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# Double-partition quantum cluster algebras

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A family of quantum cluster algebras is introduced and studied. In general, these algebras are new, but sub-classes have been studied previously by other authors. The algebras are indexed by double-partitions or double flag varieties. Equivalently, they are indexed by broken lines $L$. By grouping together neighboring mutations into quantum line mutations we can mutate from the cluster algebra of one broken line to another. Compatible pairs can be written down. The algebras are equal to their upper cluster algebras. The variables of the quantum seeds are given by elements of the dual canonical basis.



## 1. Introduction

A cluster algebra, as invented by Fomin and Zelevinsky, is a commutative algebra generated by a family of generators called cluster variables. The generators are grouped into clusters and the cluster variables can be computed recursively from the initial cluster.

The theory of cluster algebras is related to a wide range of subjects such as Poisson geometry, integrable systems, higher Teichmüller spaces, combinatorics, commutative and non-commutative algebraic geometry, and the representation theory of quivers and finite-dimensional algebras.

In [19], it is proved that the coordinate rings of $SL(n,\mathbb{C})$ and its maximal double Bruhat cell $SL(n,\mathbb{C})^{w_0,w_0}$ are cluster algebras. This is generalized in the recent work [4] where it is proved that the coordinate ring of any double Bruhat cell $G^{u,v}$ of any semi-simple algebraic group is a cluster algebra.

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Quantum cluster algebras were introduced and studied by Berenstein and Zelevinsky [2]. A main motivation was to understand the dual canonical basis. Following Lusztig [15], the dual canonical basis for the coordinate algebra $\mathcal{O}_q(M(n))$ of an $n \times n$ quantum matrix was shown to exist in [11]. This construction can be carried over to the algebra $\mathcal{O}_q(M(m,n))$ for all $m$ and $n$ verbatim.

A quantum mutation is governed by a pair of matrices, called a compatible pair, with certain favorable properties. To construct a quantum cluster, one of the main difficulties is to construct the compatible pairs. In the present paper we construct a family of quasi-commuting quantum minors of the algebra $\mathcal{O}_q(M(m,n))$ associated to each so-called broken line $L$, and construct a corresponding compatible pair $(\Lambda_L, B_L)$.

The set of broken lines has a natural partial ordering with unique biggest and smallest elements.

Let us be more specific: A **broken line** from $(1,n)$ to $(m,1)$ is a path in $\mathbb{N}\times\mathbb{N}$ starting at $(1,n)$ and terminating at $(m,1)$ while alternating between horizontal and vertical segments and passing through smaller column numbers (in the horizontal direction) and bigger row numbers (in the vertical direction). To each broken line we construct in Section 6 a family of $nm$ $q$-commuting quantum minors. With the line fixed, each of these quantum minors is uniquely given by a point $(i,j)\in\mathbb{N}\times\mathbb{N}$ with $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant n$. The quantum cluster algebra $\mathcal{A}_L^-$ is then determined by the quantum minors corresponding to the points on or below the line $L$. We prove that monomials in these are members of the dual canonical basis. We introduce the natural ordering on the set of broken lines and introduce some natural sub-algebras. One such is $\mathcal{O}_q(T_L\cup L)$ which denotes the sub-algebra of $\mathcal{O}_q(M(m,n))$ generated by the standard elements $Z_{i,j}$ of $\mathcal{O}_q(M(m,n))$ (cf. Section 3) for which $(i,j)$ is on the line, or below it.

We introduce a special class of mutations that are called quantum line mutations. To each triple of broken lines $L_a$, $L_b$ with $L_a, L_b\leqslant L$ we can mutate by quantum line mutations from $L_a$ to $L_b$.

For a compatible pair $(\Lambda_L, B_L)$ connected with $\mathcal{A}_L^-$ we show that we can mutate by quantum line mutations to a bigger line $L_1$ inside $\mathcal{O}_q(M(m,n))$ and, by carefully keeping track, construct a compatible pair $(\Lambda_{L_1}, B_{L_1})$ connected with $\mathcal{A}_{L_1}^-$ in the process. Starting at a particularly simple broken line, namely the one corresponding to the smallest broken line $L^-$, we can, by repeated quantum line mutations, construct a compatible pair for $\mathcal{A}_L^-$. Thus, we obtain compatible pairs for all broken lines. At first they are just compatible pairs for the smaller algebras. The algebra $\mathcal{O}_q(M(m,n))$ corresponds to the unique maximal broken line $L^+$. However, mutating in the opposite direction, we get a compatible pair for this bigger algebra for any line. Or, indeed, mutating backwards from any bigger line algebra to a smaller, we get a quantum seed $\mathcal{Q}_{L_1,L}$ for the bigger line algebra $L$ indexed by the smaller line algebra $L_1$.

Instances of such algebras have been studied in [13,14,6].

The main technical result is the following: Let $A$ be an $n\times n$ matrix whose entries are non-negative integers and let $b(A)$ be the element of the dual canonical basis of $\mathcal{O}_q(M(m,n))$ corresponding to this. Let $\det_q$ denote the quantum determinant. If $I$ denotes the $n\times n$ identity matrix then

$$\det_q = b(I)$$

and (this is (4.2))

$$b(A)\det_q = b(A+I).$$

Once this has been established, it can be generalized to several other configurations involving quantum minors.

After this introduction, the article continues in Section 2 with a review of quantum cluster algebras, followed in Section 3 by basic facts and structures relating to the quantized matrix algebra. The matters concerning the technical result (4.2) and its generalizations, take up Sections 4 and 5.

In Section 6, using the results on dual canonical bases, we strengthen a result of Parshall and Wang considerably. In so doing, we obtain a crucial commutation identity; Theorem 6.17. This result then makes it possible to introduce the class of mutations called quantum line mutations. We also include an observation relating this to totally positive matrices.

In Section 7, we construct compatible pairs $(\Lambda_L^0, B_L^0)$ and $(\Lambda_L, B_L)$. At first just for the algebra $\mathcal{O}_q(T_L \cup L)$, but later also for the full algebra $\mathcal{O}_q(M(m,n))$.

Finally, in Section 8, we extend slightly a result of Goodearl and Lenagan [8] saying that the $q$-determinantal ideal is prime. We then use quantum line mutations to give an inductive proof of the following, where $\mathcal{C}_L^-$ are the non-mutable (covariant) elements, and $\mathcal{U}_L^-$ is the upper cluster algebra:

**Theorem.** *Let $\mathcal{C}_L^- = \{Y_1, \dots, Y_s\}$. Then,*

$$\mathcal{U}_L^- = \mathcal{O}_q(T_L \cup L)\left[Y_1^{\pm 1}, \dots, Y_s^{\pm 1}\right] = \mathcal{A}_L^-.$$

This result is Theorem 8.5. As a consequence, we conclude that in the case of $\mathcal{O}_q(M(m,n))$, the quantum cluster algebra is equal to its upper cluster algebra.

## 2. Basics of quantum cluster algebras

Throughout the paper, the base field is $K = \mathbb{Q}(q)$, where $q$ is an indeterminate over the rational numbers. To avoid terms involving $q^{\frac{1}{2}}$, we work with the square root of the $q$ used by Berenstein and Zelevinsky; $q^2 = q_{our}^2 = q_{BZ}$.

Given an integral skew-symmetric matrix $\Lambda = (\lambda_{ij}) \in M_{\mathfrak{m}}(\mathbb{Z})$, the quasi-polynomial algebra $\mathcal{L}(\Lambda)$ associated to the matrix $\Lambda$ is an associative algebra generated by $x_1, x_2, \dots, x_{\mathfrak{m}}; x_1^{-1}, x_2^{-1}, \dots, x_{\mathfrak{m}}^{-1}$ with the defining relations

$$x_i x_j = q^{2\lambda_{ij}} x_j x_i. \tag{2.1}$$

Conversely, given such relations, the matrix $\Lambda = (\lambda_{ij}) \in M_{\mathfrak{m}}(\mathbb{Z})$ will be called the $\Lambda$-matrix of the variables $x_1, \dots, x_{\mathfrak{m}}$.

The set of ordered monomials

$$\left\{x^{\underline{a}} := x_1^{a_1} x_2^{a_2} \cdots x_{\mathfrak{m}}^{a_{\mathfrak{m}}} \;\middle|\; \underline{a} = (a_1, a_2, \dots, a_{\mathfrak{m}}) \in \mathbb{Z}^{\mathfrak{m}}\right\}$$

is a basis of $\mathcal{L}(\Lambda)$. It is well known that $\mathcal{L}(\Lambda)$ is a Noetherian domain and one can talk about its skew field of fractions which is denoted by $\mathcal{F}(\Lambda)$. Using $\Lambda$, one can define a bilinear form on $\mathbb{Z}^{\mathfrak{m}}$ as follows:

$$\begin{aligned} &\Lambda : \mathbb{Z}^{\mathfrak{m}} \times \mathbb{Z}^{\mathfrak{m}} \to \mathbb{Z}, \\ &\Lambda(\underline{a}, \underline{b}) = \underline{a} \Lambda \underline{b}^T. \end{aligned} \tag{2.2}$$

For any $\underline{a} \in \mathbb{Z}^{\mathfrak{m}}$, the normalized monomial is defined as

$$x(\underline{a}) = q^{\sum_{i<j} \lambda_{ji} a_i a_j} x^{\underline{a}}.$$

The map

$$\forall i = 1, \dots, \mathfrak{m}: \quad x_i \mapsto x_i, \qquad q \mapsto q^{-1} \tag{2.3}$$

extends to a $\mathbb{Q}$-algebra anti-automorphism, denoted by $\ell \mapsto \bar{\ell}$, which actually does not depend on the ordering. Then

$$\overline{x(\underline{a})} = x(\underline{a}). \tag{2.4}$$

It is easy to check that

$$x(\underline{a})x(\underline{b}) = q^{\Lambda(\underline{a},\underline{b})}x(\underline{a}+\underline{b}),$$

which, of course, is equivalent to the commutation relations (2.1).

Denote by $K^* := \mathbb{Q}(q) - \{0\}$ the multiplicative group of non-zero elements. The group $(K^*)^{\mathfrak{m}}$ acts on $\mathcal{L}(\Lambda)$ as an automorphism group. Explicitly, for any $\underline{h} = (h_1, h_2, \dots, h_{\mathfrak{m}}) \in (K^*)^{\mathfrak{m}}$, it acts on $\mathcal{L}(\Lambda)$ according to the formulae

$$\underline{h}(x_i) = h_i x_i \quad \text{for all } i.$$

**Remark 2.1.** If a subspace $S \subset \mathcal{A}(\Lambda)$ is invariant under the action of the group $(K^*)^{\mathfrak{m}}$, then it is spanned by the monomials that it contains.

In [2], the notion of a quantum cluster algebra was introduced. Let us recall the definition.

**Definition 2.2.** Let $B$ be an $\mathfrak{m} \times \mathfrak{n}$ integer matrix with rows labeled by $[1, \mathfrak{m}]$ and columns labeled by an $\mathfrak{n}$-element subset $ex \subset [1, \mathfrak{m}]$. Let $\Lambda$ be a skew-symmetric $\mathfrak{m} \times \mathfrak{m}$ integer matrix with rows and columns labeled by $[1, \mathfrak{m}]$. We say that a pair $(\Lambda, B)$ is *compatible* if, for every $j \in ex$ and $i \in [1, \mathfrak{m}]$, we have

$$\sum_{k=1}^{\mathfrak{m}} b_{kj}\lambda_{ki} = \delta_{ij} d_j$$

for some positive integers $d_j$ $(j \in ex)$. The $\mathfrak{n} \times \mathfrak{n}$ sub-matrix of $B$ corresponding to the subset $ex$ is called the principal part of $B$. We insist throughout this article, that $\forall j$: $d_j = 2$.

If one arranges the symbols such that $ex = \{1, 2, \dots, \mathfrak{n}\}$, the compatibility condition states that the $\mathfrak{n} \times \mathfrak{m}$ matrix $\tilde{D} = B^T \Lambda$ consists of the two blocks: the $\mathfrak{n} \times \mathfrak{n}$ diagonal matrix $D$ with positive integer diagonal entries $d_j$, and the $\mathfrak{n} \times (\mathfrak{m} - \mathfrak{n})$ zero block.

With the above setup, the triple $(\{x_1, x_2, \dots, x_{\mathfrak{m}}\}, \Lambda, B)$ is an example of a **quantum seed** of $\mathcal{F}(\Lambda)$. The notion of a quantum seed is more general than the one presented here, but ours suffices for the purposes below. The variables $x_i$ are called *quantum cluster variables*. The variables $x_i$, $i \in ex$ are called *mutable variables* and the set of these is called *the cluster*. The variables $x_j$, $j \notin ex$ are called *non-mutable variables*.

Notice that if $\underline{a} = (a_1, a_2, \dots, a_{\mathfrak{m}})$ and $\underline{f} = (f_1, f_2, \dots, f_{\mathfrak{m}})$ are vectors then

**Lemma 2.3.**

$$\Lambda(\underline{a})^T = (\underline{f})^T \quad \Leftrightarrow \quad \forall i: \quad x_i x^{\underline{a}} = q^{2f_i} x^{\underline{a}} x_i. \tag{2.5}$$

*In particular, if there exists a $j$ such that $\forall i$: $f_i = -\delta_{i,j}$ then the column vector $\underline{a}$ can be the $j$th column in the matrix $B$ of a compatible pair.*

However simple this actually is, it will have a great importance later on.

Denote by $e_1, e_2, \dots, e_{\mathfrak{m}}$ the standard basis of $\mathbb{Z}^{\mathfrak{m}}$. For a given compatible pair $(\Lambda, B = (b_{ki}))$, one can *mutate* the cluster in the direction of $i \in ex$, thereby obtaining a new cluster whose variables are $x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_{\mathfrak{m}}$. The unique new variable is defined by

$$x_i' = x\left(\sum_{b_{ki}>0} b_{ki}e_k - e_i\right) + x\left(\sum_{b_{ki}<0} -b_{ki}e_k - e_i\right). \tag{2.6}$$

One can check that $x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_{\mathfrak{m}}$ is a $q$-commuting family.

We will extend matrix mutations to those of compatible pairs. Fix an index $i \in ex$. The matrix $B_i' = \mu_i(B)$ can be written as

$$B_i' = E_i\, B\, F_i, \tag{2.7}$$

where

- $E_i$ is the $\mathfrak{m} \times \mathfrak{m}$ matrix with entries

$$e_{ab} = \begin{cases} \delta_{ab} & \text{if } b \neq i; \\ -1 & \text{if } a = b = i; \\ \max(0, -b_{ai}) & \text{if } a \neq b = i. \end{cases} \tag{2.8}$$

- $F_i$ is the $\mathfrak{n} \times \mathfrak{n}$ matrix with rows and columns labeled by $ex$, and entries given by

$$f_{ab} = \begin{cases} \delta_{ab} & \text{if } a \neq i; \\ -1 & \text{if } a = b = i; \\ \max(0, b_{ib}) & \text{if } a = i \neq b. \end{cases} \tag{2.9}$$

The triple $(\{x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_{\mathfrak{m}}\},\ \Lambda_i = E_i^T \Lambda E_i, B_i')$ is also a quantum seed. The above process of passing from a quantum seed to another is called a quantum mutation in the direction $i$. We say that two quantum seeds are mutation equivalent if they can be obtained from each other by a sequence of quantum mutations. In the general definition of [2] there is an additional parameter $\varepsilon = \pm 1$ in the definition of the matrices $E_i$, $F_i$. Throughout this article we restrict to $\varepsilon = 1$ and for this reason we suppress it.

Given a quantum seed, let $\mathcal{S}$ be the set of all quantum seeds which are mutation equivalent to the given one. The quantum cluster algebra $\mathcal{A}(\mathcal{S})$ associated to the given quantum seed is the $\mathbb{Q}(q)$ sub-algebra of $\mathcal{F}(\Lambda)$ generated by all quantum cluster variables contained in $\mathcal{S}$.

## 3. The quantum matrices and the dual canonical basis

The coordinate algebra $\mathcal{O}_q(M(m,n))$ of the quantum $m \times n$ matrix is an associative algebra, generated by elements $Z_{ij}$, $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$, subject to the following defining relations:

$$Z_{ij} Z_{ik} = q^2 Z_{ik} Z_{ij} \quad \text{if } j < k, \tag{3.1}$$

$$Z_{ij} Z_{kj} = q^2 Z_{kj} Z_{ij} \quad \text{if } i < k, \tag{3.2}$$

$$Z_{ij} Z_{st} = Z_{st} Z_{ij} \quad \text{if } i > s,\ j < t, \tag{3.3}$$

$$Z_{ij} Z_{st} = Z_{st} Z_{ij} + \left(q^2 - q^{-2}\right) Z_{it} Z_{sj} \quad \text{if } i < s,\ j < t. \tag{3.4}$$

The associated quasi-polynomial algebra $o_q(M(m,n))$ of the quantum $m \times n$ matrix is an associative algebra, generated by elements $z_{ij}$, $i = 1, 2, \dots,\ m$; $j = 1, 2, \dots, n$, subject to the following defining relations:

$$z_{ij} z_{ik} = q^2 z_{ik} z_{ij} \quad \text{if } j < k, \tag{3.5}$$

$$z_{ij} z_{kj} = q^2 z_{kj} z_{ij} \quad \text{if } i < k, \tag{3.6}$$

$$z_{ij} z_{st} = z_{st} z_{ij} \quad \text{in all other cases.} \tag{3.7}$$

For any matrix $A=(a_{ij})_{1\leqslant i\leqslant m,\,1\leqslant j\leqslant n}\in M_{m,n}(\mathbb{Z}_+)$, where $\mathbb{Z}_+=\{0,1,\dots\}$, we define a monomial $Z^A$ by

$$Z^A=\prod_{i,j=1}^{n} Z_{ij}^{a_{ij}}, \tag{3.8}$$

where the factors are arranged in the descending lexicographic order on $I(m,n)=\{(i,j)\mid i=1,2,\dots,m;\ \ j=1,\dots,n\}$ given by $(1,1)>(1,2)>\cdots>(1,n)>(2,1)>\cdots$. We define similar elements $z^A\in o_q$. It is well known that the set $\{Z^A\mid A\in M_{m,n}(\mathbb{Z}_+)\}$ is a basis of the algebra $\mathcal{O}_q(M(m,n))$.

From the defining relations (3.1)–(3.7) of the algebras $\mathcal{O}_q(M(m,n))$ and $o_q(M(m,n))$ it is easy to show the following lemma. The last statement in the lemma, though trivial, is included for its usefulness.

**Lemma 3.1.** *The mapping*

$$\begin{aligned} {}^{-}: Z_{ij}&\mapsto Z_{ij},\\ q&\mapsto q^{-1} \end{aligned} \tag{3.9}$$

*extends to an anti-automorphism of the algebra $\mathcal{O}_q(M(m,n))$ as an algebra over $\mathbb{Q}$. The mapping*

$$\begin{aligned} {}^{-}: z_{ij}&\mapsto z_{ij},\\ q&\mapsto q^{-1} \end{aligned} \tag{3.10}$$

*extends to an anti-automorphism of the algebra $o_q(M(m,n))$ as an algebra over $\mathbb{Q}$.*

*There is an obvious anti-automorphism of the tensor algebra over the vector space $M(m,n)$ of which the given anti-automorphism of $\mathcal{O}_q(M(m,n))$ is the quotient map.*

*A similar statement holds in $o_q(M(m,n))$.*

For any $A=(a_{ij})\in M_{m,n}(\mathbb{Z}_+)$,

$$\underline{ro}(A):=\Bigl(\sum_j a_{1j},\dots,\sum_j a_{mj}\Bigr):=(\underline{ro}_1,\underline{ro}_2,\dots,\underline{ro}_m).$$

This is called the row sum of $A$.

$$\underline{co}(A):=\Bigl(\sum_j a_{j1},\dots,\sum_j a_{jn}\Bigr):=(\underline{co}_1,\underline{co}_2,\dots,\underline{co}_n).$$

This is called the column sum of $A$.

The following result follows easily from the defining relations (3.1)–(3.4):

**Lemma 3.2.** *Let*

$$Z^AZ^B=\sum_C a_C^{A,B}Z^C$$

*where $a_C^{A,B}\in\mathbb{Z}[q^2,q^{-2}]$. Then $\forall a_C^{A,B}\neq 0$:*

$$\underline{ro}(C) = \underline{ro}(A) + \underline{ro}(B),$$
$$\underline{co}(C) = \underline{co}(A) + \underline{co}(B).$$

From the defining relations we also have

$$\overline{Z^A} = E(A)Z^A + \sum_{B<A} c_B(A)Z^B, \tag{3.11}$$

where

$$E(A) = q^{-2(\sum_i \sum_{j>k} a_{ij}a_{ik} + \sum_i \sum_{j>k} a_{ji}a_{ki})}$$

and $\forall B < A$: $c_B(A) \neq 0 \Rightarrow \underline{ro}(B) = \underline{ro}(A)$, and $\underline{co}(B) = \underline{co}(A)$. Here, $c_B(A) \in \mathbb{Z}[q^2, q^{-2}]$, and the lexicographic order on $M_{m,n}(\mathbb{Z}_+)$, obtained by augmenting the previous order on $I(m,n)$ by the natural order on $\mathbb{Z}_+$, is denoted $\leqslant$.

Let

$$N(A) = q^{-\sum_i \sum_{j>k} a_{ij}a_{ik} - \sum_i \sum_{j>k} a_{ji}a_{ki}} \quad \text{and} \quad Z(A) = N(A)Z^A. \tag{3.12}$$

From (3.11) we trivially have

$$\overline{Z(A)} = Z(A) \text{ modulo lower order terms.} \tag{3.13}$$

In lack of better words we introduce:

**Definition 3.3.** We call $Z(A)$ the normalized form of $Z^A$. We call $N(A)$ the normalization factor.

Let $i < s$ and $j < t$. Set $E_{i,j,s,t} = E_{i,j} + E_{s,t} - E_{i,t} - E_{s,j}$, where for any of the mentioned pairs $(a,b)$, $E_{a,b}$ is the $(a,b)$th matrix unit. Upon rewriting $\overline{Z^A}$ according to our lexicographic order as in (3.11), one picks up terms $c_{A'}Z^{A'}$, where $A'$ is obtained from $A$ by subtraction of elements of the form $E_{i,j,s,t}$. The next result follows directly from (3.12).

**Lemma 3.4.** *If $A' = A - E_{i,j,s,t}$, then*

$$N(A') = N(A)q^{4-2(a_{ij}+a_{st}-a_{it}-a_{sj})}.$$

To facilitate the following proofs, we introduce a notion of a level in $M_{m,n}(\mathbb{Z}_+)$:

**Definition 3.5.** Let

$$\mathcal{D} = \{E_{i,j,s,t} \mid i < s \text{ and } j < t\}. \tag{3.14}$$

The matrix $A \in M_{m,n}(\mathbb{Z}_+)$ is of level $L(A) = 0$ if there are no elements $D \in \mathcal{D}$ and $A_1 \in M_{m,n}(\mathbb{Z}_+)$ such that $A = D + A_1$. Let $\mathcal{L}_0$ denote the set of matrices of level 0. We define the level $L(A)$ of any $A$ not of level zero by

$$L(A) := \max\{r \in \mathbb{N} \mid \exists D_1, \dots, D_r \in \mathcal{D},\ \exists A_0 \in \mathcal{L}_0\colon A = D_1 + \dots + D_k + A_0\}.$$

(It is easy to see that this maximum is finite.)

Notice that if $L(A) = 0$ then one can reorder $\overline{Z(A)}$ without invoking the relation (3.4). Thus, $\overline{Z(A)} = Z(A)$.

**Lemma 3.6.** *In Eq. (3.11), if $B < A$ and $c_B(A) \neq 0$ then $L(B) < L(A)$.*

**Proof.** View the right hand side of (3.11) as the result of the reordering of $\overline{Z^A}$ according to our chosen ordering. The terms with $B < A$ must then have their origins in the application of relation (3.4) at least once since otherwise we can get to $Z^A$ using solely the other three relations. In this case, $\overline{Z(A)} = Z(A)$. Any application of (3.4) clearly leads, modulo terms proportional to $Z^A$, to terms of lower level. □

Set

$$L^* = \bigoplus_{A \in M_{m,n}(\mathbb{Z}_+)} \mathbb{Z}[q] Z(A).$$

**Proposition 3.7.** *There is a unique $\mathbb{Z}[q]$-basis $B^* = \{b(A) \mid A \in M_{m,n}(\mathbb{Z}_+)\}$ of $L^*$ in which each element $b(A)$ is determined uniquely by the following conditions:*

(1) $\overline{b(A)} = b(A)$.
(2) $b(A) = Z(A) + \sum_{B<A} h_B(A) Z(B)$ *where* $h_B(A) \in q^2\mathbb{Z}[q^2]$ *and* $\underline{ro}(B) = \underline{ro}(A)$, $\underline{co}(B) = \underline{co}(A)$.

*The basis $B^*$ is called the dual canonical basis of $\mathcal{O}_q(M(m,n))$.*

**Corollary 3.8.** *If we number our basis vectors in the two bases $\mathcal{B}_1 = \{b(B) \mid B \in M_{m,n}(\mathbb{Z}_+)\}$ and $\mathcal{B}_2 = \{Z(B) \mid B \in M_{m,n}(\mathbb{Z}_+)\}$ according to the lexicographic ordering then the change of basis matrices are lower triangular with 1's in the diagonal and elements from $q^2\mathbb{Z}[q^2]$ in all other non-zero positions.*

**Proof of Proposition 3.7 and Corollary 3.8.** Noticing the $q^2$ factors in Lemma 3.4, this can be proved in analogy with Lusztig [15, 2. Proposition], see also [11, Theorem 3.5]. However, we will sketch a proof for clarity: We proceed to prove Proposition 3.7 by induction on the level $k$, utilizing that if the proposition holds up to level $k$ then so does Corollary 3.8. The case of level 0 is trivial since if $L(A) = 0$ then $b(A) = Z(A)$. Suppose then that the result holds up to, and including level $k$ and let $A$ be of level $k+1$. It follows from (3.11) and Lemma 3.6 together with Corollary 3.8 (up to level $k$) that

$$\overline{Z(A)} - Z(A) = \sum_{B<A;\, L(B)<L(A)} \frac{c_B(A)}{N(A)N(B)} Z(B) = \sum_{B<A;\, L(B)<L(A)} h_B b(B) \tag{3.15}$$

with elements $h_B \in \mathbb{Z}[q^2, q^{-2}]$. Since the left hand side of (3.15) is skew under the bar operator, each $h_B$ can be decomposed as $h_B = h_B^+ + h_B^-$ with $h_B^+ \in q^2\mathbb{Z}[q^2]$ and $h_B^- = -\overline{h_B^+}$. Then

$$b(A) = Z(A) + \sum_{B<A} h_B^+ b(B) \tag{3.16}$$

is the unique solution. Invoking Corollary 3.8 (up to $k$) once again, the proof is complete. □

The following simple principle is very useful:

**Proposition 3.9.** *If $m_1 \leqslant m$ and $n_1 \leqslant n$ we may view $\mathcal{O}_q(M(m_1,n_1))$ as the sub-algebra of $\mathcal{O}_q(M(m,n))$ generated by the elements of (some) $m_1$ rows and $n_1$ columns. If, correspondingly, we consider $M_{m_1,n_1}(\mathbb{Z}_+) \subseteq$*

*$M_{m,n}(\mathbb{Z}_+)$ then for any $A \in M_{m_1,n_1}(\mathbb{Z}_+)$, upon these identifications, the basis vector $b(A) \in \mathcal{O}_q(M(m_1,n_1))$ is also a basis vector in $\mathcal{O}_q(M(m,n))$.*

*If, under such identifications, $\mathcal{O}_q(M(m_1,n_1))$ and $\mathcal{O}_q(M(m_2,n_2))$ are two* commuting *sub-algebras of $\mathcal{O}_q(M(m,n))$ and if $b(A_i) \in \mathcal{O}_q(M(m_i,n_i))$, $i = 1,2$, are members of the respective dual canonical bases, then $b(A_1 + A_2) = b(A_1)b(A_2)$ is in the dual canonical basis of $\mathcal{O}_q(M(m,n))$.*

**Proof.** The relations, the bar operator, and the order on the sub-algebras are restrictions of the relations, the bar operator, and the order on the full algebra. The result then follows by the uniqueness. □

**Remark 3.10.** The commutativity condition in Proposition 3.9 is equivalent to having all canonical generators of one sub-algebra positioned $NE$ of the other.

If $m = n$, one may define the quantum determinant $\det_q$ as follows:

$$\det_q(n) = \det_q = \sum_{\sigma \in S_n} \left(-q^2\right)^{\ell(\sigma)} Z_{1,\sigma(1)} Z_{2,\sigma(2)} \cdots Z_{n,\sigma(n)} \tag{3.17}$$

$$= \sum_{\delta \in S_n} \left(-q^2\right)^{\ell(\delta)} Z_{\delta(1),1} Z_{\delta(2),2} \cdots Z_{\delta(n),n}. \tag{3.18}$$

We recall some results from [16] regarding the Quantum Laplace Expansion:
Suppose $I = \{i_1 < 1_2 < \cdots < i_r\}$ and $J = \{j_1 < j_2 < \cdots < j_r\}$ are subsets of $I = \{1,2,\ldots,n\}$. Define

$$\xi^I_J = \sum_{\sigma \in S_r} \left(-q^2\right)^{\ell(\sigma)} Z_{i_1,j_{\sigma(1)}} Z_{i_2,j_{\sigma(2)}} \cdots Z_{i_r,j_{\sigma(r)}} \tag{3.19}$$

$$= \sum_{\tau \in S_r} \left(-q^2\right)^{\ell(\tau)} Z_{i_{\tau(1)},j_1} Z_{i_{\tau(2)},j_2} \cdots Z_{i_{\tau(r)},j_r}. \tag{3.20}$$

These elements are called quantum minors. Notice that they are only defined if $\#J = \#I$. For two subsets $I, J \subseteq \{1,2,\ldots,n\}$, the symbol $sgn_q(I;J)$ is defined by

$$sgn_q(I;J) = \begin{cases} 0 & \text{if } I \cap J \neq \emptyset, \\ (-q^2)^{\ell(I;J)} & \text{if } I \cap J = \emptyset, \end{cases} \tag{3.21}$$

where $\ell(I;J) = \#\{(i,j) \mid i \in I,\ j \in J,\ i > j\}$. Then,

$$Sgn_q(J_1;J_2)\xi^I_J = \sum_{I_1 \cup I_2 = I} \xi^{I_1}_{J_1} \xi^{I_2}_{J_2} Sgn_q(I_1;I_2), \tag{3.22}$$

$$Sgn_q(J_1,J_2)\xi^J_I = \sum_{I_1 \cup I_2 = I} \xi^{J_1}_{I_1} \xi^{J_2}_{I_2} Sgn_q(I_1;I_2). \tag{3.23}$$

If $m = n$ and $I = \{1,2,\ldots,n\} \setminus \{i\}$, $J = \{1,2,\ldots,n\} \setminus \{j\}$, $\xi^I_J$ will (occasionally) be denoted by $A(i,j)$.
The following was proved by Parshall and Wang in [17]:

**Proposition 3.11.** $\det_q$ *is central. Furthermore, let $i,k \leqslant n$ be fixed integers. Then*

$$\delta_{i,k}\det_q = \sum_{j=1}^{n} \left(-q^2\right)^{j-k} Z_{i,j} A(k,j) = \sum_j \left(-q^2\right)^{i-j} A(i,j) Z_{k,j} \tag{3.24}$$

$$= \sum_j \left(-q^2\right)^{j-k} Z_{j,i} A(j,k) = \sum_j \left(-q^2\right)^{i-j} A(j,i) Z_{j,k}. \tag{3.25}$$

It is of key importance for the rest of the article to note the following which is proved by an easy induction argument using (3.24) while invoking the uniqueness of the dual canonical basis:

**Corollary 3.12.**

$$\overline{\det_q} = \det_q = b(I).$$

*Thus, all quantum minors are members of the dual canonical basis.*

**Definition 3.13.** An element $x \in \mathcal{O}_q(M(m,n))$ is called covariant if for any $Z_{ij}$ there exists an integer $n_{i,j}$ such that

$$xZ_{i,j} = q^{2n_{i,j}} Z_{i,j}x. \tag{3.26}$$

Clearly, $Z_{1,n}$ and $Z_{m,1}$ are covariant. Two elements $x, y \in \mathcal{O}_q(M(m,n))$ are said to $q$-commute if there exists an integer $p$ such that

$$xy = q^{2p} yx.$$

Let $\det_q(t) = \xi^{\{1,\dots,t\}}_{\{n-t+1,\dots,n\}}$, for $t = 1, 2, \dots, \min\{m,n\}$. It is easy to extend [10, Theorem 4.3] from the $n \times n$ case to the general rectangular case:

**Proposition 3.14.** *The element* $\det_q(t)$ *is covariant for all* $t$. *More precisely, let* $M_t^- = \{(i,j) \in \mathbb{N}^2 \mid 1 \leqslant i \leqslant t$ *and* $1 \leqslant j \leqslant n-t\}$, $M_t^+ = \{(i,j) \in \mathbb{N}^2 \mid t+1 \leqslant i \leqslant m$ *and* $n-t+1 \leqslant j \leqslant n\}$, $M_t^l = \{(i,j) \in \mathbb{N}^2 \mid t+1 \leqslant i \leqslant m$ *and* $1 \leqslant j \leqslant n-t\}$, *and* $M_t^r = \{(i,j) \in \mathbb{N}^2 \mid 1 \leqslant i \leqslant t$ *and* $n-t+1 \leqslant j \leqslant n\}$.

$$\begin{aligned} Z_{i,j}\det_q(t) &= \det_q(t) Z_{i,j} \quad \text{if } (i,j) \in M_t^l \cup M_t^r, \\ Z_{i,j}\det_q(t) &= q^2 \det_q(t) Z_{i,j} \quad \text{if } (i,j) \in M_t^-, \quad \text{and} \\ Z_{i,j}\det_q(t) &= q^{-2} \det_q(t) Z_{i,j} \quad \text{if } (i,j) \in M_t^+. \end{aligned} \tag{3.27}$$

Recall from [10, Lemma 3.3] the result for quantum $2 \times 2$ matrices:

$$\forall a \in \mathbb{N}: \quad Z_{2,2}^a Z_{1,1} = Z_{1,1} Z_{2,2}^a + q^{-2}\left(1 - q^{4a}\right) Z_{2,2}^{a-1} Z_{2,1} Z_{1,2}. \tag{3.28}$$

For later purposes, we need the following results for $n \times n$ matrices regarding $Z_{n,n}A(n,n)$: Using (3.24) we can define elements $M_1$, $M_2$ by

$$\begin{aligned} \det_q &= \sum_{j=1}^{n} \left(-q^2\right)^{j-n} Z_{n,j} A(n,j) = \sum_{j=1}^{n} \left(-q^2\right)^{n-j} A(n,j) Z_{n,j} \\ &= Z_{n,n} A(n,n) + M_1 \\ &= A(n,n) Z_{n,n} + M_2. \end{aligned}$$

If one removes the variables in the $j$th row while adding a new 0th row, an easy application of Proposition 3.14 gives that for $j \neq n$, $Z_{n,n}A(n,j) = q^{-2}A(n,j)Z_{n,n}$ and similarly for $A(j,n)$, and it then follows that

$$Z_{n,n} M_i = q^{-4} M_i Z_{n,n} \quad \text{for } i = 1, 2.$$

(This result also follows from [17, Lemma 4.5.1].) In the ring of fractions of $\mathcal{O}_q(M(n))$ we can write $A(n,n) = (Z_{n,n})^{-1}(\det_q - M_1) = (\det_q - M_2)(Z_{n,n})^{-1}$, which is useful since, by what we have just proved, the terms $M_1$, $M_2$ have simple $q$-relations with $Z_{n,n}$. Thus,

$$\begin{aligned} Z_{n,n}A(n,n) &= \det_q - M_1 = \det_q - q^{-4}M_2, \\ A(n,n)Z_{n,n} &= \det_q - q^4 M_1 = \det_q - M_2, \quad \text{and hence} \\ \big[Z_{n,n}, A(n,n)\big] &= \big(q^4 - 1\big)M_1 \\ &= \big(1 - q^{-4}\big)M_2. \end{aligned}$$

Notice that all monomials in $M_2$ contain factors of $q^{2\ell}$ with $\ell \geqslant 1$.

More generally, it follows by induction that for all $r \in \mathbb{N}$,

$$\begin{aligned} \big[Z_{n,n}^r, A(n,n)\big] &= q^4\big(1 - q^{-4r}\big)M_1 Z_{n,n}^{r-1} \\ &= \big(1 - q^{-4r}\big)M_2 Z_{n,n}^{r-1}. \end{aligned}$$

Likewise, for all $r \in \mathbb{N}$,

$$\big[Z_{1,1}^r, A(1,1)\big] = -\big(1 - q^{-4r}\big)Z_{1,1}^{r-1}N_1, \quad \text{where} \tag{3.29}$$

$$N_1 = \sum_{j=2}^{n}\big(-q^2\big)^{j-1}Z_{1,j}A(1,j) \tag{3.30}$$

$$= \sum_{\sigma \in S_n;\, \sigma(1) \neq 1}\big(-q^2\big)^{\ell(\sigma)}Z_{1,\sigma(1)}Z_{2,\sigma(2)}\cdots Z_{n,\sigma(n)}, \tag{3.31}$$

where, for each $\sigma$ in the last sum, $\ell(\sigma) \in \mathbb{N}$. This observation is an important ingredient in the proof of Theorem 4.1 below.

## 4. $\det_q$ and dual canonical bases

In this section, $n = m$ throughout. The following result is of key importance. However simple to formulate, it is remarkably difficult to prove.

**Theorem 4.1.** *For all $A \in M_n(\mathbb{Z}_+)$ there are integers $c_B \in \{-1, 0, 1\}$ and integers $\gamma_B > 0$ such that*

$$Z(A) \cdot \det_q = Z(A + I) + \sum_{B < A+I} q^{2\gamma_B}c_B Z(B), \tag{4.1}$$

*where each $B$ furthermore has the same row and column sums as $A + I$. In particular,*

$$b(A) \cdot \det_q = b(A + I) = b(A)b(I). \tag{4.2}$$

**Proof.** By using Corollary 3.8, the second claim follows easily from the first. We proceed to prove (4.1) by induction on the number $c$ such that there are non-zero elements in at most the columns $1, \ldots, c$ of $A$. For a fixed such $c$ we proceed by induction on the number $r$ such that there are non-zero elements in at most the rows $1, \ldots, r$ in the $c$th column. Notice that the formula (4.1) holds for any $A$ with non-zero entries at most in the first column. Indeed, as follows by an elementary computation,

$$Z(A)Z(E_\sigma)\left(-q^2\right)^{\ell(\sigma)} = \left(-q^2\right)^{\ell(\sigma)} q^{2(\underline{co}_1-(a_{i,1}+a_{i+1}+\cdots+a_{n,1}))} Z(A+E_\sigma), \tag{4.3}$$

where $E_\sigma$ is the matrix of the permutation $\sigma$, $i=\sigma^{-1}(1)$, and $Z(E_\sigma)=Z^{E_\sigma}$.

It is likewise easy to see that if the theorem holds for any $A$ with non-zero entries in at most the first $c$ columns $1,2,\dots,c$, then it is also true if we replace $A$ by $A+a_{1,c+1}E_{1,c+1}$ for any $a_{1,c+1}\in\mathbb{N}$. Here it suffices to observe that $Z(A+a_{1,c+1}E_{1,c+1}) = q^{a_{1,c+1}\underline{ro}_1(A)} Z^{a_{1,c+1}E_{1,c+1}} Z(A)$. When we multiply by $\det_q$ from the left, we obtain elements of the form $q^{a_{1,c+1}\underline{ro}_1(A)} Z^{a_{1,c+1}E_{1,c+1}} Z(A')$, where $\underline{ro}(A') = (1,1,\dots,1)+\underline{ro}(A)$, and similarly for the column sums. Due to the special form of $Z^{a_{1,c+1}E_{1,c+1}}Z(A)$ it is a matter of simple bookkeeping to verify the claim here.

Now let us assume that the theorem holds up to the $r$th row in the $c$th column, where $r<n$. Let $Z^{A_0}$ correspond to a matrix $A_0$ fulfilling the requirements up to, and including, row $r$ and column $c$, and consider $A=A_0+a_{r+1,c}\cdot E_{r+1,c}$.

Before getting further into the details, let us remark that the two lexicographic orderings $(1,1)>(1,2)>\cdots>(1,n)>(2,1)>\cdots$ and $(1,1)>(2,1)>\cdots>(n,1)>(2,1)>\cdots$ have the same monomials. By this we mean that if $Z^A$ is written according to one of the orderings, then rewriting it according to the other will not create auxiliary terms. Indeed, not even a different coefficient. Let us denote the former ordering by row–column and the latter by column–row.

Consider

$$Z(A)\cdot\sum_{\sigma\in S_n}\left(-q^2\right)^{\ell(\sigma)} Z_{1,\sigma(1)}Z_{2,\sigma(2)}\cdots Z_{n,\sigma(n)}.$$

Set $\alpha = (\sum_{k=1}^{c-1} a_{(r+1),c}a_{(r+1),k} + \sum_{t=1}^{r} a_{r+1,c}a_{t,c})$. Then,

$$Z(A) = Z(A_0)\cdot q^{-\alpha} Z_{r+1,c}^{a_{r+1,c}}. \tag{4.4}$$

The task now is to order each summand in

$$Z(A)\cdot\sum_{\sigma\in S_n}(-q)^{2\ell(\sigma)} Z_{1,\sigma(1)}Z_{2,\sigma(2)}\cdots Z_{n,\sigma(n)}$$

lexicographically. To do so, we will group the terms in $\det_q$ together strategically into sums of products of quantum minors. These minors will then be ordered collectively while using their $q$-commutation relations.

We can safely assume $1\leqslant r\leqslant n-1$. Decompose

$$\left\{(i,j)\in\mathbb{N}\;\middle|\;1\leqslant i,j\leqslant n\right\} = R_1\cup R_2\cup R_3\cup R_4,$$

$R_1=\{(i,j)\mid 1\leqslant i\leqslant r;\ 1\leqslant j\leqslant c-1\}$, $R_2=\{(i,j)\mid 1\leqslant i\leqslant r;\ c\leqslant j\leqslant n\}$, $R_3=\{(i,j)\mid r+1\leqslant i\leqslant n;\ 1\leqslant j\leqslant c-1\}$, and $R_4=\{(i,j)\mid r+1\leqslant i\leqslant n;\ c\leqslant j\leqslant n\}$. Consider (3.22) applied to $\det_q$ with $J_1=\{1,\dots,c-1\}$ and $J_2=\{c,\dots,n\}$. Then apply (3.23) to each $\xi^{I_i}_{J_i}$, $i=1,2$ based on a decomposition $I_i=R^{(1)}(I_i)\cup R^{(2)}(I_i)$ of $I_i$; $R^{(1)}(I_i)\subseteq\{1,\dots,r\}$, and $R^{(2)}(I_i)\subseteq\{r+1,\dots,n\}$. The result is a formula

$$\det_q = \sum_{M_1,M_2,M_3,M_4} c_{M_1,M_2,M_3,M_4} M_1M_2M_3M_4, \tag{4.5}$$

where each $c_{M_1,M_2,M_3,M_4}$ is $\pm(q^2)^p$ for some non-negative integer $p$, and each $M_i$, $i=1,2,3,4$, is a quantum minor with entries from $R_i$, $i=1,\dots,4$. Notice that it follows from the defining relations that $M_2M_3=M_3M_2$. Not all combinations of quantum minors will occur with a non-zero coefficient, of course. For instance, no pair can share a row or a column.

Let $\mathcal{R}_{1,2,3}$ denote the set of matrices over $\mathbb{Z}_+$ with non-zero entries at most in the positions of $R_1\cup R_2\cup R_3$, let $\mathcal{R}_4$ denote the set of matrices with non-zero entries at most in the positions of $R_4$, and let $\mathcal{M}_4$ denote the set of quantum minors having entries from $R_4$.

It follows that

$$\det_q = \sum_{M_4 \in \mathcal{M}_4} \sum_{G \in \mathcal{R}_{1,2,3}} P_{G,M_4} Z^G M_4, \tag{4.6}$$

where each $P_{G,M_4} \in \mathbb{Z}[q^2]$. Then, because reordering elements from $\mathcal{R}_{1,2,3}$ does not introduce terms from $\mathcal{R}_4$,

$$Z(A_0) \det_q = \sum_{M_4 \in \mathcal{M}_4} \sum_{H \in \mathcal{R}_{1,2,3}} \hat{P}_{H,M_4} Z^H M_4 \tag{4.7}$$

for some elements $\hat{P}_{H,M_4} \in \mathbb{Z}[q^2, q^{-2}]$. At the same time, by the induction hypothesis,

$$Z(A_0) \det_q = \sum_{L \in \mathcal{R}_4} \sum_{K \in \mathcal{R}_{1,2,3}} \tilde{c}_{L,K} Z(K+L), \tag{4.8}$$

where $\tilde{c}_{L,K} = 1$ for the unique configuration corresponding to $Z(A_0 + I)$ and in all other cases, if non-zero, $\tilde{c}_{L,K} = \pm q^{2\gamma_{K,L}}$ where $\gamma_{K,L} \in \mathbb{N}$. Here, each $K+L$ has the same row and column sums as $A_0 + I$, and $K + L \leqslant A_0 + I$. The expression $Z^H M_4$ in (4.7) is a sum of monomials $\pm q^{2p_i} Z^H Z^{S_{4,i}}$ corresponding to $M_4 = \sum_i \pm q^{2p_i} Z^{S_{4,i}}$ as a quantum minor. Furthermore, $\forall i$: $p_i \in \mathbb{Z}_+$ and $p_i \in \mathbb{N}$ for all but one $i$. Due to the configurations, the normalization factors $N(H + S_{4,i})$ are easily seen to be independent of $i$ and may thus for instance be computed for the unique $i$ for which $p_i = 0$. Thus, $N(H + S_{4,i}) = N(H) q^{-\underline{ro}(H;M_4) - \underline{co}(H;M_4)}$. The symbols $\underline{ro}(H; M_4)$ and $\underline{co}(H; M_4)$ denote the row sums, respectively column sums, of $H$ corresponding to the rows, respectively columns, of $M_4$. Thus, for each $M_4$ in (4.7) we get a sum of the form $\pm \hat{P}_{H,M_4} N(H)^{-1} q^{\underline{ro}(H;M_4) + \underline{co}(H;M_4)} q^{2p_i} Z(H + S_{4,i})$. Each of these terms must correspond to a term in (4.8) where we have full information about the positivity of the powers of $q$. Notice that $\pm \hat{P}_{H,M_4}$ is independent of $i$. Indeed, $\hat{P}_{H,M_4} = c_{H,M_4} q^{2p_{H,M_4}} q^{-\underline{ro}(H;M_4) - \underline{co}(H;M_4)} N(H)$ for some constant $c_{H,M_4} \in \{-1, 0, 1\}$ and some element $p_{H,M_4} \in \mathbb{Z}_+$. It follows from this that we have a formula

$$Z(A_0) \det_q = \sum_{M_4 \in \mathcal{M}_4} \sum_{H \in \mathcal{R}_{1,2,3}} c_{H,M_4} q^{2p_{H,M_4}} q^{-\underline{ro}(H;M_4) - \underline{co}(H;M_4)} Z(H) M_4. \tag{4.9}$$

Each summand in $q^{-\underline{ro}(H;M_4) - \underline{co}(H;M_4)} Z(H) M_4$ is normalized. With the exception of one pair $(H_s, M_{4,s})$ where $p_{H_s, M_{4,s}} = 0$, we have furthermore that $p_{H,M_4} \in \mathbb{N}$ when $c_{H,M_4} \neq 0$.

Observing that $\det_q$ is central, we can insert it in any position we prefer. Returning to (4.4) we will therefore consider $Z(A_0) \cdot \det_q \cdot q^{-\alpha} Z_{r+1,c}^{a_{r+1,c}}$. In view of (4.9) we need to focus on the rewriting of expressions of the form $M_4 Z_{r+1,c}^{a_{r+1,c}}$ and, in particular, to carefully keep track of the $q$ factors we pick up. This is the only place where negative exponents might originate.

Four different situations may occur:

(1) $M_4$ has all row numbers greater than $r+1$ and all column numbers greater than $c$.
(2) $M_4$ has all row numbers greater than $r+1$ but a column number equal to $c$.
(3) $M_4$ has a row number equal to $r+1$ but all column numbers greater than $c$.
(4) $Z_{r+1,c}$ occurs in $M_4$ (and hence commutes with $M_4$).

These cases can be dealt with, and this is, indeed, the reason for the chosen decomposition: In all cases, what has to be considered are the terms

$$(\star) = c_{H,M_4} q^{2p_{H,M_4}} q^{-\underline{ro}(H;M_4) - \underline{co}(H;M_4)} Z(H) M_4 q^{-\alpha} Z_{r+1,c}^{a}. \tag{4.10}$$

In cases (2), (3), and (4), $M_4$ quasi-commutes with $Z^a_{r+1,c}$, and $(\star)$ becomes one of the following:

$$(\star)_2 = (\star)_3 = c_{H,M_4} q^{2p_{H,M_4}} q^{-\underline{ro}(H;M_4)-\underline{co}(H;M_4)} q^{-\alpha} q^{-2a} Z(H) Z^a_{r+1,c} M_4,$$
$$(\star)_4 = c_{H,M_4} q^{2p_{H,M_4}} q^{-\underline{ro}(H;M_4)-\underline{co}(H;M_4)} q^{-\alpha} Z(H) Z^a_{r+1,c} M_4.$$

The quantum minor $M_4$ is a linear combination of monomials with coefficients which are non-negative powers of $(-q^2)$. These powers are no problem (they just become part of the coefficients $q^{2\gamma_B} c_B$ in (4.1)), so what need to be dealt with are the expressions resulting from replacing $M_4$ in $(\star)_2$, $(\star)_3$, and $(\star)_4$ by any of the monomials $A$ from $M_4$.

As already noted, the non-zero positions in $Z^a_{r+1,c}A$ are ordered correctly according to the lexicographic ordering, so $Z^a_{r+1,c}A = Z^B$ for some $B \in M_n(\mathbb{Z}_+)$.

In case (2), $N(B) = q^{-a}$, so $Z^a_{r+1,c}A = q^a Z(B)$ and the term from $(\star)_2$ we need to analyze is

$$(\star\star)_2 = c_{H,M_4} q^{2p_{H,M_4}} q^{-\underline{ro}(H;M_4)-\underline{co}(H;M_4)} q^{-\alpha} q^{-a} Z(H) Z(B). \tag{4.11}$$

There are no positions where $H$ and $B$ both have non-zero entries. Moreover, if $H$ and $B$ have non-zero entries in the respective positions $(i,j)$ and $(k,l)$ with $(k,l) > (i,j)$, then the corresponding powers $Z^h_{ij}$ in $Z^H$ and $Z^b_{kl}$ in $Z^B$ commute. This means $Z^H Z^B = Z^C$, where $C = H + B$. Observe that

$$N(C) = N(H)N(B) q^{-\underline{ro}(H;M_4)-\underline{co}(H;M_4)} q^{-a\underline{ro}_{r+1}(H)-a\underline{co}_c(H)}. \tag{4.12}$$

The matrix $H$ consists of $A_0$ plus the complementary part of a permutation matrix whose other part is a permutation matrix corresponding to a term in $M_4$. So, $\underline{ro}_i(H) = \underline{ro}_i(A_0)$ for rows $i$ that occur in $M_4$ and $\underline{ro}_i(H) = \underline{ro}_i(A_0) + 1$ otherwise. Similarly for the columns. In case (2), this means $\underline{ro}_{r+1}(H) = \underline{ro}_{r+1}(A_0) + 1$ and $\underline{co}_c(H) = \underline{co}_c(A_0)$. Now, $a\underline{ro}_{r+1}(H) + a\underline{co}_c(H) = \alpha + a$, so

$$N(C) = N(H)N(B) q^{-\alpha-a-\underline{ro}(H;M_4)-\underline{co}(H;M_4)} \tag{4.13}$$

and $Z(C) = q^{-\alpha-a-\underline{ro}(H;M_4)-\underline{co}(H;M_4)} Z(H)Z(B)$. Therefore

$$(\star\star)_2 = c_{H,M_4} q^{p_{H,M_4}} Z(C), \tag{4.14}$$

the correct form for the right hand side of (4.1).

Case (3) is symmetric.

In case (4), $N(B)$ is either 1 or $q^{-2a}$, depending on whether or not $A$ has a 1 in position $(r+1,c)$. This time, $\underline{ro}_{r+1}(H) = \underline{ro}_{r+1}(A_0)$ and $\underline{co}_c(H) = \underline{co}_c(A_0)$, so $a\underline{ro}_{r+1}(H) + a\underline{co}_c(H) = \alpha$ and we end up with, setting $C = H + B$,

$$(\star\star)_4 = c_{H,M_4} q^{2p_{H,M_4}} \cdot \left[1 \text{ or } q^{4a}\right] \cdot Z(C), \tag{4.15}$$

in the correct form.

In case (1) we have a reinterpretation of (3.29):

$$\left[Z^a_{r+1,c}, M_4\right] = -q^2\left(1 - q^{-4a}\right) Z^{a-1}_{r+1,c} T, \tag{4.16}$$

where $Z_{r+1,c}T = q^4 T Z_{r+1,c}$, and $a = a_{r+1,c}$. The factor $T$ will be discussed shortly.

Notice that the left hand side of (4.16) evidently is skew under the bar operator. Thus, it follows that $\overline{Z^{a-1}_{r+1,c}T} = q^{-4a+4} Z^{a-1}_{r+1,c}T$. We have $T = q^{-2}N_1$ in terms of (3.29), so each monomial $Z^{Y_i} = Z(Y_i)$ in $T$ occurs with a factor $\pm q^{2p_i}$ with $p_i \in \mathbb{Z}_+$. However, we must utilize even finer details of $T$.

Specifically, we may assume that the monomial summands of $T$ each have a contribution $Z_{x,c}$ with $x > r+1$ and a contribution $Z_{r+1,y}$ with $y > c$. Furthermore, $T$ is ordered according to the lexicographic ordering column–row and a factor of $q^2$ is taken out of the original determinantal expression which involves expressions $(-q^2)^\ell$, where $\ell \geqslant 1$. It is clearly the term with $q^{2-4a}$ we must be able to handle. Before addressing this, we remark that the term $Z^a_{r+1,c}M_4$ from the commutator is handled by the same argument as in cases (2), (3), and (4).

We know from the construction that each $K+L$ in (4.8), appearing with a non-zero coefficient, compared to $A_0$ has an additional element in each row and column coming from the various summands in the determinant. With the given $M_4$ we then know that the extra element $W_{r+1,u}$ in the $(r+1)$th row must have $u < c$ and the extra element $W_{v,c}$ in the $c$th column must have $v < r+1$.

The above observations easily imply that

$$\begin{aligned}\overline{Z(H)Z^{a-1}_{r+1,c}T} &= q^{-4a+4}Z^{a-1}_{r+1,c}T\,\overline{Z(H)}\\ &= q^{-4a+4}q^{-2\alpha}q^{-4a}q^{-2\underline{ro}(H;M_4)-2\underline{co}(H;M_4)}Z(H)Z^{a-1}_{r+1,c}T\\ &\quad + \text{lower order terms.}\end{aligned} \tag{4.17}$$

The term we have to control is

$$X = q^2q^{-4a}q^{-\alpha}q^{-\underline{ro}(H;M_4)-\underline{co}(H;M_4)}Z(H)Z^{a-1}_{r+1,c}T.$$

Eq. (4.17) implies that the term in $X$ coming from the leading term in $T$ is normalized. The other terms are then positive powers of $q^2$ times normalized elements as follows by arguments similar to those for the cases (2), (3), and (4).

This completes the proof. □

## 5. Covariant minors and the dual canonical basis

Let us consider an $n \times n$ matrix $X \in M_n(\mathbb{Z}_+)$ decomposed into

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{5.1}$$

We assume furthermore that $C$ is square of size $s$. We denote the $s \times s$ quantum minor corresponding to the lower left corner by $I_{s,ll}$.

**Lemma 5.1.** *Let $b(X)$ be an element of the dual canonical basis with $X$ given as in (5.1). Then*

$$b(X)I_{s,ll} = q^{(S(A)-S(D))}b(\tilde{X}), \tag{5.2}$$

*with $\tilde{X} = \left(\begin{smallmatrix} A & B \\ C+I_s & D \end{smallmatrix}\right)$. Here, $I_s$ is the $s \times s$ identity matrix, while $S(A)$ and $S(D)$ denote the sum of all entries in $A$ and $D$, respectively.*

**Proof.** It is easy to see that $Z^X = Z^AZ^BZ^CZ^D$. Suppose then, by Proposition 3.7, that

$$b(X) = Z(X) + \sum_{X' < X} c_{X'}(X)Z\big(X'\big), \tag{5.3}$$

with $c_{X'}(X) \in q^2\mathbb{Z}[q^2]$. Furthermore, each $X'$ has the same row and column sums as $X$. Here we set $X' = \left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right)$ and then

$$Z(X') = C_{A',B',C',D'} Z^{A'} Z^{B'} Z^{C'} Z^{D'}$$

where

$$C_{A',B',C',D'} = N(A')N(B')N(C')N(D')N(A',B')N(A',C')N(B',D')N(C',D').$$

The factors $N(A')$, $N(B')$, $N(C')$, and $N(D')$ are given by (3.12) as are the factors of mixed summands; $N(A',B') = q^{-\sum_j \underline{ro}_j(A')\underline{ro}_j(B')}$, $N(A',C') = q^{-\sum_j \underline{co}_j(A')\underline{co}_j(C')}$, $N(B',D')$, and $N(C',D')$ (there will be no non-trivial factors $N(A',D')$ and $N(B',C')$).

Looking at row sums we have $S(A') + S(B') = S(A) + S(B)$, and looking at column sums we have $S(B') + S(D') = S(B) + S(D)$. Thus, for the matrices in the right hand side of (5.3) we have $-S(A') + S(D') = -S(A) + S(D)$. Then,

$$q^{(-S(A)+S(D))} b(X) I_{s,ll} = \sum_{A',B',C',D'} q^{(-S(A')+S(D'))} c_{X'}(X) C_{A',B',C',D'} Z^{A'} Z^{B'} Z^{C'} Z^{D'} I_{s,ll}. \tag{5.4}$$

It follows from Proposition 3.14 (a transposed version thereof) that

$$I_{s,ll} Z^{A'} Z^{B'} Z^{C'} Z^{D'} = q^{-2S(A)+2S(D)} Z^{A'} Z^{B'} Z^{C'} Z^{D'} I_{s,ll}.$$

Thus, the left hand side, and hence both sides of (5.4) are bar invariant. Now consider a term in (5.4) of the form

$$\begin{aligned} &q^{(-S(A')+S(D'))} C_{A',B',C',D'} Z^{A'} Z^{B'} Z^{C'} Z^{D'} I_{s,ll} \\ &\quad = q^{(-S(A')-S(D'))} C_{A',B',C',D'} Z^{A'} Z^{B'} Z^{C'} I_{s,ll} Z^{D'}. \end{aligned}$$

Here, $Z(C') I_{s,ll} = N(C') Z^{C'} I_{s,ll}$. By Theorem 4.1 this equals $Z(C' + I_s) + \sum_{C'' < C' + I_s} f_{C''} Z(C'')$ and for all $C''$, $f_{C''}$ a polynomial in $q^2\mathbb{Z}[q^2]$. Notice that each $C''$ has the same row and column sums as $C' + I_s$ and that $\forall i$: $\underline{ro}_i(C' + I_s) = \underline{ro}_i(C') + 1$ and, similarly, $\forall j$: $\underline{co}_j(C' + I_s) = \underline{co}_j(C') + 1$. But then $q^{-S(A')-S(D')} N(A',C')N(C',D') = N(A',C'')N(C'',D)$. Thus, the right hand side of (5.4) is a sum of terms $g_i Z(Y_i)$ with $Y_i = \left(\begin{smallmatrix} A' & B' \\ C'' & D' \end{smallmatrix}\right)$ and $g_i \in \mathbb{Z}[q^2]$. Precisely the term with $A' = A$, $B' = B$, $C' = C$, and $D' = D$ has a coefficient $g_i = 1$, all other coefficients are in $q^2\mathbb{Z}[q^2]$. Thus the right hand side has the right expansion properties, hence is a member of the dual canonical basis corresponding to the stated element $\tilde{X}$. □

Let us instead consider an $n \times n$ matrix $X \in M_n(\mathbb{Z}_+)$ decomposed into

$$X = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}, \tag{5.5}$$

where we now assume that $D$ is square of size $s$. We denote the $s \times s$ quantum minor corresponding to the lower right hand corner (as occupied by $D$) by $I_{s,lr}$.

**Lemma 5.2.** *Let $b(X)$ be an element of the dual canonical basis with $X$ given as in (5.5). Then*

$$b(X) I_{s,lr} = q^{(S(B)+S(C))} b(\tilde{X}), \tag{5.6}$$

*with $\tilde{X} = \left(\begin{smallmatrix} 0 & B \\ C & D+I_s \end{smallmatrix}\right)$. As before, $I_s$ is the $s \times s$ identity matrix, while $S(B)$ and $S(C)$ denote the sum of all entries in $B$ and $C$, respectively.*

The proof follows the same lines as that of Lemma 5.1 and is omitted. By Proposition 3.9 the following case encompasses the two former. The proof is omitted for similar reasons.

**Lemma 5.3.** *Let $b(X)$ be an element of the dual canonical basis with*

$$X = \begin{pmatrix} 0 & B_1 & B_2 \\ C_1 & D & C_2 \\ G_1 & G_2 & 0 \end{pmatrix}$$

*where $D$ is $s \times s$. Then, if $I_{s,cc}$ denotes the $s \times s$ quantum minor corresponding to the position of $D$,*

$$b(X) I_{s,cc} = q^{S(B_1)+S(C_1)-S(C_2)-S(G_2)} b(\tilde{X}), \tag{5.7}$$

*with $\tilde{X} = \begin{pmatrix} 0 & B_1 & B_2 \\ C_1 & D+I_s & C_2 \\ G_1 & G_2 & 0 \end{pmatrix}$. As before, $I_s$ is the $s \times s$ identity matrix.*

**Remark 5.4.** Using Proposition 3.9 it follows that analogous results hold for the configurations

$$X = \begin{pmatrix} C_1 & D & C_2 \\ G_1 & G_2 & 0 \end{pmatrix}, \qquad \tilde{X} = \begin{pmatrix} C_1 & D + I_s & C_2 \\ G_1 & G_2 & 0 \end{pmatrix}$$

and

$$X = \begin{pmatrix} 0 & B_2 \\ D & C_2 \\ G_1 & G_2 \end{pmatrix}, \qquad \tilde{X} = \begin{pmatrix} 0 & B_2 \\ D + I_s & C_2 \\ G_1 & G_2 \end{pmatrix},$$

in which the matrix $X$ is not necessarily square. Similarly, the transposed cases, where the 0 matrix is in the opposite corner, are covered.

## 6. Broken line constructions

Consider the $m \times n$ quantum matrix algebra $\mathcal{O}_q(M(m,n))$. In this section, all elements $Z_{i,j}$ and all quantum minors are elements of this algebra.

**Definition 6.1. A broken line in** $M_{m,n}(\mathbb{Z}_+)$ is a path in $\mathbb{N} \times \mathbb{N}$ starting at $(1,n)$ and terminating at $(m,1)$. We will occasionally also refer to this as a broken line from $(1,n)$ to $(m,1)$. It must satisfy furthermore that it alternates between horizontal and vertical segments while passing through smaller column numbers (in the horizontal direction) and bigger row numbers (in the vertical direction).

Unless we are in the extreme cases $(1,n) \mapsto (1,1) \mapsto (m,1)$ or $(1,n) \mapsto (m,n) \mapsto (m,1)$, this will divide the indices $(i,j)$ into 3 disjoint sets $S_L$, $L$, and $T_L$. Here, $S_L$ is the set of points above the line (when there are 3 subsets, we have that $(1,1)$ is above the line), $L$ is the line itself, and $T_L$ is the set of points below the line.

**Remark 6.2.** A broken line is determined by a double-partition

$$1 = i_1 \leqslant i_2 \leqslant i_3 \leqslant \cdots \leqslant i_s = m \quad \text{and} \quad n = j_1 \geqslant j_2 \geqslant j_3 \geqslant \cdots \geqslant j_s = 1,$$

such that the corners in the line $L$ are $(i_t, j_t)$; $t = 1, 2, \ldots, s$. This, naturally, dictates that in the partitions, precisely every second inequality is sharp. Furthermore, if in a given position, one is sharp, then the other is not, and vice versa.

In a similar vein, the broken line is given by a double flag variety.

For a given broken line $L$, we now construct a family $\mathcal{V}_L$ with $mn$ elements consisting of certain quantum minors: (It will be proved below that all members $q$-commute.) For points in $(i,j) \in T_L \cup L$ we take the biggest quantum minor having its bottom right corner in $(i,j)$ and completely contained in $T_L \cup L$. One can also say that it is the biggest quantum minor consisting of adjacent rows and columns (we call such a quantum minor **solid**) and which contains $(i,j)$ as well as points from $L$ but no points from $S_L$. The line $L$ is thus represented by points, that is, $1 \times 1$ matrices. For the points in $S_L$ we do something else: For $(i,j) \in S_L$ we take the biggest quantum minor consisting of adjacent rows and columns and which contains $(i,j)$ in the upper left corner (all other rows have numbers bigger than $i$ and all columns have numbers bigger than $j$). Notice that with $L$ fixed, each quantum minor in $\mathcal{V}_L$ corresponds uniquely to a point $(i,j)$. By *the quantum minor corresponding to a point* we then mean this quantum minor.

The first important observation is:

**Proposition 6.3.** *Any quantum minor corresponding to a point in $S_L$ $q$-commutes with any $Z_{i,j}$ for which $(i,j) \notin S_L$.*

This follows immediately from Proposition 3.14.

**Proposition 6.4.** *Let $M = M_{a,b}(k)$ be a $k \times k$ quantum minor with upper left corner in $(a,b)$ and lower right hand corner in $(a+k-1, b+k-1)$, and such that $M$ is inside the $m \times n$ quantum matrices. Refer to the* 9 *different positions of a pair $(i,j)$ relative to $M$ as $NW(M), N(M), NE(M), \dots, I(M), \dots, SE(M)$ such that $NW(M)$ is $(a > i$ and $b > j)$ and $SE(M)$ is $(i > a+k-1$ and $j > b+k-1)$. Here $I(M)$ denotes the position of the indices of $M$. Then $Z_{i,j}$ $q$-commutes with $M$ unless $(i,j)$ is in $NW(M)$ or in $SE(M)$. For the remaining pairs, in the $q$-commutation formulas $Z_{i,j}M = q^{2p_{i,j}}MZ_{i,j}$, $p_{i,j}$ depends only on the relative positions. Indeed, $p_{i,j} = 1$ for $(i,j)$ in $W(M) \cup N(M)$, $p_{i,j} = 0$ for $(i,j)$ in $I(M) \cup SW(M) \cup NE(M)$, and $p_{i,j} = -1$ for $(i,j)$ in $S(M) \cup E(M)$.*

**Proof.** With the exception of $NW(M)$ and $SE(M)$, the $q$-commutation relation may be seen as taking place inside a smaller matrix algebra in which $M$ is a covariant quantum minor. □

**Proposition 6.5.** *All members of $\mathcal{V}_L$ $q$-commute.*

**Proof.** Let $A, B \in \mathcal{V}_L$. Let $\mathcal{E}(A)$ denote the entries of $A$. It follows by inspection that, after possibly interchanging $A$ and $B$, only the following cases may occur: $\mathcal{E}(A) \subseteq S(B) \cup SW(B) \cup W(B) \cup I(B)$, $\mathcal{E}(A) \subseteq N(B) \cup NE(B) \cup E(B) \cup I(B)$, or $\mathcal{E}(B) \subseteq \mathcal{E}(A)$. Some situations involving fewer sets like $\mathcal{E}(A) \subseteq S(B) \cup I(B)$ or $\mathcal{E}(A) \subseteq S(B)$ may also occur, while others may be prohibited due to the configuration at hand. All non-trivial cases are treated in the same way and it suffices to consider the very first of these. Let $B$ be fixed and consider the expansion of $A$ into a linear combination of monomials of the form $Z_{i_1+1, j_1+\sigma(1)} \cdots Z_{i_1+r, j_1+\sigma(r)}$ for some $\sigma \in S_r$. Using Proposition 6.3 and Proposition 6.4 we obtain the following: If $W_\sigma(B)$ and $S_\sigma(B)$ denote the number of terms $Z_{i_1+i, \sigma(i_1)+i}$ to the west, respectively to the south, of $B$, the given monomial will $q$-commute with $B$ with a factor $q^{2(W_\sigma(B) - S_\sigma(B))}$. It is easily seen that $W_\sigma(B) - S_\sigma(B)$ is independent of $\sigma$, and thus the claim follows. □

This result also follows from [18, Theorem 1].

**Remark 6.6.** One gets a similar family by interchanging $S_L$ and $T_L$. Colloquially speaking, if one allows $L$ to vary, this can be accomplished by a reflection mapping $m \times n$ matrices to $n \times m$ matrices while interchanging rows and columns.

**Definition 6.7.** We introduce a partial ordering of the broken lines:

$$L_1 \leqslant L_2 \quad \Leftrightarrow \quad S_{L_2} \subseteq S_{L_1}.$$

In this ordering, the line $L^+$ corresponding to the empty set: $(1,n) \to (1,1) \to (m,1)$, is the unique maximal element, and the line $L^-$ corresponding to $T_L = \emptyset$: $(1,n) \to (m,n) \to (m,1)$, is the unique smallest element.

In the extreme case of $L^+$, the $q$-commuting quantum minors in the corresponding family are the following:

(1) For $i \geqslant j$, $\xi^{\{i-j+1,i-j+2,\dots,i\}}_{\{1,2,\dots,j\}}$.
(2) For $j > i$, $\xi^{\{1,2,\dots,i\}}_{\{j-i+1,j-i+2,\dots,j\}}$.

In the sequel, we shall consider the following more general family $\mathcal{V}_{L_1,L_2} \subset \mathcal{V}_{L_1}$ where, clearly, $\mathcal{V}_L = \mathcal{V}_{L,L^+}$:

**Definition 6.8.** Let $L_1$, $L_2$ be broken lines with $L_1 \leqslant L_2$. The family $\mathcal{V}_{L_1,L_2}$ is the subfamily of $\mathcal{V}_{L_1}$ that corresponds to the points in $T_{L_2} \cup L_2$.

**Definition 6.9.** Given a broken line $L$, let $\mathcal{O}_q(T_L \cup L)$ denote the sub-algebra of $\mathcal{O}_q(M(m,n))$ generated by the $Z_{i,j}$ for which $(i,j) \in T_L \cup L$. We will refer to the members of $\mathcal{V}_L$ as *variables*. Let $\mathcal{V}_L^-$ denote the set of variables in $\mathcal{V}_L$ corresponding to the points in $L \cup T_L$, and let $\mathcal{C}_L^-$ denote the set of variables in $\mathcal{V}_L$ for the points in $L^-$ $(\subseteq (T_L \cup L))$. Analogously, let $\mathcal{V}_L^+$ denote the set of variables in $\mathcal{V}_L$ corresponding to the points in $S_L$.

In the following we will consider cluster algebra constructions inside an ambient space which is either (i) the skew field of fractions $\mathcal{F}_L$ constructed from $\mathcal{O}_q(M(m,n))$ (or, equivalently, $\mathcal{V}_L$) and where $\mathcal{V}_L$ is part of an initial seed or (ii) the skew field of fractions $\mathcal{F}_L^-$ constructed from $\mathcal{O}_q(T_L \cup L)$ (or, equivalently, $\mathcal{V}_L^-$) and where $\mathcal{V}_L^-$ is part of an initial seed.

Proposition 6.3 can be stated as the fact that any variable in $\mathcal{V}_L^+$ is covariant with respect to the full sub-algebra $\mathcal{O}_q(T_L \cup L)$. The algebra $\mathcal{O}_q(T_L \cup L)$ has previously been studied in [14, Section 3].

**Theorem 6.10.** *Let $\mathcal{V}_L$ denote the family of mn q-commuting quantum minors constructed from a broken line as above. Then up to multiplication by a power of q, any monomial in the members of $\mathcal{V}_L$ is a member of the dual canonical basis.*

**Proof.** The main tool is Lemma 5.3, but Proposition 3.9 is also important, cf. the remark following Lemma 5.3. Consider then a monomial. Rewrite it, if necessary, in such a way that the factors coming from the points on $L$ are furthest to the left. Then place to the right of these the $2 \times 2$-minors corresponding to the points one step below the line. Continue in this way until all the factors corresponding to the points on, or below, the line are positioned. While continuing to add from the right, order the factors coming from $S_L$ in a similar fashion and such that the factor corresponding to the position $(1,1)$ is furthest to the right. The finer order is not important. We view the monomial as the result of a sequence of multiplications from the right by minors according to this ordering. Inductively, we may at each step $r$ in the sequence assume that what we are multiplying the minor onto is some $q^{2p_r}b(X_r)$. The start is clearly trivial. When we add minors below the line, $X_r$ is all the time of the form as given in Lemma 5.3. After that the zero in the lower right corner disappears and we apply Remark 5.4 instead. The result follows. □

**Definition 6.11.** For a given line $L$, we say that the line $L_1$ is a closest bigger line to $L$ if $L < L_1$ and there is no other line $L_2$ such that $L < L_2 < L_1$. In this case, if $L = (1,n) \to \cdots \to (f,d) \to (c,d) \to (c,g) \to \cdots \to (m,1)$, then $L_1 = (1,n) \to \cdots \to (f,d) \to (c-1,d) \to (c-1,d-1) \to (c,d-1) \to (c,g) \to \cdots \to (m,1)$ for some such "corner" $(f,d) \to (c,d) \to (c,g)$, where we, naturally, also allow $f = c-1$ and $g = d-1$.

We will call the given corner of $L$ **convex** and the resulting corner of $L_1$ **concave**. We will also write

$$L_1 = L \uparrow (c,d) \quad \text{or} \quad L = L_1 \downarrow (c-1, d-1).$$

*6.1. Key technical results*

Focus on a position $(i_0, j_0)$ inside the quantum matrix algebra $\mathcal{O}_q(M(n_0, r_0))$. Consider the sub-algebra $M = \mathcal{O}_q^{i_0,j_0}(M(s))$ generated by the variables $Z_{i_0+a,j_0+b}$ with $0 \leqslant a, b \leqslant s-1$ where $s$ is the biggest positive integer such that $Z_{i_0+s-1,j_0+s-1} \in \mathcal{O}_q(M(n_0, r_0))$. Naturally, this sub-algebra is isomorphic to $\mathcal{O}_q(M(s))$ in which we number the rows and columns as $0, 1, \ldots, s-1$. Assume $s \geqslant 2$. Inside $M$ are the quantum minors $Y_r = Y_r^{(s-2)} = \xi_{\{1,\ldots,s-1\}}^{\{0,1,\ldots,s-2\}}$, $Y_l = Y_l^{(s-2)} = \xi_{\{0,\ldots,s-2\}}^{\{1,\ldots,s-1\}}$, $X_o = X_o^{(s-2)} = \xi_{\{1,\ldots,s-2\}}^{\{1,\ldots,s-2\}}$, $X_t = X_t^{(s-2)} = \xi_{\{0,\ldots,s-2\}}^{\{0,\ldots,s-2\}}$, $X_b = X_b^{(s-2)} = \xi_{\{1,\ldots,s-1\}}^{\{1,\ldots,s-1\}}$, and $D = D^{(s-2)} = \xi_{\{0,1,\ldots,s-1\}}^{\{0,1,\ldots,s-1\}}$. The last is just the full quantum determinant in $M$. $X_o^{(0)}$ is defined as the constant 1.

**Definition 6.12.** A set $\{X_t, X_b, D, X_o, Y_l, Y_r\} \subset \mathcal{O}_q(M(n_0, r_0))$ whose elements are given as above for some $i_0$, $j_0$, $s \in \mathbb{N}$ will be called an $\mathcal{M}$-set.

We have the following facts which follow by direct computation:

**Lemma 6.13.** *The elements $Y_\ell$ and $Y_r$ commute. The elements $D$, $X_o$, and $X_b$ commute, and we have*

$$\overline{X_o D X_b^{-1}} = X_o D X_b^{-1} \quad \text{and} \quad \overline{q^2 Y_r Y_\ell X_b^{-1}} = q^2 Y_r Y_\ell X_b^{-1}. \tag{6.1}$$

**Corollary 6.14.** *The element $X_o^{-1} D^{-1} Y_r Y_l$ commutes with all elements $Z_{i,j} \in M$ with the exception of $Z_{i_0,j_0}$ and $Z_{i_0+s-1,j_0+s-1}$. In particular, it commutes with the quantum minors $X_o^{(a)} = \xi_{\{1,\ldots,a\}}^{\{1,\ldots,a\}}$ for $a = 1, \ldots, s-2$. It $q$-commutes with the quantum minors $X_b$, and $X_t$ according to*

$$\begin{aligned} X_b X_o^{-1} D^{-1} Y_r Y_l &= q^{-4} X_o^{-1} D^{-1} Y_r Y_l X_b \quad \text{and} \\ X_t X_o^{-1} D^{-1} Y_r Y_l &= q^{4} X_o^{-1} D^{-1} Y_r Y_l X_t. \end{aligned} \tag{6.2}$$

The following result will allow us to construct the $B$ matrices of the compatible pairs. It follows by easy computation.

**Corollary 6.15.** *Let $\Lambda$ be defined as the $\Lambda$-matrix of Laurent quasi-polynomial algebra generated by the variables $X_b$, $X_o$, $D$, $Y_r$, $Y_l$ and their inverses. Then,*

$$\Lambda \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The following was proved by Parshall and Wang in [17, Theorem 5.2.1] but is also a special case of [7, Theorem 6.2].

**Proposition 6.16** *(Parshall and Wang).*

$$X_t X_b - X_b X_t = \left(q^2 - q^{-2}\right) Y_r Y_l.$$

We wish to strengthen this result considerably, namely to the following equation which will play an important role later when we consider the quantum mutations in certain directions.

**Theorem 6.17.**

$$X_tX_b = X_oD + q^2Y_rY_l.$$

**Proof.** We first observe that by Theorem 4.1 and Lemma 5.1, $X_oD$ and $Y_rY_l$ are members of the dual canonical basis; $X_oD = b(A_1)$ and $Y_rY_l = b(A_2)$ for some specific matrices $A_1$, $A_2$ with $A_2 < A_1$. Notice that when cut down to the $s \times s$ block where it is located,

$$A_1 = \begin{pmatrix} 1 & & \cdots & & 0 \\ & 2 & & & \\ \vdots & & \ddots & & \vdots \\ & & & 2 & \\ 0 & & \cdots & & 1 \end{pmatrix}. \tag{6.3}$$

We consider the expansion of $X_tX_b$ onto the dual canonical basis;

$$X_tX_b = \sum_i c_i(q)b(C_i). \tag{6.4}$$

The coefficients $c_i$ are in $\mathbb{Z}[q^2, q^{-2}]$. Since $X_tX_b = Z^{A_1}$ plus lower order terms, the leading term must be $b(A_1)$ with coefficient 1. If we can prove that the other coefficients actually are polynomials in $q^2$ without constant term then the proof follows from Proposition 6.16. We proceed to prove this: First we expand

$$X_t = Z_{0,0}X_o + \sum_j \left(-q^2\right)^{\ell_{\sigma_j}} Z(C_{\sigma_j}), \tag{6.5}$$

where the powers of $q^2$ are strictly positive and where $Z(C_{\sigma_j})$ is a normalized monomial without contribution from $Z_{0,0}$. Thus, $Z(C_{\sigma_j}) = b(C_{\sigma_j}) + \sum_i \ell_i$, where the terms $\ell_i$ are of lower order and have coefficients in $q^2\mathbb{Z}[q^2]$. According to Lemma 5.2,

$$\left(\sum_j \left(-q^2\right)^{\ell_{\sigma_j}} Z(C_{\sigma_j})\right) X_b \tag{6.6}$$

then fulfills the requested condition. It remains to consider $Z_{0,0}X_oX_b$. But here we notice that $X_oX_b$ again is a member of the dual canonical basis; $X_oX_b = b(A_3)$ with $A_3$, appropriately cut down, given by

$$A_3 = \begin{pmatrix} 0 & & \cdots & & 0 \\ & 2 & & & \\ \vdots & & \ddots & & \vdots \\ & & & 2 & \\ 0 & & \cdots & & 1 \end{pmatrix}. \tag{6.7}$$

This matrix is without contributions from the first row and the first column. Now notice that

$$b(A_3) = Z(A_3) + q^2 \sum_{G_k < A_3} d_k(q) Z(G_k), \tag{6.8}$$

where the coefficients are in $\mathbb{Z}[q^2]$ and the first row and column in each $G_k$ are zero. It then follows from the above remarks that

$$Z_{0,0} b(A_3) = Z(A_3 + E_{0,0}) + q^2 \sum_{G_k < A_3} d_k(q) Z(G_k + E_{0,0}). \tag{6.9}$$

Expanding the right hand side in terms of the canonical basis, we get the requested result about the coefficients $c_i$ since the basis change matrix is lower diagonal with 1's in the diagonal and all non-diagonal terms have zero constant terms. Also notice that $A_3 + E_{0,0} = A_1$.

Proposition 6.16 implies that $X_t X_b - q^2 Y_r Y_l$ is bar invariant and therefore coincides with the dual canonical basis elements with the same leading term which is $X_o D$. □

**Remark 6.18.** The referee has kindly informed us that another, in a way easier, proof may be obtained as follows: Consider the equation $Z_{1,1} Z_{n,n} = q^2 Z_{1,n} Z_{n,1} + \xi_{1,n}^{1,n}$ in $\mathcal{O}_q(M(n))$ and apply the anti-endomorphism $\Gamma$ given in [17, Corollary 5.2.2] to it. The effect of $\Gamma$ on quantum minors has been computed in [12, Lemma 4.1], and from this one can see that the formula is obtained.

The following result is a variation of [9, Lemma 4.1, Proposition 4.5]:

**Lemma 6.19.** *Let $\Lambda_{L^+}$ be the $\Lambda$-matrix of the Laurent quasi-polynomial algebra generated by the elements of the family $\mathcal{V}_{L^+}$ and let $\mathcal{C} = \mathcal{C}_{L^+}^-$ be the set of $n + m - 1$ covariant quantum minors determined as the variables in $\mathcal{V}_{L^+}$ corresponding to the points in $L^-$. Let $s = \operatorname{corank}(\Lambda_{L^+})$. The kernel of $\Lambda_{L^+}$ is then generated by $s$ elements that have non-zero coefficients at most at the positions of $\mathcal{C}$.*

**Proof.** We consider here $m \times n$ matrices; in [9] it is $n \times r$. We assume $n \leqslant m$ and $s > 0$. Then $s = g.c.d.(m,n)$. Furthermore, if $m = y \cdot s$, and $n = x \cdot s$, then $x$ and $y$ are odd. Let us label the covariant elements by integers $j = 1, 2, \dots, n+m-1$ corresponding to the broken line starting at $(1,n)$, passing through $(m,n)$, and terminating at $(m,1)$. The covariant element $\Psi_j$ then has its lower right corner at the $j$th point on the line. Set $\Psi_0 = 1$. For each $a = 1, \dots, s$ the element in the Laurent quasi-polynomial algebra generated by the covariant elements,

$$\Phi_a = \prod_{\ell=-x}^{y-1} \Psi_{a+\ell\cdot s+n-1}^{(-1)^\ell},$$

commutes with all elements $Z_{i,j}$, and hence with all elements in $\mathcal{V}_{L^+}$. The proof of this is in [9]. Using Lemma 2.3 one easily constructs elements $v(\Phi_1), \dots, v(\Phi_s)$ in the kernel of $\Lambda_{L^+}$. The element $v(\Phi_a)$ is given with zeros everywhere except at the covariant elements $\Psi_k$ where the coordinate is $\sum_{\ell=-x}^{y-1} (-1)^\ell \delta_{k.a+s\cdot\ell+n-1}$. As these elements evidently are linearly independent, the claim follows. □

**Theorem 6.20.** *Consider two broken lines $L_1 < L_2$ in $M_{m,n}(\mathbb{Z}_+)$ such that $L_2$ is a closest bigger line to $L_1$. Let $\mathcal{Q}_{L_1} = (\mathcal{V}_{L_1}, \Lambda_{L_1}, B_{L_1})$ and $\mathcal{Q}_{L_2} = (\mathcal{V}_{L_2}, \Lambda_{L_2}, B_{L_2})$ be quantum seeds corresponding to these such that the set of non-mutable elements in both cases is $\mathcal{C} = \mathcal{C}_{L^+}^-$. Then $\mathcal{Q}_{L_1}$ can be obtained from $\mathcal{Q}_{L_2}$ through a sequence of moves which alternate between modifications to the $B$ matrix at a given step and quantum mutations. The modifications to the $B$ matrices do not affect their principal parts.*

**Proof.** Let $L_2 \neq L^-$ be an otherwise arbitrary broken line and let $(i, j)$ be a concave corner of $L_2$, specifically, assume that $(i, j+1), (i, j), (i+1, j)$ are points on the broken line $L_2$. If we replace $(i, j)$ by $(i+1, j+1)$ while keeping the other points, we get a broken line $L_1 < L_2$ such that $L_2$ is a closest bigger line to $L_1$. Consider quantum seeds $\mathcal{Q}_{L_2}$ and $\mathcal{Q}_{L_1}$ as in the statement of the theorem corresponding to this configuration. We will construct a sequence of interim quantum seeds $\mathcal{Q}_a = (\mathcal{V}_a, \Lambda_a, B_a)$ and $\tilde{\mathcal{Q}}_a = (\mathcal{V}_a, \Lambda_a, \tilde{B}_a)$ such that

$$\mathcal{Q}_{L_2} = \mathcal{Q}_0 \to \tilde{\mathcal{Q}}_0 \Rightarrow \mathcal{Q}_1 \to \tilde{\mathcal{Q}}_1 \Rightarrow \cdots \to \tilde{\mathcal{Q}}_{m-i} = \mathcal{Q}_{L_1}. \tag{6.10}$$

The double arrows are quantum mutations while the single arrows indicate some change on the level of the $B$ matrix.

Without loss of generality, we may assume that $j+m-i \leqslant n$. By construction, the quantum minor $\xi^{\{i,i+1,\dots,m\}}_{\{j,j+1,\dots,j+m-i\}}$ is both a quantum cluster variable for the quantum seeds associated to $L_2$ and to $L_1$, but labeled by different points, namely, labeled by $(m, j+m-i)$ in the quantum seed associated to $L_2$ and labeled by $(i, j)$ in the quantum seed associated to $L_1$. This quantum minor is not affected by the following manipulations. The quantum minors $\xi^{\{i\}}_{\{j\}}, \xi^{\{i,i+1\}}_{\{j,j+1\}}, \dots, \xi^{\{i,\dots,m-1\}}_{\{j,\dots,j+m-1-i\}}$ are changed, step by step, into $\xi^{\{i+1\}}_{\{j+1\}}, \xi^{\{i+1,i+2\}}_{\{j+1,j+2\}}, \dots, \xi^{\{i+1,\dots,m\}}_{\{j+1,\dots,j+m-i\}}$, and all other quantum cluster variables stay unchanged.

Specifically, we do the following sequence of replacements:

$$\begin{aligned}
\xi^{\{i\}}_{\{j\}} &\mapsto \xi^{\{i+1\}}_{\{j+1\}}, \\
\xi^{\{i,i+1\}}_{\{j,j+1\}} &\mapsto \xi^{\{i+1,i+2\}}_{\{j+1,j+2\}}, \\
&\ \vdots \\
\xi^{\{i,\dots,i+a\}}_{\{j,\dots,j+a\}} &\mapsto \xi^{\{i+1,\dots,i+1+a\}}_{\{j+1,\dots,j+1+a\}}, \\
&\ \vdots \\
\xi^{\{i,\dots,m-1\}}_{\{j,\dots,j+m-1-i\}} &\mapsto \xi^{\{i+1,\dots,m\}}_{\{j+1,\dots,j+m-i\}}.
\end{aligned} \tag{6.11}$$

We claim that each replacement is a quantum mutation in the sense of Berenstein, Zelevinsky. Indeed, quantum mutations are determined by $B$ matrices and we know, modulo the kernel of $\Lambda_a$, the relevant column of $B_a$ by Corollary 6.14: At any level $a < m-i$, the quantum minors $\xi^{\{i,\dots,i+a\}}_{\{j,\dots,j+a\}} = X^{(a)}_t$, $\xi^{\{i+1,\dots,i+1+a\}}_{\{j+1,\dots,j+1+a\}} = X^{(a)}_b$, $\xi^{\{i,\dots,i+1+a\}}_{\{j,\dots,j+1+a\}} = D^{(a)}$, $\xi^{\{i+1,\dots,i+a\}}_{\{j+1,\dots,j+a\}} = X^{(a)}_o$, $\xi^{\{i+1,\dots,i+1+a\}}_{\{j,\dots,j+a\}} = Y^{(a)}_l$, and $\xi^{\{i,\dots,i+a\}}_{\{j+1,\dots,j+1+a\}} = Y^{(a)}_r$ constitute an $\mathcal{M}$-set. The elements $X^{(a)}_t$, $D^{(a)}$, $X^{(a)}_o$, $Y^{(a)}_l$, and $Y^{(a)}_r$ are all quantum cluster variables in $\mathcal{V}_a$. On the other hand, for $a \geqslant 1$, $X^{(0)}_t, \dots, X^{(a-1)}_t$ are not variables in $\mathcal{V}_a$. By easy checking of which variables from $\mathcal{V}_a$ contain one or the other of $Z_{i,j}$ and $Z_{i+a+1,j+a+1}$, or both, it follows from Corollary 6.14 that the element $X^{(a)}_o D^{(a)} (Y^{(a)}_l)^{-1} (Y^{(a)}_r)^{-1}$ commutes with all the quantum cluster variables in $\mathcal{V}_a$ except $X^{(a)}_t$. In fact,

$$X^{(a)}_t X^{(a)}_o D^{(a)} \big(Y^{(a)}_l\big)^{-1} \big(Y^{(a)}_r\big)^{-1} = q^{-4} X^{(a)}_o D^{(a)} \big(Y^{(a)}_l\big)^{-1} \big(Y^{(a)}_r\big)^{-1} X^{(a)}_t.$$

By Lemma 2.3 this implies that, modulo the kernel of $\Lambda_a$, the only non-zero entries in the column of $B_a$ corresponding to the variable $X^{(a)}_t$ are at the row positions of the variables $Y^{(a)}_l$ and $Y^{(a)}_r$ where it is 1, and at the row positions of the variables $X^{(a)}_o$ and $D^{(a)}$ where it is $-1$. We then change (if needed) $B_a$ into $\tilde{B}_a$ such that the column in the latter corresponding to $X^{(a)}_t$ is non-zero precisely at the mentioned 4 places. Notice that this change only involves the kernel of $\Lambda_a$. By considering $B^T_a \Lambda_a$ it follows that any element in the kernel has non-zero coefficients at most at the places of the non-mutable elements $\mathcal{C}$. With this, the quantum mutation of $X^{(a)}_t$ to some new element $(X^{(a)}_t)'$ in the sense of (2.6) can be performed; and by combining Lemma 6.13 with Theorem 6.17, it follows that $X^{(a)}_b = (X^{(a)}_t)'$ is the target of this mutation. We then perform this quantum mutation and obtain a new interim quantum seed $\mathcal{Q}_{a+1} = (\mathcal{V}_{a+1}, \Lambda_{a+1}, B_{a+1})$. The variables $Y_l^{(a)}$, $Y_r^{(a)}$, $a = 0, 1, \ldots$ are both in $\mathcal{V}_a$ and in $\mathcal{V}_{a+1}$. In the step $a+1$, $D^{(a)} = X_t^{(a+1)}$ and, most importantly, $X_o^{(a+1)} = X_b^{(a)}$ which is now a variable in $\mathcal{V}_{a+1}$. In this way we can carry out the entire transition from $L_2$ to $L_1$. Hence our changing of the set of variables for a broken line at a concave point is obtained through a sequence of steps alternating between quantum mutations in the sense of Berenstein and Zelevinsky, and changes to the $B$ matrix. □

[As an aside, we observe that we, starting at the top, could break off the above replacements at any lower level, but we shall not find it useful to do so.]

**Remark 6.21.** In case $\Lambda_{L_2}$ is invertible all modifications to the $B$ matrices in Theorem 6.20 are trivial. Indeed, a modification makes changes involving only the kernel of $\Lambda_{L_2}$. Hence, in this case the two quantum seeds are equivalent by quantum mutations. If we can embed our algebra $\mathcal{O}_q(M(m,n))$ into a bigger algebra $\mathcal{O}_q(M(m_1,n_1))$ such that $\mathcal{Q}_{L_2}$ and $\mathcal{Q}_{L_1}$ are the restrictions of quantum seeds $\mathcal{Q}^E_{L_2} = (\mathcal{V}^E_{L_2}, \Lambda^E_{L_2}, B^E_{L_2})$ and $\mathcal{Q}^E_{L_1} = (\mathcal{V}^E_{L_1}, \Lambda^E_{L_1}, B^E_{L_1})$ with $\Lambda^E_{L_2}$ invertible, then we need at most make modifications to $B_{L_2}$ and $B_{L_1}$.

We shall see later that this can always be accomplished.

Now, for each broken line $L$, we have a family of $mn$ $q$-commuting quantum minors which by construction is a generating set of the fraction field of the Noetherian domain $\mathcal{O}_q(M(m,n))$.

**Corollary 6.22.** *Let $L$ be an arbitrary broken line and let $\xi_I^J$ be any solid quantum minor. Then $\xi_I^J$ can be written as a $q$-Laurent polynomial with coefficient in $\mathbb{Z}_+[q^2, q^{-2}]$ of the cluster variables associated to $L$.*

**Proof.** By our construction using broken lines, one can see that the solid quantum minor $\xi_I^J$ belongs to some quantum seed associated to a broken line $L'$. By the above theorem, $\xi_I^J$ can be obtained through a sequence of quantum mutations from the quantum cluster variables associated to $L$. Now the statement follows from the quantum Laurent phenomenon established in [2, Corollary 5.2]. □

**Remark 6.23.** Recall that a real matrix $A$ is totally positive (resp. totally non-negative) if all of its minors are positive (resp. non-negative). In [5], it is shown that a matrix is totally positive if all of its solid minors are positive. Moreover, in [3], it is shown that a matrix is totally positive if some specially chosen minors (in fact a cluster) are positive. The above result is related to the totally positivity of real matrices. Specializing $q$ to 1, we obtain a family of seeds (associated to broken lines) which are mutation equivalent to each other. To test if a matrix is totally positive one only needs to check if the minors in an arbitrary cluster associated to a broken line are positive.

### *6.2. Quantum line mutations*

**Definition 6.24.** In the general setting of $\mathcal{F}_L = \mathcal{F}_{L^+}$, let $L_1$ be a closest bigger line to the line $L$. Assume the configurations are as in Definition 6.11. The restricted quantum line mutation $\mu_R(L_1, L)$ is the map $\mathcal{V}_{L_1} \to \mathcal{V}_L$ given as the composite map (6.11) where $(i,j)$ is replaced by $(c-1, d-1)$. Assume that the set of non-mutable elements is $\mathcal{C}^-_{L^+}$.

If $\mathcal{Q}_{L_1} = (\mathcal{V}_{L_1}, \Lambda_{L_1}, B_{L_1})$ and $\mathcal{Q}_L = (\mathcal{V}_L, \Lambda_L, B_L)$ are quantum seeds, the **quantum line mutation** $\mu(L_1, L) : \mathcal{Q}_{L_1} \to \mathcal{Q}_L$ is a process as given by the analogue of (6.10) but where it furthermore is demanded that at each level $i$, $\tilde{\mathcal{Q}}_i = \mathcal{Q}_i$. The existence of this will be established in Proposition 7.1. For practical purposes, we also consider the trivial quantum mutation as a quantum line mutation and denote it by $\mu(L, L)$.

**Definition 6.25.** In the general setting of $\mathcal{F}^-_L$, let $L_2 \leqslant L_3 \leqslant L$ be broken lines such that $L_3$ is a closest bigger line to the line $L_2$. The quantum line mutation $\mu^L(L_3, L_2)$ is the process $\mathcal{Q}_{L_3,L} =$

$(\mathcal{V}_{L_3,L}, \Lambda_{L_3,L}, B_{L_3,L}) \to \mathcal{Q}_{L_2,L} = (\mathcal{V}_{L_2,L}, \Lambda_{L_2,L}, B_{L_2,L})$ defined in analogy with Definition 6.24. In particular, $\mu(L_1, L) = \mu^{L^+}(L_1, L)$. We denote the inverse of $\mu^L(L_3, L_2)$ by $\mu^L(L_2, L_3)$.

We have the following diamond lemma for quantum line mutations, cf. [1]:

**Lemma 6.26.** *Let $L_1 \leqslant L$. Let $\mu^L(L_1, L_1 \downarrow (c_1, d_1))$ and $\mu^L(L_1, L_1 \downarrow (c_2, d_2))$ be quantum line mutations. Then $\mu^L(L_1 \downarrow (c_1, d_1), (L_1 \downarrow (c_1, d_1)) \downarrow (c_2, d_2))$ and $\mu^L(L_1 \downarrow (c_2, d_2), (L_1 \downarrow (c_2, d_2)) \downarrow (c_1, d_1))$ are quantum line mutations. Furthermore,*

$$\mu^L\big(L_1 \downarrow (c_1, d_1), \big(L_1 \downarrow (c_1, d_1)\big) \downarrow (c_2, d_2)\big) \circ \mu^L\big(L_1, L_1 \downarrow (c_1, d_1)\big)$$
$$= \mu^L\big(L_1 \downarrow (c_2, d_2), \big(L_1 \downarrow (c_2, d_2)\big) \downarrow (c_1, d_1)\big) \circ \mu^L\big(L_1, L_1 \downarrow (c_2, d_2)\big).$$

**Proof.** The key to this lemma is Corollary 6.15 as well as the explicit formulas (2.8) and (2.9). The mutation which does the replacement $X_t^{(a)} \mapsto X_b^{(a)}$ makes changes to the rows in $B$ corresponding to the quantum minors $D^{(a)}$, $X_o^{(a)}$, $Y_l^{(a)}$, and $Y_r^{(a)}$. It follows by direct inspection that if an entry in the row of $X_t^{(a)}$ in some position $v$ is zero then the column of $v$ stays unchanged under the quantum mutation. At this level the entry at the position of $X_t^{(a)}$ of course is zero. Clearly, $X_t^{(a)}$ is not a member of any of the subsequent sets $D^{(a+p)}$, $X_o^{(a+p)}$, $Y_l^{(a+p)}$, $Y_r^{(a+p)}$, where $p = 1, \dots, p_0$ for some specific positive integer $p_0$. It follows that the positions in the row of $X_t^{(a)}$ corresponding to the later values $X_t^{(a+p)}$ must be zero since we know precisely what the column of $X_t^{(a+p)}$ looks like. These considerations can easily be extended to include the case of a second quantum line mutation since none of the variables $X_t^{(a)}$ take part in any way in the second quantum line mutation. In the case where $(c_2, d_2) = (c_1 - 1, d_1 + 1)$ there is an overlap of variables in the sense that the $Y_r$ variables belonging to $(c_1, d_1)$ play the role of $Y_l$ variables belonging to $(c_2, d_2)$, but this is easily taken care of: They are not the sources or targets of mutations and then the effects of the two different quantum line mutations on the rows of such elements are independent of each other. The crucial observation is that neither of the two quantum line mutations affects the rows of the variables involved in the other. □

The following result concerning independence of paths, follows easily since one may fill in diamonds as in Lemma 6.26:

**Corollary 6.27.** *If $L_1 \leqslant L_2 \leqslant \cdots \leqslant L_{n-1} \leqslant L_n \leqslant L$ and $L_1 \leqslant L_2' \leqslant \cdots \leqslant L_{n-1}' \leqslant L_n$ are broken lines such that at each level the bigger line is a closest bigger line to the neighboring smaller one. Then*

$$\mu^L(L_2, L_1) \circ \cdots \circ \mu^L(L_n, L_{n-1}) = \mu^L\big(L_2', L_1\big) \circ \cdots \circ \mu^L\big(L_n, L_{n-1}'\big).$$

In view of Corollary 6.27 we extend our definition of a quantum line mutation to the following

**Definition 6.28.** Let $L_1 \leqslant L_n \leqslant L$ be broken lines. The quantum line mutation $\mu^L(L_n, L_1)$ is the composite of any sequence as in Corollary 6.27 between $L_1$ and $L_n$.

Let $L_a \leqslant L$ and $L_b \leqslant L$ be broken lines. The quantum line mutation $\mu^L(L_a, L_b)$ is defined in terms of any broken line $L_c \leqslant L_a, L_b$ as

$$\mu^L(L_a, L_b) = \mu^L(L_b, L_c)^{-1} \circ \mu^L(L_a, L_c).$$

We shall also need

**Definition 6.29.** A position $(c,d) \in T_L \cup L$ is called attractive with respect to $T_L \cup L$ if either there exist $i > 0$, $j > 0$ such that $(c-i,d) \in T_L \cup L$ and $(c,d+j) \in T_L \cup L$ or if there exist $i > 0$, $j > 0$ such that $(c+i,d) \in T_L \cup L$ and $(c,d-j) \in T_L \cup L$. Clearly, if $(c,d)$ satisfies the first condition of attraction then $(c-i,d+j)$ satisfies the second, and vice versa. If $(c,d)$ is not attractive we call it repulsive.

The following is obvious

**Lemma 6.30.** *The concave corners of $L$ are repulsive. The point $(m,n)$ is also repulsive.*

## 6.3. Covariant elements

We extend Definition 3.13 in the obvious way to $\mathcal{O}_q(T_L \cup L)$. The next observation we wish to make is that the seeds we construct are minimal in the following sense:

**Proposition 6.31.** *The set of covariant elements for $\mathcal{O}_q(T_L \cup L)$ is contained in the sub-algebra generated by the $m+n-1$ elements in $\mathcal{C}_L^-$.*

**Proof.** First of all it is clear that the elements in $\mathcal{C}_L^-$ are covariant, and hence, so is any monomial in these.

Since there is a unique smallest element in the set of broken lines, this may be proved by induction. For the line $L^-$ it is clear that we have a quasi-polynomial algebra so here, the claim is trivial. Consider then a line $L$ for which the claim is true and let $L_1$ be a closest bigger line. Assume the configurations are as in Definition 6.11. (Thus, $(i,j) = (c-1,d-1)$.) It is clear that $\mathcal{O}_q(T_L \cup L_1)$ is obtained by adjoining $Z_{c-1,d-1}$ to $\mathcal{O}_q(T_L \cup L)$. There is a unique element $X_b$ from $\mathcal{C}_L^-$ having its upper left corner in $(c,d)$. Depending on the configuration, $X_b = X_b^{(m-c)}$, or $X_b = X_b^{(n-d)}$. This is the largest solid quantum minor with its upper left corner in this position and completely contained in $\mathcal{O}_q(T_L \cup L)$. It is clear from Proposition 6.4 that this is the only element from $\mathcal{C}_L^-$ which does not $q$-commute with $Z_{c-1,d-1}$. On the other hand, when $(c-1,d-1)$ is viewed as an element in $S_L$, the variable $D = D^{(m-c)}$, respectively $D = D^{(n-d)}$, does, by Proposition 6.3, $q$-commute with all the $Z_{i,j}$ in $\mathcal{O}_q(T_L \cup L)$ – and clearly also with $Z_{c-1,d-1}$. Next observe that evidently $D \in \mathcal{C}_{L_1}^-$.

Suppose then that $C \in \mathcal{O}_q(T_{L_1} \cup L_1)$ is covariant. It is clear that

$$\mathcal{O}_q(T_{L_1} \cup L_1) \subseteq \mathcal{O}_q(T_L \cup L)\big[D, X_b^{-1}\big].$$

Both adjoined elements are covariant as far as $\mathcal{O}_q(T_L \cup L)$ is concerned, and it follows easily that $C$ must be a polynomial in the variables from $\mathcal{C}_L^-$ together with $D$ and $X_b^{-1}$. The element $Z_{c-1,d-1}$ $q$-commutes with all these generators except $X_b$ and this easily implies that $X_b^{\pm 1}$ cannot appear. □

**Proposition 6.32.** *Consider the quadratic algebra $\mathcal{O}_q(T_L \cup L) \subseteq \mathcal{O}_q(M(m,n))$. Then there is a non-trivial center if and only if $m = n$ and $L = L^+$. This center is generated by $\det_q(n)$.*

**Proof.** Consider the covariant element $M = M_{m,n} \in \mathcal{C}_L^-$ corresponding to the position $(m,n)$. If $N$ is any other covariant element and $MN = q^{\delta} NM$ then $\delta \leqslant 0$. This follows by easy inspection. Hence, since any central element must be a polynomial in the elements in $\mathcal{C}_L^-$, the central element must a polynomial in those elements from $\mathcal{C}_L^-$ that are quantum minors of $M$. If there are elements $Z_{i,j}$ in the algebra not occurring in $M$, then there will be non-trivial commutation relations between these and the quantum minors from $M$. Thus, there can be no positions outside $M$ if $M$ is to be central. The remaining details now follow from the classical result of Parshall and Wang [17]. □

## 7. On compatible pairs

We now settle the existential questions implicitly raised in Theorem 6.20, Definition 6.24, and Definition 6.25.

**Proposition 7.1.** *To a given broken line $L$ in $M_{m,n}(\mathbb{Z}_+)$ one can construct the following data $\mathcal{D}_L$:*

- *An ordering of the set of variables $\mathcal{V}_L = \mathcal{V}_L^+ \cup \mathcal{V}_L^-$ of the broken line such that the variables in $\mathcal{V}_L^-$ are assigned the numbers from 1 to $N_L$ and the remaining variables the numbers from $N_L + 1$ to $mn$. Let $c_L^- = \#\mathcal{C}_L^-$. ($\mathcal{C}_L^-$ is the set of non-mutable variables when $\mathcal{V}_L^-$ is considered in the ambient space $\mathcal{F}_L^-$. We have, of course, that $c_L^- = m + n - 1$ but it is convenient to have the additional notation.) We will even assume that the mutable variables are assigned the numbers from 1 to $\tilde{N}_L = N_L - c_L^-$.*
- *The $mn \times mn$ matrix $\Lambda_L$ corresponding to the variables $\mathcal{V}_L$ of the broken line.*
- *A matrix $B_L$ of size $mn \times (m-1)(n-1)$ such that $(\Lambda_L, B_L)$ is a compatible pair.*
- *The $N_L \times N_L$ matrix $\Lambda_L^0$ of the variables in $\mathcal{V}_L^-$. We view this as a sub-matrix of $\Lambda_L$.*
- *An $N_L \times \tilde{N}_L$ matrix $B_L^0$ such that $(\Lambda_L^0, B_L^0)$ is a compatible pair for the set of variables $\mathcal{V}_L^-$. If one defines an $mn \times \tilde{N}_L$ matrix $B_L^R$ by adding $mn - N_L$ rows of zeros to $B_L^0$ such that $B_L^0$ occupies the top rows of $B_L$, the following holds in addition:*
  - $\to$ *$\Lambda_L B_L^R = -2D_L$, where $D_L = I_{\tilde{N}_L} \oplus 0$ is the $mn \times \tilde{N}_L$ matrix consisting of $I_{\tilde{N}_L}$ in the top $\tilde{N}_L \times \tilde{N}_L$ corner augmented by an appropriate number of rows of zeros. Here, $I_{\tilde{N}_L}$ is the $\tilde{N}_L \times \tilde{N}_L$ identity matrix.*
  - $\to$ *$B_L^R$ is a sub-matrix of $B_L$.*

*In the case of ambient space $\mathcal{F}_L^-$, consider the quantum seed $\mathcal{Q}_L^- = (\mathcal{V}_L^-, \Lambda_L^0, B_L^0)$ with the set non-mutable elements given as $\mathcal{C}_L^-$. If $L_1$ is a broken line in $M_{m,n}(\mathbb{Z}_+)$ and $L_1 < L$, there exists a quantum seed $\mathcal{Q}_{L_1,L} = (\mathcal{V}_{L_1,L}, \Lambda_{L_1,L}, B_{L_1,L})$ which is equivalent by the quantum line mutations $\mu^L(L, L_1)$ to $\mathcal{Q}_L^-$ and where the pairs $(\Lambda_{L_1,L}, B_{L_1,L})$ and $(\Lambda_{L_1}^0, B_{L_1}^0)$ are related in a way that generalizes in an obvious manner the way $(\Lambda_{L_1}, B_{L_1})$ is related to $(\Lambda_{L_1}^0, B_{L_1}^0)$.*

**Proof.** A short proof would be to say that this follows by bootstrapping. We give here a more detailed proof using the same principle: First assume that $n = m + 1$ (or, analogously, $n = m - 1$). The existence of $B_L$ will follow from the first parts of the proof. We prove the claims involving $B_L^R$ and $B_L^0$ by induction on the partial order on the set of broken lines. The induction starts with the line $L^-$. There are no mutable elements in $\mathcal{V}_{L^-}^-$ so $B_{L^-}^R$ and $B_{L^-}^0$ are empty. The other structure we start with is $\mathcal{V}_{L^-}$ and a compatible pair $(\Lambda_{L^-}, B_{L^-})$ connected with this set of variables. The set of non-mutable elements is $\mathcal{C} = \mathcal{C}_{L^+}^-$ as before. The matrix $\Lambda_{L^-}$ can, naturally, be explicitly written down. It is known from [9, Proposition 4.5] that $\Lambda_{L^-}$ is invertible as a real matrix. Furthermore it follows from Proposition 4.11 therein that there is a block decomposition into $2 \times 2$ skew integer matrices. It follows then from the discussion on p. 85 in [9] that there exists an integer matrix $A$ with $\det A = 1$ such that $A(\Lambda_{L^-})A^t = \tilde{D}$ where $\tilde{D}$ is a block diagonal matrix consisting of $\frac{1}{2}m(m-1)$ $2 \times 2$ blocks $\left(\begin{smallmatrix} 0 & 2 \\ -2 & 0 \end{smallmatrix}\right)$ and $m$ $2 \times 2$ blocks $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The results in [9] are obtained where $q$ is a root of unity. The end results on the structure of $\Lambda_{L^-}$ are, however, independent of special choices of $q$, and hence apply to the block diagonalization of the integer matrix $\Lambda_{L^-}$ as in our case. It follows that $2\Lambda_{L^-}^{-1} = 2A^T(A\Lambda_{L^-}A^T)^{-1}A$ is an integer matrix. The existence of $B_{L^-}$ follows easily from this as the $m(m+1) \times (m(m+1) - c_L^-)$ sub-matrix of $-2\Lambda_{L^-}^{-1}$ consisting of the first $(m(m+1) - c_L^-)$ columns. At the moment, it is only the existence and uniqueness of $B_{L^-}$ (up to multiplication by a positive integer) that matters. After we have presented the induction step, we encourage the reader to take it right at the start as a simple exercise.

Suppose now that we have a line $L$ with data $\mathcal{D}_L$. Let $L_1 > L$ be a broken line closest to $L$. We must now construct the data $\mathcal{D}_{L_1}$ for $L_1$. For this purpose we consider the inverse, $\mu(L, L_1)$, of the quantum line mutation $\mu(L_1, L)$. We view these mutations as taking place inside the full $m(m+1) \times m(m+1)$

matrix algebra, but we shall keep a keen eye on its relation to the ambient spaces $\mathcal{F}_L^-$ and $\mathcal{F}_{L_1}^-$. Specifically, how $\mathcal{Q}_{L_1}^-$ grows from $\mathcal{Q}_L^-$.

The mutation $\mu(L, L_1)$ begins with a mutation of the form of the reverse of the bottom line in (6.11) and where the element we mutate from, $X_b^{(m-i-1)}$, is a covariant, viz. non-mutable, element of $\mathcal{V}_L^-$. This is not represented in the matrices $B_L^0$ and $B_L^R$, so we define a new matrix $\tilde{B}_L^R$ by joining one new column $c_{b,m-i-1} = c(X_b^{(m-i-1)})$ to $B_L^R$ in the position $\tilde{N}_L + 1$ and labeled by $X_b^{(m-i-1)}$. Denote by $\tilde{B}_L^0$ the sub-matrix of $\tilde{B}_L^R$ with the same column numbers and having rows corresponding to those of $B_L^0$ together with an additional row labeled by $D^{(m-i-1)}$. If the approach is possible, it follows from Corollary 6.15 in combination with Definition 2.2 and Lemma 2.3 what the added column must look like: There should be the value $-1$ at the positions corresponding to $X_o^{(m-i-1)}$ and $D^{(m-i-1)}$ and the value 1 at the positions of $Y_r^{(m-i-1)}$, and $Y_l^{(m-i-1)}$. All other entries must be zero. $\Lambda_L(c_{b,m-i-1})$ is a column with exactly one non-zero coefficient $-4$. This occurs at the position of $X_b^{(m-i-1)}$. We have that $X_o^{(m-i-1)}$, $Y_r^{(m-i-1)}$, and $Y_l^{(m-i-1)}$ are variables of $\mathcal{V}_L^-$. The element $D^{(m-i-1)}$ is a variable in $\mathcal{V}_L$ but is not a variable of $\mathcal{V}_L^-$. It is a covariant (non-mutable) element of $\mathcal{V}_{L_1}^-$. To begin with we consider the set of variables $\mathcal{V}_{L,D}^- = \mathcal{V}_L^- \cup \{D^{(m-i-1)}\}$. Corresponding to this we have a $\Lambda$-matrix $\tilde{\Lambda}_L^0$ with one more column and row than $\Lambda_L^0$. Since $-\frac{1}{2}B_L$ is a part of the inverse matrix of $\Lambda_L$ it is clear that the matrix $\tilde{B}_L^R$ is part of the matrix $B_L$. Furthermore, by construction, $(\tilde{\Lambda}_L^0, \tilde{B}_L^0)$ is a compatible pair for $\mathcal{V}_{L,D}^-$. It is also obvious, using the equations involved in (6.11) in the proof of Theorem 6.20, that the skew field of fractions generated by $\mathcal{V}_{L,D}^-$ is exactly $\mathcal{F}_{\Lambda_1}^-$. We now perform the mutation inside the full matrix algebra at the variable $X_b^{(m-i-1)}$. On the level of variables, with the given column of $B_L$, this is, according to Theorem 6.17, exactly the mutation

$$X_b^{(m-i-1)} \mapsto X_t^{(m-i-1)}.$$

Correspondingly, we obtain an interim pair $(\Lambda_L^{(m-i-1)}, B_L^{(m-i-1)})$ which corresponds to an interim set of variables $\mathcal{V}_{L,(m-i-1)}$. At the same time we perform a mutation in $\mathcal{F}_{L_1}^-$ at the same variable but using the pair $(\tilde{\Lambda}_L^0, \tilde{B}_L^0)$. Here we obtain an interim set of variables $\mathcal{V}_{L,(m-i-1)}^-$ which is a part of $\mathcal{V}_{L,(m-i-1)}$. The two mutations do not differ in what they do to the variable $X_b^{(m-i-1)}$. The difference lies entirely on the level of compatible pairs. The parts of the mutations which involve the matrix $E_i$ in (2.8) affect the rows corresponding to the variables $X_o$, $D$ and we get the same contributions for both mutations as far as $\tilde{B}_L^R$ is concerned. It is less transparent at first what happens at the place of the matrix $F_i$. Here there are two different matrices for the two mutations. The difference relies on the fact that $D^{(m-i-1)}$, and one or both of $Y_r^{(m-i-1)}$, $Y_l^{(m-i-1)}$ are mutable in the full algebra but non-mutable in the small algebra. At this moment of the proof we are only concerned with what happens to the columns of $\tilde{B}_L^R$ inside $B_L^{(m-i-1)}$. The remaining columns of $B_L^{(m-i-1)}$, though, are known in principle. The two mutations differ only in what happens to the columns outside $B_L^{(m-i-1)}$ where, according to (2.9), a multiple of the column corresponding to $X_b$ may be added. Furthermore, if one or both of $Y_r^{(m-i-1)}$, $Y_l^{(m-i-1)}$ are non-mutable in $\mathcal{V}_L^-$ they stay so in $\mathcal{V}_{L,D}^-$. So, since the columns of the non-mutable variables of $\mathcal{V}_{L,D}^-$ are not under discussion, it is clear that the analogous new compatible pair $(\Lambda_L^0, B_L^0)^{(m-i-1)}$, obtained from $(\tilde{\Lambda}_L^0, \tilde{B}_L^0)$ by mutation in the ambient space $\mathcal{F}_{L_1}^-$, is a sub-pair of $(\Lambda_{L_1}, B_{L_1})$. Now perform the remaining mutations in the quantum line mutation. These only involve mutable variables and are easily seen to preserve the general form. Finally, one can reshuffle the variables to obtain the wanted ordering. Again, this does not change the general form. Thereby the induction step is completed.

In this way we build up $B$ matrices with more and more columns. In the end we reach $\mathcal{Q}_{L^+} = (\mathcal{V}_{L^+}, \Lambda_{L^+}, B_{L^+})$ of the extremal line $L^+$. Once we have that, we can mutate back, by quantum line mutations, to any quantum seed $\mathcal{Q}_L = (\mathcal{V}_L, \Lambda_L, B_L)$. We can also stop the growing process at an earlier point, where we have obtained a seed $\mathcal{Q}_L^- = (\mathcal{V}_L^-, \Lambda_L^0, B_L^0)$ and use mutations $\mu^L$ inside the ambient space $\mathcal{F}_L^-$ to obtain the seeds $\mathcal{Q}_{L_1,L}$ mentioned in the proposition. Finally recall the independence of path result Corollary 6.27.

The proof applies to any pair $(m,n)$ for which $\Lambda_{L^-}$ is invertible. Let us then consider the general situation of $\mathcal{O}_q(M(m,n))$. Suppose for simplicity that $m = n + r$ with $r \geqslant 2$. We can view this as the sub-algebra of $\mathcal{O}_q(M(m,n+r+1))$ generated by the elements $Z_{i,j}$ with $1 \leqslant i \leqslant m$ and $2 + r \leqslant j \leqslant n + r + 1$. Any broken line $L : (1,n) \to \cdots \to (m,1)$ in $M_{m,n}(\mathbb{Z}_+)$ is similarly considered as a line $\tilde{L} : (1, n + r + 1) \to \cdots \to (m, r + 2)$ in $M_{m,m+1}(\mathbb{Z}_+)$ which is then extended by the segment $\tilde{L} \to (m,1)$. This corresponds to adding the non-mutable covariant variables $Z_{m,1}, \ldots, Z_{m,r+1}$ to all sets of variables in all quantum seeds. If we stipulate that the mutations and other operations in $\mathcal{O}_q(M(m,n+r+1))$ should never involve these we clearly get the result as a sub-case of the full case based on $(m,m+1)$. Finally, the case $m = n$ follows by analogous considerations. $\square$

**Remark 7.2.** Also for the remaining mutations in the quantum line mutation $\mu(L, L_1)$ we can write down explicitly the values in the $B$-column which we mutate from simply by using Corollary 6.15 repeatedly. In this way one can in fact "explicitly" write down the compatible pairs at each step.

## 8. The quantum (upper) cluster algebra of a broken line

We define now some algebras connected with a broken line. Our terminology may seem a bit unfortunate since the notions of a cluster algebra and an upper cluster algebra already have been introduced by Berenstein and Zelevinsky in terms of all mutations. We only use quantum line mutations which form a proper subset of the set of all quantum mutations. However, it will be a corollary to what follows that the two notions in fact coincide, and for this reason we do not introduce some auxiliary notation.

**Definition 8.1.** The cluster algebra $\mathcal{A}_L^-$ connected with a broken line $L$ in $M_{m,n}(\mathbb{Z}_+)$ is the $\mathbb{Z}[q]$-algebra generated in the space $\mathcal{F}_L^-$ by the inverses of the non-mutable elements $\mathcal{C}_L^-$ together with the union of the sets of all variables obtainable from the initial seed $\mathcal{Q}_L^-$ by composites of quantum line mutations $\mu^L(L, L_1)$ with $L_1 \leqslant L$.

Observe that we include $\mathcal{C}_L^-$ in the set of variables.

**Definition 8.2.** The upper cluster algebra $\mathcal{U}_L^-$ connected with a broken line $L$ in $M_{m,n}(\mathbb{Z}_+)$ is the $\mathbb{Z}[q]$-algebra in $\mathcal{F}_L^-$ given as the intersection of all the Laurent quasi-polynomial algebras of the sets of variables obtainable from the initial seed $\mathcal{Q}_L^-$ by composites of quantum line mutations $\mu^L(L, L_1)$ with $L_1 \leqslant L$.

**Remark 8.3.** The results we obtain below are independent of the pairs $(\Lambda, B)$ entering into the quantum seeds. What enters into the proofs are mutations as in Theorem 6.20.

**Remark 8.4.** The algebras $\mathcal{A}_L^-$ and $\mathcal{U}_L^-$ of a broken line $L$ are defined in terms of some $\mathcal{O}_q(M(m,n))$, but of course, if the line has a segment $(m,1) \leftarrow (m,u)$, with $(m,u)$ denoting a corner, and $u > 1$, then the elements $Z_{m,1}, \ldots, Z_{m,u-1}$ are all covariant. Thus, $\mathcal{O}_q(T_L \cup L) = \mathcal{O}_q(T_{L_1} \cup L_1)[Z_{m,1}, \ldots, Z_{m,u-1}]$ and $\mathcal{A}_L^- = \mathcal{A}_{L_1}^-[Z_{m,1}^{\pm 1}, \ldots, Z_{m,u-1}^{\pm 1}]$, where $L_1$ is what remains of $L$ after these elements have been removed. Similarly with segments $(1,n) \to (u,n)$. In the same spirit, covariant elements may be added if it is convenient to view $\mathcal{A}_L^-$ as a part of a quantum cluster algebra based on some other $\mathcal{O}_q(M(m_1,n_1))$ with $m \leqslant m_1$ and $n \leqslant n_1$. See also the last part of the proof of Proposition 7.1.

It is clear that $\mathcal{O}_q(T_L \cup L) \subseteq \mathcal{U}_L^-$ and that $Y_i^{\pm 1} \in \mathcal{U}_L^-$ for all $Y_i \in \mathcal{C}_L^-$. Indeed, by the $q$-Laurent Phenomenon [2, Corollary 5.2], $\mathcal{O}_q(T_L \cup L) \subseteq \mathcal{A}_L^- \subseteq \mathcal{U}_L^-$.

**Theorem 8.5.** *Let $\mathcal{C}_L^- = \{Y_1, \ldots, Y_s\}$. Then,*

$$\mathcal{U}_L^- = \mathcal{O}_q(T_L \cup L)\big[Y_1^{\pm 1}, \ldots, Y_s^{\pm 1}\big] = \mathcal{A}_L^-.$$

We need only establish that $\mathcal{U}_L^- \subseteq \mathcal{O}_q(T_L \cup L)[Y_1^{\pm 1}, \ldots, Y_s^{\pm 1}]$. We will in the proof of that use the following

**Proposition 8.6.** *A quantum minor $M \in \mathcal{C}_L^-$ generates a completely prime ideal of $\mathcal{O}_q(T_L \cup L)$. Specifically, it satisfies the following crucial property*:

*If $p_1 M = p_2 p_3$ in $\mathcal{O}_q(T_L \cup L)$*

*then $p_2 = p_4 M$ or $p_3 = p_5 M$ for some $p_4, p_5$ in $\mathcal{O}_q(T_L \cup L)$.*

**Proof.** Goodearl and Lenegan proved in [8, Theorem 2.5] that the determinantal ideal is prime. We can reduce our case, in which $M$ is a covariant quantum minor, to theirs by using a PBW basis of the full set of variables $\{Z_{ij}\}$ in which the variables of the rows and columns of $M$, henceforth referred to as the variables of $M$, are written to the right. The elements $p_2$ and $p_3$ may then be written as sums of polynomials in the variables of $M$ with coefficients (to the left) that are monomials in the variables that are not variables of $M$. Let us be specific and say that $M = \xi^{\{i,i+1,\ldots,m\}}_{\{j,j+1,\ldots,j+m-i\}}$. Let us order the monomials in the variables not in $M$ so that the points with column number less than $j$ are biggest, and ordered lexicographically with the biggest being the point with smallest row and column number. The finer details are irrelevant. Next in the ordering we take those points having a column number between $j$ and $j+m-i$ with a similar lexicographical ordering. Finally we take those with a column number bigger than $j+m-i$. Here we chose an opposite ordering. We can then focus on the monomials that are the biggest in $p_2$ and $p_3$. The point of the chosen ordering is that one does not pick up bigger terms via (3.4) while rewriting a product. Let $v_2^0 p_2^0$ be the summand in $p_2$ corresponding to the biggest monomial $v_2^0$. Here $p_2^0$ is a polynomial in the variables of $M$. Let $v_3^0 p_3^0$ be the analogous summand for $p_3$. Regrouping in the product $p_2 p_3$ according to our total ordering results in a unique highest term (up to a factor of $q$ to some power) $w p_2^0 p_3^0$, where $w$ is the highest order element of $v_2^0 v_3^0$. This must then match a term $v_1^0 p_1^0 M$ in $p_1 M$. Specifically, $v_1^0 = w$. By [8], $p_2^0 = p_4^0 M$ or $p_3^0 = p_5^0 M$. Say it is $p_2^0 = p_4^0 M$. Since $M$ is covariant with respect to all the variables of $\mathcal{O}_q(T_L \cup L)$ we can just drop the expression $v_2^0 p_2^0$ from $p_2$. Indeed, by looking at the biggest elements, we can assume from the beginning that neither $p_2$ nor $p_3$ contains a summand of the form $pM$ and then argue by contradiction. □

**Proof of Theorem 8.5.** We prove this by induction. For the unique smallest line $L^-$ the algebra $\mathcal{O}_q(T_L \cup L^-) = \mathcal{O}_q(L^-)$ is generated by the covariant elements in $\mathcal{C}_{L^-}^-$. The algebra is quasi-polynomial and there are no quantum line mutations except the trivial one. Thus the claim is trivially true. [Actually, there is also a unique line $L_1$ closest to $L^-$ and the situation here essentially corresponds to $\mathcal{O}_q(M(2,2))$. This case is also true and well known.] Let us then consider a line $L$ and let $L_1$ be a closest line with $L < L_1$. Let the notation be as in the proof of Proposition 6.31. Then $\mathcal{C}_{L_1}^-$ is obtained from $\mathcal{C}_L^-$ by replacing $X_b$ by $D$. Suppose that $u \in \mathcal{U}_{L_1}^-$. Since $\mathcal{U}_{L_1}^-$ is an algebra it is clear that we may assume that when $u$ is expressed as a $q$-Laurent polynomial of some set of variables, all powers of the covariant elements $\mathcal{C}_{L_1}^-$ are non-negative. Then it is true for all allowed sets of variables as in Definition 8.2. Moreover, $L$ is obtained from $L_1$ by a quantum line mutation and all subsequent quantum line mutations of $L$ are thus also quantum line mutations of $L_1$. In all these mutations $D$ stays non-mutable. For all lines $L_2 \leqslant L$, the algebra generated by the variables from $\mathcal{V}_{L_2,L_1}$ and their inverses is contained in the algebra generated by the variables from $\mathcal{V}_{L_2,L}$ and their inverses together with $D^{\pm 1}$. It is then clear that $\mathcal{U}_{L_1}^- \subseteq \mathcal{U}_L^-[D^{\pm 1}]$. Now, the non-mutable variables of $L$ are the same as those of $L_1$ with the exception of $X_b$. By the argument about the positivity of the non-mutable variables we can then assume that all these variables except the latter occur with a non-negative power when $u$ is expanded in one of the allowed quasi-Laurent algebras. Thus, we can assume $u \in \mathcal{U}_L^-[X_b^{\pm 1}, D]$. By the induction hypothesis we then have

$$u \in \mathcal{O}_q(T_L \cup L)\big[X_b^{-1}, D\big], \tag{8.1}$$

and to meet our goal, we only need to be concerned about the elements with a strictly negative power of $X_b$ in each summand.

Naturally, $\mathcal{O}_q(T_L \cup L)$ can be viewed as a sub-algebra of $\mathcal{O}_q(T_{L_1} \cup L_1)$.

Let us denote the initial variables of $L_1$ by $D = X_t^{(m-c+1)}, X_t^{(m-c)}, \dots, X_t^{(0)}, W_1, \dots, W_N$. The initial variables of $L$ are then $X_b = X_b^{(m-c)}, X_b^{(m-c-1)}, \dots, X_b^{(0)}, \dots, W_1, \dots, W_N$. Let us look at the element $u$. This can be written as a $q$-Laurent polynomial in the given initial variables of $L_1$, one of which is $D$:

$$u = \sum_{\underline{\alpha}, \underline{\beta}} c_{\underline{\alpha}, \underline{\beta}} W^{\underline{\beta}} \prod_{i=0}^{m+1-c} \big(X_t^{(m-c+1-i)}\big)^{\alpha_i}.$$

We can factor out the biggest non-positive powers such that

$$u = p_{top} \cdot W^{-\underline{\beta}^0} \prod_{i=0}^{m+1-c} \big(X_t^{(m-c+1-i)}\big)^{-\alpha_i^0}, \tag{8.2}$$

where $p_{top} \in \mathcal{O}_q(T_{L_1} \cup L_1)$ and, in particular, $p_{top}$ contains no overall factor of $D$. We wish to argue by contradiction and thus assume that the multi-indices $\underline{\alpha}^0$, $\underline{\beta}^0$ are non-negative, and at least one $\alpha_r^0$ or $\beta_s^0$ is positive.

Set $Z = Z_{c-1,d-1}$. Then $D = ZX_b$ modulo $\mathcal{O}_q(T_L \cup L)$. By (8.1) we have

$$u = \bigg(\sum_i Z^i p_i X_b^{k_i}\bigg) X_b^{-\rho}, \tag{8.3}$$

where $\forall i$: $0 \leqslant k_i < \rho$ and where, furthermore, the elements $p_i \in \mathcal{O}_q(T_L \cup L)$ are neither divisible by $Z$ nor by $X_b$. Combining (8.2) and (8.3), we have

$$\bigg(\sum_i Z^i p_i X_b^{k_i}\bigg) X_b^{-\rho} = p_{top} \cdot W^{-\underline{\beta_0}} \prod_{i=0}^{m+1-c} \big(X_t^{(m-c+1-i)}\big)^{-\alpha_i^0}.$$

Now, in the $q$-Laurent algebra we clearly get

$$\bigg(\sum_i Z^i p_i X_b^{k_i}\bigg) \prod_{i=0}^{m+1-c} \big(X_t^{(m-c+1-i)}\big)^{\alpha_i^0} W^{\underline{\beta}^0} = q^{2\gamma} p_{top} X_b^{\rho}, \tag{8.4}$$

where $q^{2\gamma}$ is an irrelevant factor stemming from the $q$-commutativity between $X_b$ and the elements $W_i$. We ignore this and similar factors in the following.

The first crucial observation is that by Proposition 8.6, $\alpha_0^0 = 0$ since the right hand side of (8.4) clearly does not contain a positive power of $D$.

The next important fact is that the position $(c-1, d-1)$ is repulsive with respect to $T_{L_1} \cup L_1$. This implies that it is straightforward to look at the highest order terms of $Z$ in (8.4). In the right hand side we simply write $p_{top} = Z^S u_S + \ell$ where $\ell$ is of lower order, and where $u_S$ is a polynomial in the variables of $T_L \cup L$. In the left hand side, let us say that $Z^K p_K X_b^{k_K}$ is the term containing the highest $Z$ exponent $K$. We then get additional $Z$ terms from $\prod_{i=1}^{m+1-c}(X_t^{(m-c+1-i)})^{\alpha_i^0}$, and here the highest $Z$ term is $Z^{\alpha_S^0}\prod_{i=1}^{m-c}(X_b^{(m-c-i)})^{\alpha_i^0}$ where $\alpha_S^0 = \sum_{i=1}^{m-c+1}\alpha_i^0$.

Using the repulsiveness again, we get

$$Z^{K+\alpha_S^0} p_K \prod_{i=1}^{m-c}\left(X_b^{(m-c-i)}\right)^{\alpha_i^0} X_b^{k_K} = Z^S u_S X_b^{\rho}.$$

And thus,

$$p_K \prod_{i=1}^{m-c}\left(X_b^{(m-c-i)}\right)^{\alpha_i^0} = u_S X_b^{\rho-k_K}.$$

This identity holds in $\mathcal{O}_q(T_L \cup L)$. Since $\rho - k_k > 0$ it follows by Proposition 8.6 that $X_b$ must be a right divisor of one of the terms on the left hand side. The $X_b^{(a)}$ with $a = 0, 1, \ldots, m-c-1$ terms of course are impossible in this respect. Thus, $p_K = \hat{p}X_b$ for some $\hat{p}$. This is a contradiction to the way $p_K$ was defined. Hence, there can be no negative power $X_b^{-\rho}$ in (8.3). Thus, $u \in \mathcal{O}_1(T_L \cup L)[D] \subseteq \mathcal{O}_q(T_{L_1} \cup L_1)$. $\square$

Since we have established the more restrictive inclusion $\mathcal{U}_L^- \subseteq \mathcal{A}_L^-$ we get

**Corollary 8.7.** *The algebras $\mathcal{A}_L^-$ and $\mathcal{U}_L^-$ coincide with the cluster algebra and upper cluster algebra of $\mathcal{Q}_L^-$ in the sense of [2].*

**Corollary 8.8.** *For the case of the quantum $n \times r$ matrix algebra, the quantum cluster algebra is equal to its upper bound. This holds irrespective of which $B$ we use in our initial seed.*

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