 6A_1$
	$A_2 + D_3^{(3)}$	$\mathbf{D}_5^{(3)},\ D_6^{(3)},\ 2D_3^{(3)},\ A_2 + D_4^{(3)}$
	$A_1 + D_4$	$\mathbf{K}_5,\ A_1 + K_5,\ D_6,\ 2A_1 + D_4$
	$2A_1 + A_3$	$\mathbf{D}_5,\ A_1 + K_5,\ D_6,\ 2A_1 + D_4,\ A_1 + A_5,\ 2A_3$
	$A_1 + 2A_2$	$\mathbf{A}_1 + \mathbf{D}_4^{(3)},\ E_6,\ A_2 + D_4^{(3)},\ A_1 + A_5,\ 3A_2$
	$A_5.2$	$\mathbf{K}_5,\ E_6,\ D_6^{(3)},\ A_1 + A_5$
℘	$A_5.1$	$K_6,\ D_6,\ D_6^{(3)},\ A_6.1$
℘	$A_5.3$	$K_6,\ D_6,\ D_6^{(3)},\ A_6.2$
℘	$A_2 + A_3$	$K_6,\ D_6,\ D_6^{(3)},\ A_2 + D_4^{(3)},\ 2A_3,\ A_6.1,\ A_6.2$
℘	$A_1 + A_4$	$K_6,\ E_6,\ A_1 + K_5,\ A_1 + A_5,\ A_6.1,\ A_6.2$
℘	$A_1 + D_4^{(3)}$	$K_6,\ A_1 + K_5,\ D_6^{(3)},\ A_2 + D_4^{(3)}$
℘	D_5	$K_6,\ E_6,\ D_6$
℘	$D_5^{(3)}$	$K_6,\ D_6^{(3)}$
℘	K_5	$K_6,\ A_1 + K_5$
	$6A_1$	$2A_1 + D_4$
	$2A_1 + D_4$	$A_1 + K_5,\ D_6$
	$3A_2$	$E_6,\ A_2 + D_4^{(3)}$
	$2D_3^{(3)}$	$D_6^{(3)}$
	$A_1 + A_5$	$\mathbf{K}_6,\ A_1 + K_5,\ E_6$
	$A_2 + D_4^{(3)}$	$\mathbf{K}_6,\ D_6^{(3)}$
	$2A_3$	$\mathbf{K}_6,\ D_6$
	$A_6.1$	$\mathbf{K}_6$
	$A_6.2$	$\mathbf{K}_6$
	$A_1 + K_5$	$\mathbf{K}_6$
	$D_6^{(3)}$	$\mathbf{K}_6$
	D_6	$\mathbf{K}_6$
	E_6	$\mathbf{K}_6$

Table 14
Reflection subgroup classes of $G_{35} = E_6$.

	Class	Simple extensions
℘	A_1	$A_2,\ 2A_1$
℘	$2A_1$	$A_1 + A_2,\ A_3,\ 3A_1$
℘	A_2	$A_1 + A_2,\ A_3$
℘	$A_1 + A_2$	$2A_1 + A_2,\ A_1 + A_3,\ A_4,\ 2A_2$
℘	A_3	$D_4,\ A_1 + A_3,\ A_4$
℘	$3A_1$	$D_4,\ A_1 + A_3,\ 2A_1 + A_2,\ 4A_1$
	$4A_1$	$\mathbf{D}_4,\ 2A_1 + A_3$
℘	$2A_1 + A_2$	$D_5,\ A_1 + A_4,\ 2A_1 + A_3,\ A_1 + 2A_2$
℘	$A_1 + A_3$	$D_5,\ A_1 + A_4,\ 2A_1 + A_3,\ A_5$
℘	A_4	$D_5,\ A_1 + A_4,\ A_5$
℘	$2A_2$	$A_1 + 2A_2,\ A_5$
℘	D_4	D_5
	$2A_1 + A_3$	$\mathbf{D}_5,\ A_1 + A_5$
℘	$A_1 + 2A_2$	$E_6,\ A_1 + A_5,\ 3A_2$
℘	$A_1 + A_4$	$E_6,\ A_1 + A_5$
℘	A_5	$E_6,\ A_1 + A_5$
℘	D_5	E_6
	$A_1 + A_5$	$\mathbf{E}_6$
	$3A_2$	$\mathbf{E}_6$

Table 15
Reflection subgroup classes of $G_{36} = E_7$.

	Class	Simple extensions (ranks 1 to 5)
℘	A_1	$2A_1,\ A_2$
℘	$2A_1$	$A_1 + A_2,\ A_3,\ 3A_1.1,\ 3A_1.2$
℘	A_2	$A_3,\ A_1 + A_2$
℘	$A_1 + A_2$	$(A_1 + A_3).1,\ (A_1 + A_3).2,\ 2A_1 + A_2,\ A_4,\ 2A_2$
℘	A_3	$D_4,\ (A_1 + A_3).1,\ (A_1 + A_3).2,\ A_4$
℘	$3A_1.1$	$D_4,\ 2A_1 + A_2,\ (A_1 + A_3).1,\ 4A_1.1,\ 4A_1.2$
℘	$3A_1.2$	$(A_1 + A_3).2,\ 4A_1.2$
	$4A_1.1$	$\mathbf{D}_4,\ (2A_1 + A_3).1,\ 5A_1$
℘	$4A_1.2$	$A_1 + D_4,\ (2A_1 + A_3).2,\ 3A_1 + A_2,\ 5A_1$
℘	$(A_1 + A_3).1$	$D_5,\ A_1 + D_4,\ A_1 + A_4,\ A_2 + A_3, (2A_1 + A_3).1,\ (2A_1 + A_3).2,\ A_5.1$
℘	$(A_1 + A_3).2$	$A_1 + D_4,\ (2A_1 + A_3).2,\ A_5.2$
℘	$2A_1 + A_2$	$D_5,\ A_1 + A_4,\ A_1 + 2A_2,\ A_2 + A_3,\ 3A_1 + A_2,\ (2A_1 + A_3).1,\ (2A_1 + A_3).2$
℘	A_4	$D_5,\ A_1 + A_4,\ A_5.1,\ A_5.2$
℘	$2A_2$	$A_2 + A_3,\ A_1 + 2A_2,\ A_5.1,\ A_5.2$
℘	D_4	$D_5,\ A_1 + D_4$
	$5A_1$	$\mathbf{A}_1 + \mathbf{D}_4,\ 2A_1 + D_4,\ 3A_1 + A_3,\ 6A_1$
	$(2A_1 + A_3).1$	$\mathbf{D}_5,\ 2A_1 + D_4,\ 3A_1 + A_3,\ (A_1 + A_5).1,\ 2A_3$
℘	$(2A_1 + A_3).2$	$D_6,\ A_1 + D_5,\ 2A_1 + D_4,\ (A_1 + A_5).2,\ A_1 + A_2 + A_3,\ 3A_1 + A_3$
℘	$A_1 + D_4$	$D_6,\ A_1 + D_5,\ 2A_1 + D_4$
℘	$A_1 + A_4$	$E_6,\ A_2 + A_4,\ A_1 + D_5,\ A_6,\ (A_1 + A_5).1,\ (A_1 + A_5).2$
℘	$A_2 + A_3$	$D_6,\ A_6,\ A_1 + A_2 + A_3,\ A_2 + A_4,\ 2A_3$
℘	$A_1 + 2A_2$	$E_6,\ A_1 + A_2 + A_3,\ (A_1 + A_5).1,\ (A_1 + A_5).2,\ A_2 + A_4,\ 3A_2$
℘	$3A_1 + A_2$	$A_1 + D_5,\ 3A_1 + A_3,\ A_1 + A_2 + A_3$
℘	$A_5.1$	$E_6,\ A_6,\ D_6,\ (A_1 + A_5).1$
℘	$A_5.2$	$D_6,\ (A_1 + A_5).2$
℘	D_5	$E_6,\ D_6,\ A_1 + D_5$

Table 16
Reflection subgroup classes of $G_{36} = E_7$ (continued).

	Class	Simple extensions (ranks 6 and 7)
	$6A_1$	$2A_1 + D_4,\ 3A_1 + D_4,\ 7A_1$
	$2A_1 + D_4$	$\mathbf{D}_6,\ A_1 + D_6,\ 3A_1 + D_4$
	$2A_3$	$\mathbf{D}_6,\ A_7,\ A_1 + 2A_3$
	$3A_1 + A_3$	$\mathbf{A}_1 + \mathbf{D}_5,\ A_1 + D_6,\ A_1 + 2A_3,\ 3A_1 + D_4$
	$3A_2$	$\mathbf{E}_6,\ A_2 + A_5$
	$(A_1 + A_5).1$	$\mathbf{E}_6,\ A_1 + D_6,\ A_7$
℘	$(A_1 + A_5).2$	$E_7,\ A_2 + A_5,\ A_1 + D_6$
℘	$A_1 + D_5$	$E_7,\ A_1 + D_6$
℘	$A_2 + A_4$	$E_7,\ A_2 + A_5,\ A_7$
℘	$A_1 + A_2 + A_3$	$E_7,\ A_1 + D_6,\ A_2 + A_5,\ A_1 + 2A_3$
℘	A_6	$E_7,\ A_7$
℘	D_6	$E_7,\ A_1 + D_6$
℘	E_6	E_7
	$7A_1$	$3A_1 + D_4$
	$3A_1 + D_4$	$A_1 + D_6$
	$A_1 + 2A_3$	$\mathbf{E}_7,\ A_1 + D_6$
	$A_1 + D_6$	$\mathbf{E}_7$
	$A_2 + A_5$	$\mathbf{E}_7$
	A_7	$\mathbf{E}_7$

Table 17
Reflection subgroup classes of $G_{37} = E_8$.

	Class	Simple extensions (ranks 1 to 4)
℘	A_1	$2A_1,\ A_2$
℘	$2A_1$	$A_1+A_2,\ 3A_1,\ A_3$
℘	A_2	$A_1+A_2,\ A_3$
℘	A_1+A_2	$2A_1+A_2,\ 2A_2,\ A_4,\ A_1+A_3$
℘	A_3	$D_4,\ A_4,\ A_1+A_3$
℘	$3A_1$	$D_4,\ 2A_1+A_2,\ A_1+A_3,\ 4A_1.1,\ 4A_1.2$
	$4A_1.1$	$\mathbf{D}_4,\ (2A_1+A_3).1,\ 5A_1$
℘	$4A_1.2$	$A_1+D_4,\ (2A_1+A_3).2,\ 3A_1+A_2,\ 5A_1$
℘	$2A_1+A_2$	$D_5,\ A_1+A_4,\ A_1+2A_2,\ A_2+A_3,\ 3A_1+A_2,$ $(2A_1+A_3).1,\ (2A_1+A_3).2$
℘	$2A_2$	$A_2+A_3,\ A_5,\ A_1+2A_2$
℘	A_4	$D_5,\ A_1+A_4,\ A_5$
℘	A_1+A_3	$D_5,\ A_1+D_4,\ A_1+A_4,\ A_2+A_3,\ A_5,$ $(2A_1+A_3).1,\ (2A_1+A_3).2$
℘	D_4	$D_5,\ A_1+D_4$

Table 18
Reflection subgroup classes of $G_{37} = E_8$ (continued).

	Class	Simple extensions (ranks 5 and 6)
	$5A_1$	$\mathbf{A}_1+\mathbf{D}_4,\ 3A_1+A_3,\ 2A_1+D_4,\ 4A_1+A_2,\ 6A_1$
	$(2A_1+A_3).1$	$\mathbf{D}_5,\ (A_1+A_5).1,\ 2A_3.1,\ 2A_1+D_4,\ 3A_1+A_3$
℘	$(2A_1+A_3).2$	$D_6,\ (A_1+A_5).2,\ 2A_3.2,\ 2A_1+D_4,\ A_1+D_5,\ 3A_1+A_3,$ $2A_1+A_4,\ A_1+A_2+A_3$
℘	A_1+A_4	$E_6,\ A_1+D_5,\ (A_1+A_5).1,\ (A_1+A_5).2,\ A_2+A_4,\ 2A_1+A_4,\ A_6$
℘	A_1+2A_2	$E_6,\ (A_1+A_5).1,\ (A_1+A_5).2,\ A_2+A_4,\ A_1+A_2+A_3,\ 2A_1+2A_2,\ 3A_2$
℘	A_2+A_3	$D_6,\ A_2+D_4,\ A_6,\ A_2+A_4,\ 2A_3.1,\ 2A_3.2,\ A_1+A_2+A_3$
℘	$3A_1+A_2$	$A_1+D_5,\ A_2+D_4,\ A_1+A_2+A_3,\ 2A_1+A_4,\ 3A_1+A_3,\ 2A_1+2A_2,\ 4A_1+A_2$
℘	D_5	$E_6,\ D_6,\ A_1+D_5$
℘	A_5	$E_6,\ D_6,\ A_6,\ (A_1+A_5).1,\ (A_1+A_5).2$
℘	A_1+D_4	$D_6,\ A_2+D_4,\ A_1+D_5,\ 2A_1+D_4$
	$6A_1$	$2A_1+D_4,\ 3A_1+D_4,\ 4A_1+A_3,\ 7A_1$
	$2A_1+D_4$	$\mathbf{D}_6,\ A_1+D_6,\ A_3+D_4,\ 2A_1+D_5,\ 3A_1+D_4$
	$3A_1+A_3$	$\mathbf{A}_1+\mathbf{D}_5,\ 2A_1+A_5,\ A_1+D_6,\ 2A_1+D_5,\ A_3+D_4,$ $A_1+2A_3,\ 3A_1+D_4,\ 2A_1+A_2+A_3,\ 4A_1+A_3$
	$4A_1+A_2$	$\mathbf{A}_2+\mathbf{D}_4,\ 2A_1+D_5,\ 2A_1+A_2+A_3,\ 4A_1+A_3$
	$3A_2$	$\mathbf{E}_6,\ A_2+A_5,\ A_1+3A_2$
	$2A_3.1$	$\mathbf{D}_6,\ A_7.1,\ A_3+D_4,\ A_1+2A_3$
	$(A_1+A_5).1$	$\mathbf{E}_6,\ A_7.1,\ A_1+D_6,\ 2A_1+A_5$
℘	$2A_3.2$	$A_7.2,\ D_7,\ A_3+A_4,\ A_3+D_4$
℘	$(A_1+A_5).2$	$E_7,\ A_7.2,\ A_1+D_6,\ 2A_1+A_5,\ A_1+E_6,\ A_1+A_6,\ A_2+A_5$
℘	E_6	$E_7,\ A_1+E_6$
℘	D_6	$E_7,\ D_7,\ A_1+D_6$
℘	A_6	$E_7,\ D_7,\ A_7.1,\ A_7.2,\ A_1+A_6$
℘	A_2+A_4	$E_7,\ A_7.1,\ A_7.2,\ A_3+A_4,\ A_2+D_5,\ A_2+A_5,\ A_1+A_2+A_4$
℘	A_1+D_5	$E_7,\ A_1+D_6,\ D_7,\ A_1+E_6,\ 2A_1+D_5,\ A_2+D_5$
℘	$A_1+A_2+A_3$	$E_7,\ A_1+D_6,\ A_2+D_5,\ A_3+A_4,\ A_1+A_6,\ 2A_1+A_2+A_3,$ $A_1+A_2+A_4,\ A_2+A_5,\ A_1+2A_3$
℘	$2A_1+A_4$	$D_7,\ 2A_1+A_5,\ A_1+A_6,\ A_3+A_4,\ 2A_1+D_5,\ A_1+E_6,\ A_1+A_2+A_4$
℘	$2A_1+2A_2$	$2A_1+A_5,\ A_2+D_5,\ A_1+A_2+A_4,\ A_1+E_6,\ A_1+3A_2,\ 2A_1+A_2+A_3$
℘	A_2+D_4	$D_7,\ A_2+D_5,\ A_3+D_4$

Table 19
Reflection subgroup classes of $G_{37} = E_8$ (continued).

Class	Simple extensions (ranks 7 and 8)
$7A_1$	$3A_1 + D_4,\ 4A_1 + D_4,\ 8A_1$
$4A_1 + A_3$	$2A_1 + D_5,\ A_3 + D_4,\ 2A_1 + D_6,\ 2A_1 + 2A_3,\ 4A_1 + D_4$
$3A_1 + D_4$	$\mathbf{A}_1 + \mathbf{D}_6,\ 2A_1 + D_6,\ 2D_4,\ 4A_1 + D_4$
$2A_1 + D_5$	$\mathbf{D}_7,\ A_1 + E_7,\ 2A_1 + D_6,\ A_3 + D_5$
$A_3 + D_4$	$\mathbf{D}_7,\ D_8,\ A_3 + D_5,\ 2D_4$
$2A_1 + A_2 + A_3$	$\mathbf{A}_2 + \mathbf{D}_5,\ 2A_1 + D_6,\ A_1 + E_7,\ A_3 + D_5,\ A_1 + A_2 + A_5,\ 2A_1 + 2A_3$
$A_1 + 3A_2$	$\mathbf{A}_1 + \mathbf{E}_6,\ A_1 + A_2 + A_5,\ A_2 + E_6,\ 4A_2$
$2A_1 + A_5$	$\mathbf{A}_1 + \mathbf{E}_6,\ D_8,\ A_1 + A_7,\ 2A_1 + D_6,\ A_1 + E_7,\ A_1 + A_2 + A_5$
$A_1 + 2A_3$	$\mathbf{E}_7,\ A_1 + D_6,\ A_3 + D_5,\ A_1 + A_7,\ 2A_1 + 2A_3$
$A_2 + A_5$	$\mathbf{E}_7,\ A_8,\ A_2 + E_6,\ A_1 + A_2 + A_5$
$A_1 + D_6$	$\mathbf{E}_7,\ D_8,\ A_1 + E_7,\ 2A_1 + D_6$
$A_7.1$	$\mathbf{E}_7,\ D_8,\ A_1 + A_7$
℘ $A_7.2$	$E_8,\ D_8,\ A_8$
℘ D_7	$E_8,\ D_8$
℘ E_7	$E_8,\ A_1 + E_7$
℘ $A_1 + A_6$	$E_8,\ A_1 + A_7,\ A_8,\ A_1 + E_7$
℘ $A_1 + E_6$	$E_8,\ A_1 + E_7,\ A_2 + E_6$
℘ $A_3 + A_4$	$E_8,\ D_8,\ A_8,\ A_3 + D_5,\ 2A_4$
℘ $A_2 + D_5$	$E_8,\ D_8,\ A_2 + E_6,\ A_3 + D_5$
℘ $A_1 + A_2 + A_4$	$E_8,\ A_1 + E_7,\ A_2 + E_6,\ A_1 + A_7,\ A_1 + A_2 + A_5,\ 2A_4$
$2A_1 + 2A_3$	$2A_1 + D_6,\ A_1 + E_7,\ A_3 + D_5$
$8A_1$	$4A_1 + D_4$
$4A_1 + D_4$	$2D_4,\ 2A_1 + D_6$
$2A_1 + D_6$	$D_8,\ A_1 + E_7$
$4A_2$	$A_2 + E_6$
$2D_4$	D_8
$A_1 + A_2 + A_5$	$\mathbf{E}_8,\ A_1 + E_7,\ A_2 + E_6$
$A_3 + D_5$	$\mathbf{E}_8,\ D_8$
$A_1 + A_7$	$\mathbf{E}_8,\ A_1 + E_7$
$2A_4$	$\mathbf{E}_8$
$A_2 + E_6$	$\mathbf{E}_8$
A_8	$\mathbf{E}_8$
$A_1 + E_7$	$\mathbf{E}_8$
D_8	$\mathbf{E}_8$

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