The Dirichlet Problem for Singularly Perturbed Quasilinear Elliptic Equations

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Received February 5, 1980; revised July 10, 1980

1. INTRODUCTION

The study of boundary value problems for singularly perturbed nonlinear equations of elliptic type has a relatively brief history. Most work has centered on the Dirichlet problem for the semilinear equation

$$\varepsilon Lu = h(z, u)$$

(1.1)

where $L$ is a second-order, uniformly elliptic operator, $z \in \mathbb{R}^n$ and $\varepsilon$ is a small positive parameter. Some of the basic references for this problem are Berger and Fraenkel [1, 2], Fife [5], Fife and Greenlee [6] and Howes [11]. Howes [12] has also examined (1.1) with Robin and Neumann boundary conditions.

Much less is known about the Dirichlet problem for the quasilinear equation

$$\varepsilon Lu + \sum_{i=1}^{n} f_i(z, u) u_{z_i} = h(z, u).$$

(1.2)

Assuming a $C^2$ solution, Holland [9] used probabilistic methods to verify the regular and boundary layer expansions for (1.2) in case the $f_i$ are independent of $u$ and $z \in \mathbb{R}^2$. van Harten [14] proved the correctness of formal approximations for solutions of (1.2) with $f_i = 0$, $i \neq n$, in case the boundary values are given on an $n$-sphere. Howes [11, 12] has obtained existence and asymptotic estimates for solutions of various boundary value problems associated with (1.2) in case the equation $h(z, u) = 0$ has a smooth solution $u_0$ which satisfies certain stability conditions and such that the characteristic curves for the equation

$$\sum_{i=1}^{n} f_i(z, u_0) u_{z_i} = h(z, u)$$

are everywhere outgoing on the boundary.
For the linear analogue of (1.2) with $z \in R^2$, Eckhaus and deJager [4] treated the case for which the characteristics of a suitable reduced problem do not intersect in the interior of the domain and have two points of tangency with the boundary. They developed asymptotic expansions which are valid outside neighborhoods of the points of tangency. The problem is complicated near such points because the derivatives of the solution of the reduced problem are generally singular there. However, more recent treatments (for example, Frankena [7] and Grasman [8]) have yielded uniformly valid approximations.

In the present paper, we study the Dirichlet problem for (1.2) with $z \in R^2$ under similar restrictions on the reduced problem. It is shown in Section 2 that under certain conditions there is a classical solution of this problem for small $\varepsilon > 0$. Also, estimates are obtained on the difference between this solution and the solution of a certain reduced problem on the entire domain. In Section 3, some special cases and examples are considered.

2. THE GENERAL PROBLEM

Let $R^n$ denote $n$-dimensional Euclidean space with Euclidean norm $|x|$ and inner product $x \cdot y$ for $x, y \in R^n$. A function defined on some subset of $R^n$ is of class $C^{k,\alpha}$ if the partial derivatives of the function through order $k$ are continuous and the $k$th partials are Hölder continuous with exponent $\alpha$, $0 < \alpha < 1$.

Let $D$ be a bounded, simply connected domain in $R^2$ with boundary $\partial D$ of class $C^{2,\alpha}$. The problem to be studied is

\[ \varepsilon Lu + f(x, y, u) u_x + g(x, y, u) u_y = h(x, y, u), \quad (x, y) \in D, \quad (x, y) \in \partial D, \]

\[ u(x, y) = \phi(x, y), \quad (x, y) \in \partial D, \]  

where $L = a(x, y) u_{xx} + 2b(x, y) u_{xy} + c(x, y) u_{yy}$ and $a, \varepsilon > 0$. Let $N(x, y)$ be an outer normal vector field for $\partial D$ and suppose there is a proper, connected subset $\Gamma$ of $\partial D$ so that

\[ (f(x, y, \phi(x, y)), g(x, y, \phi(x, y))) \cdot N(x, y) \geq 0 \]  

on $\Gamma$ with equality only at the endpoints of $\Gamma$.

The reduced problem for (2.1), (2.2) is

\[ f(x, y, u) u_x + g(x, y, u) u_y = h(x, y, u), \quad (x, y) \in D \]

\[ u(x, y) = \phi(x, y), \quad (x, y) \in \Gamma. \]
Thus $\Gamma$ has been chosen so that the characteristics of (2.4), (2.5) flow out of $D$ along $\Gamma$ and are tangent to $\partial D$ at the endpoints of $\Gamma$.

Let $\gamma(s), 0 \leq s \leq s_0$, be the arclength parametrization of $\Gamma$ in the counterclockwise direction. Assume $\gamma \in C^3[0, s_0]$. Consider the system

\begin{align*}
x_s &= -f(x, y, u), \\
y_s &= -g(x, y, u), \\
u_s &= -h(x, y, u)
\end{align*}

(2.6)

with initial conditions $[y(s, 0), y'_s(0)] = \gamma(s), u(x, 0) = \phi(\gamma(s)), 0 \leq s \leq s_0$. If $f$, $g$, $h$ and $\phi$ are of class $C^3$, then the initial value problem has unique solutions $x(s, \tau), y(s, \tau), u_0(s, \tau)$ of class $C^3$.

Suppose there is a class $C^{3,\infty}$ function $\sigma(s) > 0$, $0 \leq s < s_0$, so that $\partial D - \Gamma$ is given parametrically by $[x(s, \sigma(s))], 0 \leq s \leq s_0$, $[y(s, \tau)] \in D$ for $0 < \tau < \sigma(s)$ and $u(s, \tau)$ is defined for $0 \leq \tau \leq \sigma(s), 0 \leq s \leq s_0$. Let $\Delta(s, \tau) = x_s y_t - x_t y_s$. By (2.3), $\Delta(s, 0) > 0$ for $0 < s < s_0$ and $\Delta(0, 0) = \Delta(s_0, 0) = 0$. Assume $\Delta(s, \tau) > 0$ for $0 \leq \tau \leq \sigma(s), 0 < s < s_0$.

Let $\Omega = \{(s, \tau): 0 < \tau < \sigma(s), 0 < s < s_0\}$. Then the mapping $[y(s, \tau)]$ maps $\tilde{D}$ onto $D$, is locally one-to-one and open on $\Omega$ and maps $\partial \Omega$ homeomorphically onto $\partial D$. It follows that the mapping is a homeomorphism. This result was first proved for $R^2$ by Whyburn [15] in the context of his study of light, open maps. Thus $u_0(s(x, y), \tau(x, y))$ is defined and continuous on $D$ and is of class $C^3$ on $D$ except possibly at the endpoints of $\Gamma$.

Next, in order to simplify approximations near the boundary $\partial D - \Gamma$, we introduce the change of parameter $t = \sigma(s) - \tau$ and relabel the functions $x(s, t) = x(s, \sigma(s) - t), y(s, t) = y(s, \sigma(s) - t)$ and $u_0(s, t) = u_0(s, \sigma(s) - t)$. Define $\delta(s, t) = -\Delta(s, \sigma(s) - t) = x_s y_t - y_s x_t$. We will use the following notation: for a function such as $f$ given in $(x, y)$ coordinates, we let $\tilde{f}$ denote the function $f(x(s, t), y(s, t), u)$ and for a function such as $\delta$ given in $(s, t)$ coordinates, we let $\tilde{\delta}$ denote $\delta(s(x, y), t(x, y))$.

In $(s, t)$ coordinates, (2.1) becomes

\begin{align*}
e L'u + P(s, t, u) u_s + Q(s, t, u) u_t = \tilde{h}(s, t, u),
\end{align*}

(2.7)

where $P = (\tilde{f}_t - \tilde{g}_s) \delta^{-1}, Q = (\tilde{g}_x - \tilde{f}_x) \delta^{-1}$ and $L'u$ is given in Section 4. Note that $P(s, t, u_0) = 0$ and $Q(s, t, u_0) = 1$.

The following theorem extends a result of Coddington and Levinson [3] to elliptic equations.

**Theorem 2.1.** In addition to the above hypotheses, assume:
(1) \( a > 0, b^2 - ac < 0 \) and \( a, b, c \) are of class \( C'^0,\infty \) in \( \bar{D} \); \( f, g \) and \( h \) are of class \( C^3 \) for \( (x, y) \) in \( \bar{D} \) and \( u \in \{ u : |u - u_0(x, y)| \leq \eta(x, y) \} \), where \( \eta = \sigma(|\phi - u_0|) \) near \( \partial D - \Gamma \) and \( \eta = \alpha(1) \) elsewhere, and \( \phi \) is of class \( C'^3,\alpha \) in a neighborhood of \( \partial D \);

(2) the gradient \( \nabla \delta \) is bounded away from zero near the endpoints of \( \Gamma \) and there are numbers \( p, q \geq 1 \) and constants \( c_1, c_2 > 0 \) so that \( c_1 s^p \leq -\delta(s, t) \leq c_2 s^p \) for \( 0 \leq t \leq \sigma(s) \) and small \( s > 0 \) and \( c_1(s_0 - s)^q \leq -\delta(s, t) \leq c_2(s_0 - s)^q \) for \( 0 \leq t \leq \sigma(s) \) and small \( s_0 - s > 0 \);

(3) \( \int_0^\sigma Q(s, 0, u_0(s, 0) + u) du > 0 \) for \( \theta \) between 0 and \( \tilde{\phi}(s, 0) - u_0(s, 0) \), including \( \theta = \tilde{\phi}(s, 0) - u_0(s, 0) \), if \( u_0(s, 0) \neq \tilde{\phi}(s, 0) \), \( 0 < s < s_0 \);

(4) \( Q_u(s, t, u) \) and \( P_u(s, t, u) \) are bounded for \( |u - u_0| = \sigma(s(s_0 - s)) \), \( 0 \leq t \leq \sigma(s) \), \( 0 < s < s_0 \).

Then for sufficiently small \( \varepsilon > 0 \), (2.1), (2.2) has at least one solution \( u_\varepsilon(x, y) \in C'^3,\alpha(\bar{D}) \), and in \((s, t)\)-coordinates we have:

\[
|\tilde{u}_\varepsilon(s, t) - u_0(s, t)| = \sigma(s) \quad \text{for} \quad s \neq \sigma(e^{\varepsilon^{1/(3p+1)}}),
\]

\[
|\tilde{u}_\varepsilon(s, t) - u_0(s, t)| = \sigma(s_0 - s) \quad \text{for} \quad s_0 - s = \sigma(e^{\varepsilon^{1/(3q+1)}}),
\]

\[
|\tilde{u}_\varepsilon(s, t) - u_0(s, t)| \leq w(s, t, \varepsilon) + \sigma(e^{s-3p(s_0 - s)^{-3q}}) \quad \text{otherwise},
\]

where \( w(s, t, \varepsilon) = \sigma(|\tilde{\phi}(s, t) - u_0(s, t)|) \) for \( t = \sigma(\varepsilon) \) and \( w(s, t, \varepsilon) \to 0 \) as \( \varepsilon \to 0 \) for each fixed \( t > 0 \) and \( s \in (0, s_0) \).

**Proof.** The method of proof is the construction of upper and lower solutions for (2.1), (2.2). We will construct only the upper solution since the lower solution can be defined in a similar way.

Let \( \beta_\varepsilon(s, t) = u_0(s, t) + sH(t) \), where \( H(t) = e^{t(m - l)} - 1 \), \( m > t \) and \( l > 0 \) are constants to be chosen below. From the formulas in Section 4, we have that \( L'\beta_\varepsilon \) equals

\[
-\frac{\varepsilon}{\delta} \left[ \nabla \delta \cdot \begin{bmatrix} a & b \\ b & c \end{bmatrix} \nabla \tilde{u}_\varepsilon + \nabla \delta \cdot \begin{bmatrix} a & b \\ b & c \end{bmatrix} (\tilde{H} \nabla \tilde{\delta} + \delta \nabla \tilde{H}) + \sigma \left( \frac{1}{\delta} \right) \right].
\]

We will show this expression is negative for large \( H \) and small \( s > 0 \) by proving the dominance of the term \( \nabla \delta \cdot \begin{bmatrix} a & b \\ b & c \end{bmatrix} \tilde{H} \nabla \tilde{\delta} \).

First, note that \( \nabla \delta = \left[ \frac{\gamma(s)}{\delta^{-1}} \right] \delta^{-1} = \sigma(\delta^{-1}) \) and is everywhere orthogonal to the characteristic direction of (2.4), (2.5). In fact, \( \nabla \delta \) approaches the direction of the inner normal to \( \partial D \) as \( (x, y) \to \gamma(0) \).

Next, we claim that \( \nabla \delta \) approaches the direction of the outer normal to \( \partial D \)
as \((x, y) \rightarrow y(0)\). For the case \(p > 1\), note that \(\nabla \delta \cdot [\delta_x] = \delta_x \rightarrow 0\) as \(s \rightarrow 0\). Thus \(\nabla \delta\), which is bounded away from zero, approaches a direction orthogonal to \([\delta_x]\) as \(s \rightarrow 0\). On the other hand, \(\delta = [x, y] \cdot [\delta_x] \rightarrow 0\) as \(s \rightarrow 0\), so \(\nabla \delta\) also becomes orthogonal to \([\delta_x]\) as \(s \rightarrow 0\). For the case \(p = 1\), at least one of

\[
\left(\frac{\delta_x}{\delta_y}\right)^{\pm 1} = \left(\frac{\delta_x y_t - \delta_t y_x}{-\delta_t x_t + \delta_t x_e}\right)^{\pm 1}
\]

is continuous since \(\delta_s \neq 0\) and either \(x_t\) or \(y_t\) \(\neq 0\) near \(s = 0\). Then the angle of \(\nabla \delta\) with the \(x\)-axis is continuous near \(s = 0\). Since \(\delta(0, 0) = 0\) and \(\delta(s, t) < 0\) for \(0 < s < s_0\), it follows that \(\nabla \delta\) approaches the direction of the outer normal to \(\partial D\) as \((x, y) \rightarrow y(0)\).

Since \([\frac{\delta_x}{\delta_y}]\) is uniformly positive definite on \(\partial D\), it follows that \(\nabla \delta \cdot [\frac{\delta_x}{\delta_y}] H \nabla \delta\) is negative and is the dominant term in \(L'\beta_1\) for small \(s\) and large \(H\).

We can use (2.7) and the mean value theorem to write

\[
e L'\beta_1 + P(s, t, \beta_1) \beta_1 s + Q(s, t, \beta_1) \beta_1 s - \hat{h}(s, t, \beta_1) \leq P_u(s, t, \ast) s H(t)(u_0, + H(t)) + (1 + Q_u(s, t, \ast) s H(t))(u_0, + s H'(t)) - (\hat{h}(s, t, u_0) + \hat{h}_u(s, t, \ast) s H(t)) = s H(t)[P_u(s, t, \ast) u_0, + Q_u(s, t, \ast) u_0, - \hat{h}(s, t, \ast)] + s H'(t) + Q_u(s, t, \ast) s^2 H(t) H'(t) + P_u(s, t, \ast) H^2(t) s (2.8)
\]

since \(u_0, = \hat{h}(s, t, u_0)\), where \(\ast\) is between \(u_0(s, t)\) and \(\beta_1(s, t)\).

First, choose \(d_1 > 0\) so that \(P_u(s, t, \ast) \leq d_1\) and \(d_2 > 0\) so that \(\nabla \delta \cdot [\frac{\delta_x}{\delta_y}] H \nabla \delta\) is dominant in \(L'\beta_1\) for small \(s\) if \(H \geq d_2\). Select \(l = \max\{3d_1 d_2, 2[P_u(s, t, \ast) u_0, + Q_u(s, t, \ast) u_0, - \hat{h}(s, t, \ast)]\}\) and \(m\) so that \(l/3 < d_1 H(t) < l/2\) for \(0 \leq t \leq \sigma(s)\) and small \(s\) (recall that \(H(t) = e^{t\mu - I} - 1\)). It follows that \(H(t) \geq d_2\) and \(P_u(s, t, \ast) H(t) < l/2\) for \(0 \leq t \leq \sigma(s)\), \(s\) small. Thus (2.8) is less than or equal to

\[
s[H(t) l/2 + H'(t) + s Q_u(s, t, \ast) H'(t) + H(t) l/2]\]

\[
= s[-l + Q_u(s, t, \ast) s H'(t)] < 0
\]

for small \(s\) since \(H' + IH = 0\). We have shown that \(\beta_1\) satisfies

\[
e L'\beta_1 + P(s, t, \beta_1) \beta_1 s + Q(s, t, \beta_1) \beta_1 s - \hat{h}(s, t, \beta_1) < 0
\]
near $\gamma(0)$. Also, $\beta_1(s, t) \geq \bar{\phi}(s, t)$ on $\partial D$ for small $s$ and large $H$, so $\beta_1$ is a "local" upper solution for (2.1), (2.2) in a deleted neighborhood of $\gamma(0)$. In a similar manner, one can show that $\beta_2(s, t) = u_0(s, t) + (s_0 - s)(e^{(m-t)} - 1)$ is a local upper solution in a deleted neighborhood of $\gamma(s_0)$ for some choices of $l$ and $m$.

The next step is to construct an upper solution outside neighborhoods of $\gamma(0)$ and $\gamma(s_0)$ of magnitudes $s = \mathcal{O}(s_0^{-\eta})$ and $s_0 - s = \mathcal{O}(s_0^{-\eta})$, respectively. Since the expression $\alpha\gamma_2 - 2bx_1y_1 + cx_3^2$ is positive in $\bar{D}$, we can assume without loss of generality that it is identically one (see the formula for $L'$ in Section 4).

By assumption (4), there is an $\eta > 0$ and a $k \in (0, 1]$ so that $Q(s, 0, u_0(s, 0) + u) > k$ for $0 < |s|, |s_0 - s| < \eta$ and $|u| < |\phi(s, 0) - u_0(s, 0)|$.

There is $C^2, 0 > 0$ function $\psi(s)$ and a constant $\rho \in (0, k)$ so that $\psi(s) - \mathcal{O}(s)$ as $s \to 0$, $\psi(s) = \mathcal{O}(s_0 - s)$ and $\psi(s) > 0$ and $\psi(s) \geq \bar{\phi}(s, 0) - u_0(s, 0)$ for $0 \leq s_0 < s_0$, $\int_0^1 (-Q(s, 0, u_0(s, 0) + u) + \rho) du < 0$ for $\eta/2 \leq s \leq s_0 - \eta/2$, $0 < \lambda \leq \psi(s)$ and $Q(s, 0, u_0(s, 0) + u) > k$ for $0 < |s|, |s_0 - s| < \eta$, $0 < u < \psi(s)$.

Let functions $\theta(s)$, $\phi(s) \in C^3(0, s_0)$ be chosen so that $\theta(s) = 0$ for $0 < s < \eta/2$, $\theta(s) = 1$ for $\eta < s < s_0$, $\phi(s) = 1$ for $0 < s < s_0 - \eta$, $\phi(s) = 0$ for $s_0 - \eta/2 < s < s_0$ and $\theta(s), \phi(s) \in (0, 1)$ otherwise. Define

$$F(s, u) = (1 - \theta(s)) c_1 s_2^p (\rho - k)$$
$$+ \theta(s) \phi(s) (\rho - Q(s, 0, u_0(s, 0) + u)) c_3$$
$$+ (1 - \phi(s)) c_4 (s_0 - s)^{2q} (\rho - k),$$

for $0 \leq u \leq \psi(s)$, $0 < s < s_0$, where $c_1$ is the constant of assumption (2) and $c_3$ satisfies $0 < c_3 \leq \delta^2(t, s)$ for $0 \leq t \leq \sigma(s)$, $\eta/2 \leq s \leq s_0 - \eta/2$.

Let $w$ be the solution of

$$\epsilon \psi_{tt} = F(s, w) w_t,$$

$$w(s) = \psi(s), \quad w \to 0 \quad \text{as} \quad t \to \infty$$

given by Lemma 4.1. The estimates on the derivatives of $w$ given in that lemma are valid for $\eta/2 < s < s_0 - \eta/2$. Of course, for $s \leq \eta/2$, $w = \psi(s) \exp(c_1 s_2^p (\rho - k)(t/\epsilon))$ and for $s_0 - \eta/2 \leq s$, $w = \psi(s) \exp(c_1 (s_0 - s)^{2q} (\rho - k)(t/\epsilon))$.

Define $\beta_3(s, t) = u_0(s, t) + w(s, t, \psi) + \epsilon G(s) H(s, t)$, where $H(s, t) = e^{(m(s) - t)} - 1$, $l > 0$ and $m(s) > t$ are to be chosen below, $G(s) = s^{-3p} (s_0 - s)^{-3q}$ and $p$ and $q$ are given by assumption (2). We will show $\beta_3$ satisfies the appropriate differential inequality.

First, consider the case that $\eta/2 \leq s \leq s_0 - \eta/2$. Using the mean value theorem, we have
\[ \varepsilon L' \beta_3 + P(s, t, \beta_3) \beta_{3s} + Q(s, t, \beta_3) \beta_{3t} - \tilde{h}(s, t, \beta_3) \]

\[ = \varepsilon L' \beta_3 + P_u(s, t, *) (w + \varepsilon GH) \beta_{3s} \]

\[ + (1 + Q_u(s, t, *) (w + \varepsilon GH))(u_{0t} + \varepsilon GH_t) \]

\[ + (Q(s, t, u_0 + w) + Q_u(s, t, **) \varepsilon GH) w_t \]

\[ - \tilde{h}(s, t, u_0) - \tilde{h}_u(s, t, *) (w + \varepsilon GH), \]

where \(*\) is between \(u_0(s, t)\) and \(\beta_3(s, t)\) and \(**\) is between \(u_0(s, t) + w(s, t, \varepsilon)\) and \(\beta_3(s, t)\).

Writing

\[ Q(s, t, u_0(s, t) + w) = Q(s, 0, u_0(s, 0) + w) + (Q(s, t, u_0(s, t) + w) \]

\[ - Q(s, 0, u_0(s, 0) + w)), \]

and using the properties of \(w\) given by Lemma 4.1, we have

\[ \varepsilon L' w + Q(s, t, u_0(s, t) + w) w_t \]

\[ = \varepsilon \delta^{-2} w_{tt} + \mathcal{O}(tw) + \mathcal{O}\left(\frac{t^2 w}{\varepsilon}\right) + \mathcal{O}(w) + \mathcal{O}\left(\frac{tw}{\varepsilon}\right) \]

\[ + Q(s, 0, u_0(s, 0) + w) w_t + \mathcal{O}(tw_t) \]

\[ \leq \rho w_t + \mathcal{O}(tw) + \mathcal{O}\left(\frac{t^2 w}{\varepsilon}\right) + \mathcal{O}(w) + \mathcal{O}\left(\frac{tw}{\varepsilon}\right) + \mathcal{O}(tw_t). \]  

(2.10)

Since \(\rho > 0\), \(w_t < 0\), and \(w_t = \mathcal{O}(w/\varepsilon)\), (2.10) is negative on some interval \(0 < t < t'\), for small \(\varepsilon > 0\).

Next, we collect all terms in (2.9) involving \(w\):

\[ \Phi = \varepsilon L' w + P_u(s, t, *) \beta_{3s} + P_u(s, t, *) \varepsilon GH w_t \]

\[ + Q_u(s, t, *) w(u_0 + \varepsilon GH) \]

\[ + (Q(s, t, u_0 + w) + Q_u(s, t, **) \varepsilon GH) w_t - \tilde{h}_u(s, t, *) w. \]

Note that \(\Phi < 0\) for \(0 \leq t \leq t'\) and small \(\varepsilon\), and \(\Phi\) is exponentially small for \(t' \leq t\). Since \(u_{0t} = \tilde{h}(s, t, u_0)\), (2.9) equals

\[ \varepsilon L' u_0 + \varepsilon^2 L'(GH) + \Phi + P_u(s, t, *) \varepsilon GH(u_{0s} + \varepsilon G'H + \varepsilon GH) \]

\[ + \varepsilon GH_t + Q_u(s, t, *) \varepsilon GH(u_{0t} + \varepsilon GH_t) - \tilde{h}_u(s, t, *) \varepsilon GH \]

\[ = \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon^2) + \Phi + \varepsilon GH(P_u(s, t, *) u_{0s} \]

\[ + Q_u(s, t, *) u_{0t} - \tilde{h}_u(s, t, *)) + \varepsilon GH_t + \mathcal{O}(\varepsilon^2). \]
Let $l$ be an upper bound for $P_u(s, t, *) u_{0s} + Q_u(s, t, *) u_{0t} - \bar{h}_u(s, t, *)$. Then (2.8) is less than or equal to

$$
\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon^3) + \Phi + \varepsilon G(lH + H_t) = \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon^3) + \Phi - \varepsilon G < 0
$$

for large enough $l$ and small $\varepsilon > 0$. Thus (2.9) is negative for the case $\eta/2 \leq s \leq s_0 - \eta/2$.

Suppose now that $\eta/2 \geq s \geq \mu e^{1/(3p + 1)}$, for some $\mu > 0$. By the mean value theorem,

$$
\varepsilon L' \beta_3 + P(s, t, \beta_3) \beta_{3s} + Q(s, t, \beta_3) \beta_{3t} - \bar{h}(s, t, \beta_3)
$$

$$
= \varepsilon L' \beta_3 + P(s, t, *)(w + \varepsilon GH) \beta_{3s}
$$

$$
+ (1 + Q_u(s, t, *)(w + \varepsilon GH))(u_{0t} + \varepsilon GH_t) - \bar{h}(s, t, u_0)
$$

$$
- \bar{h}_u(s, t, *)(w + \varepsilon GH) + Q(s, t, \beta_3) w_t,
$$

(2.11)

where $*$ is between $u_0(s, t)$ and $\beta_3(s, t)$. By the calculations in Section 4 and the fact that $w = \psi(s) \exp(c_1 s^{2p}(\rho - k)(t/c))$,

$$
\varepsilon L' w + Q(s, t, \beta_3) w_t
$$

$$
\leq \varepsilon [s^{2p}(\rho - k)^2 \varepsilon^{-2} w + \mathcal{O}(w^{s-2p-1})] + \mathcal{O}(w^{s-1} t^{s-2}) + \mathcal{O}(w t^{s-2p-2} \varepsilon^{-2})
$$

$$
+ \mathcal{O}(w t^{s-1} \varepsilon^{-1}) + \mathcal{O}(w t^{s-2p-1} \varepsilon^{-2}) + \mathcal{O}(w^{s-3p-1}) + \mathcal{O}(w t^{s-1} \varepsilon^{-p-1})
$$

$$
+ \mathcal{O}(w^{s-1} \varepsilon^{-p})] + \varepsilon s^{2p}(\rho - k) w / \varepsilon
$$

$$
\leq \frac{s^{2p}(\rho - k) \rho w}{\varepsilon} + \mathcal{O}
\left(\frac{w}{s^p}\right)
$$

$$
+ \mathcal{O}(s^{2p-1} w) + \mathcal{O}\left(\frac{t^{2p-2} w}{\varepsilon}\right)
$$

(2.12)

for $s \geq \mu e^{1/(3p + 1)} \geq t$ since $Q(s, t, \beta_3) > k$ for small $\varepsilon > 0$.

Define $\Psi$ to be the collection of terms in (2.11) involving $w$; i.e.,

$$
\Psi \equiv \varepsilon L' w + Q(s, t, \beta_3) w_t + P_u(s, t, *)(w \beta_{3s} + P_u(s, t, *) \varepsilon GH w_s
$$

$$
+ Q_u(s, t, *)(w_{0t} + \varepsilon GH_t) - \bar{h}_u(s, t, *) w.
$$

From (2.12) and hypothesis (4),

$$
\Psi \leq w s^{2p}(\rho - k) \rho e^{-1} + \mathcal{O}(w^{s-p}) + \mathcal{O}(w t^{s-2p-1} \varepsilon^{-1})
$$

$$
+ \mathcal{O}(w t^{2p-2} \varepsilon^{-1}) + \mathcal{O}(w) + \mathcal{O}(w t^{2p-1} \varepsilon^{-1}) + \mathcal{O}(w e^{s-3p-1})
$$

$$
+ \mathcal{O}(w t^{s-1} \varepsilon^{-p}) + \mathcal{O}(w e^{s-3p}).
$$

Since $s^{2p}(\rho - k) \rho e^{-1} w < 0$, $\Psi < 0$ if $t = o\left(\varepsilon^{1/(3p + 1)}\right)$ and $s \geq \mu e^{1/(3p + 1)}$. Furthermore, $\Psi$ is exponentially small if $t \geq \mathcal{O}(\varepsilon^{1/3})$. 
Now (2.11) equals

\[ \varepsilon L' u_0 + \varepsilon L' (e^{GH}) + \Psi + P_u (s, t, *) \varepsilon G H (u_{0s} + \varepsilon G'H + \varepsilon G H_s) + u_{0t} \\
+ \varepsilon G H_t + Q_u (s, t, *) \varepsilon G H (u_{0s} + \varepsilon G H_t) - \bar{h}(s, t, u_0) - \bar{h}_u (s, t, *) \varepsilon G H \\
= \mathcal{O} \left( \frac{\varepsilon}{s^{3p}} \right) + \mathcal{O} \left( \frac{\varepsilon^2 H}{s^{6p+1}} \right) + \Psi + \varepsilon G H_t + \mathcal{O} \left( \frac{\varepsilon^2 H^2}{s^{6p+1}} \right) + \mathcal{O} \left( \frac{\varepsilon^3 H H_t}{s^{6p}} \right) \\
+ \varepsilon G H (P_u (s, t, *) u_{0s} + Q_u (s, t, *) u_{0t} - \bar{h}_u (s, t, *))). \]

If \( l \) is an upper bound for \( P_u (s, t, *) u_{0s} + Q_u (s, t, *) u_{0t} - \bar{h}_u (s, t, *) \), (2.11) is less than

\[ \mathcal{O} \left( \frac{\varepsilon}{s^{3p}} \right) + \mathcal{O} \left( \frac{\varepsilon^2 H}{s^{6p+1}} \right) + \Psi + \frac{\varepsilon l}{s^{3p} (s^3 - s)^3} + \mathcal{O} \left( \frac{\varepsilon^2 H^2}{s^{6p+1}} \right) + \mathcal{O} \left( \frac{\varepsilon^3}{s^{6p}} \right). \]

Recalling that \( H(s, t) = e^{l (m(s) - t) - 1} \), it is easy to check that the above expression is negative for suitable choices of \( l \) and \( m(s) \) if \( \eta/2 \geq s \geq \mu \varepsilon^{1/(3p+1)} \) and \( \varepsilon > 0 \) is sufficiently small. The proof that \( \beta_3 \) satisfied the differential inequality for \( \eta/2 \geq s_0 - s \geq \mu \varepsilon^{1/(3q+1)} \) is similar.

It remains to show that a continuous upper solution can be constructed from \( \beta_1, \beta_2, \beta_3 \). If \( s = \mathcal{O} (\varepsilon^{1/(3p+1)}) \), both \( \beta_1 \) and \( \beta_3 \) satisfy the requisite differential inequality. It is easy to check that the surfaces \( z = \beta_1 (s, t) \) and \( z = \beta_2 (s, t) \) intersect along a curve for which \( s = \mathcal{O} (\varepsilon^{1/(3p+1)}) \) and that \( \beta_{1s} > \beta_{3s} \) if the number \( l \) in \( \beta_1 \) is large enough. Thus \( \min \{ \beta_1, \beta_3 \} \) is continuous for \( s = \mathcal{O} (\varepsilon^{1/(3p+1)}) \). Similarly, \( \min \{ \beta_2, \beta_3 \} \) is continuous for \( s_0 - s = \mathcal{O} (\varepsilon^{1/(3q+1)}) \). Thus we can construct a function \( \beta \in C(\bar{D}) \) from \( \beta_1, \beta_2, \beta_3 \) which is of class \( C_{2, \alpha} \) on \( \bar{D} \) except at the intersections of \( \beta_1, \beta_3 \) and \( \beta_2, \beta_3 \) and at the endpoints of \( \Gamma \).

Similarly, a lower solution \( \tau \in C(\bar{D}) \) can be defined, and we have from standard results on differential inequalities such as Theorem 3.2 of [13] that (2.1) and (2.2) has at least one solution \( u_c \in C^{2, \alpha} (\bar{D}) \) so that \( \tau (x, y) \leq u_c (x, y) \leq \beta (x, y) \) for \( (x, y) \in \bar{D} \). Q.E.D.

All of the hypotheses of Theorem 2.1 except (2) and (4) are similar to conditions used by other authors (see \[3, 9, 10\]) to study problems of this type. We will discuss (2) and (4) in the context of the examples in the next section. Finally, note that the estimates in Theorem 2.1 yield the uniform estimate

\[ |u_c (x, y) - \bar{u}_0 (x, y)| \leq \hat{w} (x, y, \varepsilon) + \mathcal{O} (\varepsilon^{1/(3r+1)}), \]

for \( (x, y) \in \bar{D} \) and small \( \varepsilon > 0 \), where \( r = \max \{ p, q \} \).
3. Special Cases and Examples

First, consider the semilinear problem

\[ \varepsilon Lu + f(x, y) u_x + g(x, y) u_y = h(x, y, u) \quad (x, y) \in D, \]
\[ u(x, y) = \phi(x, y) \quad (x, y) \in \partial D, \]  

where \( L, f, g, h, \phi \) and \( D \) satisfy the conditions of Section 2. Assume \( \Gamma \) can be chosen so that (2.3) holds. Let \( x(s, \tau), y(s, \tau), \) and \( u_0(s, \tau) \) be the solutions of the initial value problem (2.6). Assume there is a \( C^{2,\alpha} \) function \( \sigma(s) \) so that \( \partial D - \Gamma \) is given by \( \{y < \sigma(s)\} \), \( 0 \leq s \leq s_0 \). If \( u_0(s, \tau) \) is defined and \( x_s y_t - x_t y_s \neq 0 \) for \( 0 \leq \tau \leq \sigma(s), 0 < s < s_0 \), then the reduced problem

\[ f(x, y) u_x + g(x, y) u_y = h(x, y, u) \quad (x, y) \in D, \]
\[ u(x, y) = \phi(x, y) \quad (x, y) \in \Gamma, \]

has a solution \( \tilde{u}_0(x, y) \) which is continuous on \( \bar{D} \) and of class \( C^3 \) except possibly at the endpoints of \( \Gamma \) (see the discussion preceding Theorem 2.1).

Letting \( t = \sigma(s) - \tau \), we have in \( (s, t) \) coordinates the equation

\[ \varepsilon L' u + u_t = \tilde{h}(s, t, u) \]

since \( P \equiv 0 \) and \( Q \equiv 1 \). Clearly (3), (4) of Theorem 2.1 are satisfied. Recall that \( \delta = x_s y_t - y_s x_t \). Assume numbers \( p, q \geq 1 \) exist so that the second part of (2) in Theorem 2.1 is satisfied. An easy calculation gives \( \delta_t = \delta(f_x + g_y) \). Then

\[ \nabla \delta = \frac{1}{\delta} \begin{bmatrix} \delta_y y_t - \delta_t y_s \\ -\delta_s x_t + \delta_t x_s \end{bmatrix} = \begin{bmatrix} \delta^{-1} \delta_y y_t - (f_x + g_y) y_s \\ -\delta^{-1} \delta_s x_t + (f_x + g_y) x_s \end{bmatrix}, \]

so \( |\nabla \delta| = \mathcal{O}(\delta^{-1} \delta_t) \), which is bounded away from zero near the endpoints of \( \Gamma \). We can apply Theorem 2.1 to obtain a \( C^{2,\alpha} \) solution \( u_\varepsilon \) of (3.1) for small \( \varepsilon > 0 \) and to conclude that \( u_\varepsilon \to \tilde{u}_0 \) as \( \varepsilon \to 0 \) outside a boundary layer along \( \partial D - \Gamma \). Furthermore, the results of Holland [9] yield regular and boundary layer expansions which are valid outside fixed neighborhoods of the endpoints of \( \Gamma \).

Next, consider the case where \( f \) is identically zero in (2.1):

\[ \varepsilon Lu - g(x, y, u) u_y = h(x, y, u) \quad (x, y) \in D, \]
\[ u(x, y) = \phi(x, y) \quad (x, y) \in \partial D. \]

The characteristics of the reduced problem are vertical lines. Suppose there are precisely two points where tangent lines to \( \partial D \) are vertical. Let \( y^+(x), \)
$\gamma^-(x)$ denote the functional representations of the upper and lower boundaries of $D$, respectively, $x_1 \leq x \leq x_2$. The choice of $\Gamma$ for this problem depends on the sign of $g$. For the case $g > 0$, the reduced problem is

$$-g(x, y, u) u_y = h(x, y, u) \quad (x, y) \in D,$$

$$u(x, y) = \phi(x, y) \quad (x, y) \in \Gamma,$$

where $\Gamma$ is the lower boundary of $D$.

Adapting Theorem 2.1 to this special case, we have:

**Theorem 3.1.** Assume (1) of Theorem 2.1 and

2. $\gamma^\pm(x) = C^\pm(x)(x - x_1)^{(p+1)}$ for small $x - x_1 > 0$ and $\gamma^\pm(x) = D^\pm(x)(x_2 - x)^{(q+1)}$ for small $x_2 - x > 0$, where $p, q \geq 1$, $C^\pm, D^\pm$ are of class $C^2$ and $C^\pm(x_1) \neq 0, D^\pm(x_2) \neq 0$;

3. the reduced problem (3.3) has a solution $u_0(x, y)$ continuous on $\bar{D}$ and of class $C^{2,a}$ except possibly at the endpoints of $\Gamma$;

4. $g(x, y, u_0(x, y)) > 0$ for $(x, y) \in \bar{D}$;

5. $(\phi(x, \gamma^+(x)) - u_0(x, \gamma^+(x))) \int_{u_0(x, \gamma^+(x))}^{u(x, \gamma^+(x))} g(x, y, u) \, du > 0$ for $\theta$ between $u_0(x, \gamma^+(x))$ and $\phi(x, \gamma^+(x))$, including $\phi(x, \gamma^+(x))$, if $u_0(x, \gamma^+(x)) \neq \phi(x, \gamma^+(x))$.

Then for small $\varepsilon > 0$, (3.2) has at least one solution $u_\varepsilon(x, y) \in C^{2,a}(\bar{D})$ and

- $|u_\varepsilon(x, y) - u_0(x, y)| = O((x - x_1)^{(p+1)})$ for $x - x_1 = O(\varepsilon^{1/2})$,
- $|u_\varepsilon(x, y) - u_0(x, y)| = O((x_2 - x)^{(q+1)})$ for $x_2 - x = O(\varepsilon^{1/2})$,
- $|u_\varepsilon(x, y) - u_0(x, y)| \leq w(x, y, \varepsilon) + O(\varepsilon(x - x_1)^{(2p-1)(q+1)}(x_2 - x)^{(2q-1)(q+1)})$ otherwise,

where $w$ is a boundary layer function.

It is easy to check that assumptions (1)–(5) above imply that the conditions of Theorem 2.1 are satisfied. However, the estimates given above for $|u_\varepsilon - u_0|$ are better than those obtained from Theorem 2.1 if $p > 1$ or $q > 1$. It is possible to achieve these better estimates in this case because the singularities of $L u_0(x, y)$ at the end points of $\Gamma$ are at worst $O(s^{-2p-1})$ and $O((s_0 - s)^{-2q-1})$, while for the general problem they may be as bad as $O(s^{-3p})$ and $O((s_0 - s)^{-3q})$. The estimates of Theorem 3.1 agree with those found by Frankena [7] for the linear problem.

van Harten [14] has obtained uniformly valid asymptotic expansions for solutions of (3.2) in the case that $D$ is a disk. Theorem 3.1 requires less regularity for the existence of a solution than van Harten's result.
Before giving further examples, let us examine hypotheses (2) and (4) of Theorem 2.1. The condition that $\nabla \delta$ is bounded away from zero was automatically satisfied in the above special cases. For the case $p = q = 1$, this condition is certainly true since $\nabla \delta \cdot [\delta_x, \delta_y] = \delta_x$ and $\delta_y \neq 0$ at $s = 0, s_0$. We do not know if this condition is satisfied in all cases. Concerning (4), $P_u(s, t, u) = \delta^{-1}(f_u y_t - g_u y_s)$ and $Q_u(s, t, u) = \delta^{-1}(g_u x_t - f_u y_s)$. Thus in order for (4) to hold, it is necessary that either $f_u$ and $g_u$ tend to zero or $[f_u]/[g_u]$ approach the characteristic direction as $(x, y)$ approaches either endpoint of $\Gamma$ and $u \to u_0$. The above special cases give representatives of each of these possibilities.

We close this section with two specific examples.

**Example 3.1.** Consider the problem

$$
e\nabla^2 u + uu_x + u_y = 0 \quad x^2 + y^2 < r^2, \quad (3.4)$$

$$u = \gamma \quad x^2 + y^2 = r^2, \quad (3.5)$$

where $n \geq 2$.

If $r \leq 1$, we may choose $\Gamma = \{(x, y): x^2 + y^2 = r^2, y \geq 0\}$, and the reduced problem is

$$u_{xx} + u_y = 0 \quad x^2 + y^2 < r^2,$$

$$u = \gamma \quad (x, y) \in \Gamma.$$ 

The solutions of the initial value problem (2.6) are $x(s, \tau) = -r^n \sin^n(s/r) + r \cos(s/r)$, $y(s, \tau) = -\tau + r \sin(s/r)$ and $u_o(s, \tau) = r \sin(s/r)$. The lower boundary of the disk is given parametrically by $[x(s, \sigma(s))], \ \gamma = \sigma(s) = 2r \sin(s/r) \left(1 + r^n \sin^n(s/r) \cos(s/r) \right)$, $0 \leq s \leq \pi r$.

Also,

$$\Delta(s, \tau) = \sin(s/r) \left[ 1 + r^n \sin^{n-2}(s/r) \cos(s/r) \tau \right]$$

$$+ r^n \sin^{n-1}(s/r) \cos(s/r).$$

It is easy to see that $\Delta(s, \tau) > 0$, for $0 \leq \tau \leq \sigma(s)$, $0 < s < \pi r$, if $r \leq \frac{1}{2}$. The discussion preceding Theorem 2.1 assures us that the reduced problem has a continuous solution $\hat{u}_o(x, y)$ for $x^2 + y^2 \leq r^2$ which will be of class $C^\infty$ except at the points $(-r, 0)$ and $(r, 0)$.

Letting $t = \sigma(s) - \tau$, Eq. (3.4) becomes

$$\varepsilon L'u + P(s, t, u) u_x + Q(s, t, u) u_t = 0,$$
where
\[ P(s, t, u) = \delta^{-1} [u^n - (r \sin(s/r))^n], \]
\[ Q(s, t, u) = \delta^{-1} \left( -\sin(s/r) r^{n-1} n \sin^{n-2}(s/r) \cos(s/r)(\sigma(s) - t) \right. \]
\[ \left. + r^n \sin^{n-1}(s/r) \sigma'(s) + 1 \right] - u^n(\cos(s/r) - \sigma'(s)), \]
\[ \delta(s, t) = -\Delta(s, \sigma(s) - t). \]

Checking the hypotheses of Theorem 2.1, we see that (2) is true with \( p = q = 1 \), (4) is true since \( n \geq 2 \) and (3) is true for sufficiently small \( r \). In fact, for any \( r < \frac{1}{3} \), all the conditions of the theorem are satisfied, and problem (3.4), (3.5) has a \( C^{2,\alpha} \) solution \( u(\varepsilon, x, y) \) for small \( \varepsilon > 0 \) which satisfies the inequalities given in Theorem 2.1, as well as the uniform estimate \( u(\varepsilon, x, y) - u_0(x, y) \leq \omega(x, y, \varepsilon) + \sigma(\varepsilon^{1/4}) \) for \( x^2 + y^2 \leq r^2 \).

If \( n = 1 \) in (3.4), one can show that all the hypotheses of Theorem 2.1 are valid for small \( r \) except (4). However, our method of analysis fails in this case.

**Example 3.2.** Finally, consider briefly the problem
\[ \varepsilon \Delta^2 u + yu_x + uu_y = 0 \quad x^2 + y^2 < r^2, \]
\[ u = y + 2 \quad x^2 + y^2 = r^2. \]

Again, \( \Gamma = \{(x, y): x^2 + y^2 = r^2, y \geq 0\} \). Note that \( [\frac{\varepsilon}{\delta}] = [0, 1] \) and \( u_0(x, y) = r \sin(s/r) + 2 \), so the vector field \( [\frac{\varepsilon}{\delta}] \) has the same direction as the characteristics of the reduced problem at the endpoints of \( \Gamma \) (see the remarks preceding Example 3.1). In fact, a somewhat lengthy calculation shows that this problem satisfies the hypotheses of Theorem 2.1 if \( r \) is small enough and the conclusions of that theorem are valid in this case.

4. **Appendix**

The following lemma was used in the proof of Theorem 2.1.

**Lemma 4.1.** Let \( I \) be a compact interval of real numbers, and suppose \( F: I \times R^1 \rightarrow R^1 \) and \( \psi: I \rightarrow (0, \infty) \) are continuous and \( F(s, 0) < 0 \) and \( \int_0^s F(s, u) \, du < 0 \) for \( 0 < \theta \leq \psi(s) \) and \( s \in I \). Then the equation
\[ \varepsilon u_{tt} = F(s, u) u_t \quad (4.1) \]
has for each \( \varepsilon > 0 \) and \( s \in I \) a unique decreasing solution \( w(s, t, \varepsilon) \) satisfying \( w(s, 0, \varepsilon) = \psi(s) \) and \( w(s, t, \varepsilon) \rightarrow 0 \) as \( t \rightarrow \infty \). Furthermore, \( w \) is continuous in
s. There is a constant \( \mu_1 \) so that
\[ |w_s(s, t, \varepsilon)| \leq \mu_1(w(s, t, \varepsilon)/\varepsilon) \]
for \( s \in I, \ v > 0 \) and \( t \geq 0 \), and given \( \delta > 0 \) and \( s \in I \), there are constants \( C_{\delta}(s) \) and \( D_{\delta}(s) \) so that
\[ D_{\delta}(s) e^{(F(s, 0) - \delta)(\varepsilon/\varepsilon)} \leq w(s, t, \varepsilon) \leq C_{\delta}(s) e^{(F(s, 0) + \delta)(\varepsilon/\varepsilon)} \]
for \( t \geq 0 \) and \( \varepsilon > 0 \). If \( F \) and \( \psi \) have continuous second derivatives with respect to \( s \), then \( w \) has continuous second partials, and there are constants \( |\mu_{i\mid}^2 \) so that
\[ |w_{ss}(s, t, \varepsilon)| \leq (\mu_2 + \mu_3(t/\varepsilon)) w(s, t, \varepsilon), \]
\[ |w_{st}(s, t, \varepsilon)| \leq (\mu_4 + \mu_5(t/\varepsilon)) (w(s, t, \varepsilon)/\varepsilon) \]
for \( s \in I, t \geq 0 \) and \( \varepsilon > 0 \).

**Proof.** The unique solution \( w \) of (4.1) is given implicitly by
\[ t/\varepsilon = \int_{G(s, \theta)}^{\infty} \frac{d\theta}{G(s, \theta)} , \quad G(s, \theta) = \int_{0}^{F(s, \theta)} du. \]

Note that \( w \) is decreasing in \( t \) and continuous in \( s \). Differentiating (4.3) with respect to \( t \), we find \( w_t = ((G(s, w))/\varepsilon) \), so there is a constant \( \mu \), for which
\[ |w_s(s, t, \varepsilon)| \leq \mu_1((w(s, t, \varepsilon))/\varepsilon) \]
for \( s \in I \) and \( t \geq 0 \). The estimates (4.2) follow easily from (4.3) (see the proof of Lemma 2.1 in [5]).

Suppose that \( F \) and \( \psi \) have continuous second derivatives with respect to \( s \). Differentiating (4.3) with respect to \( s \), we obtain
\[ w_s = \frac{\psi'}{G(s, \psi)} G(s, w) + \frac{G(s, \psi)}{G^2(s, \psi)} \int_{G(s, \psi)}^{\infty} G_s(s, \theta) d\theta. \]

Let \( H(s, t, \varepsilon) = \int_{G(s, \psi)}^{\infty} \frac{G_s(s, \theta) G^2(s, \theta)}{G(s, \psi)} d\theta. \) Then \( H(s, 0, \varepsilon) = 0 \) and \( H_t = \frac{(G_s(s, w)/G^2(s, w)) w_t (G(s, w)/G(s, w))(1/\varepsilon)}{(G(s, w))} \) and \( \lim_{w \to 0} (G_s(s, w))/G(s, w)) = ((F_s(s, 0))/(F(s, 0))) < \infty. \) Thus \( |H(s, t, \varepsilon)| \leq (Nt)/\varepsilon \) for some constant \( N \) and \( t > 0, \ v > 0 \), so there are constants \( \mu_2, \mu_3 \) so that
\[ |w_s(s, t, \varepsilon)| \leq (\mu_2 + \mu_3(t/\varepsilon)) w(s, t, \varepsilon). \]
The estimates for the other partials follow in a similar manner. Q.E.D.

Next, for the convenience of the reader, we list some formulas used in the discussion of Section 2. Let \( x(s, t), y(s, t) \) and \( \delta(s, t) \) be defined as in Section 2. If \( u(x, y) \) is of class \( C^2 \), we have \( u_x = (u_x y_i - u_i y_x) \delta^{-1} \), and \( u_y = (-u_x x_i + u_i x_x) \delta^{-1} \). Further, the operator \( L \) in (2.1) becomes
\[ L'u = (a_y^2 - 2b x_y + c x_x) \delta^{-2} u_{tt} + (a_y^2 - 2b x_y y_t + c x_t^2) \delta^{-2} u_{ss} - \frac{1}{2} a_y y_t - b(y_x y_t + y_x^2) + 2c x_x y_x \delta^{-2} u_{st} \]
\[ + [a(y_y y_y y_t - y_y y_t y_t) + 2b(x_y y_t y_t - x_t y_t) + c(x_t x_t y_t - x_t x_t)] \delta^{-2} u_{tt} \]
\[ + [a(y_y y_y y_t - y_y y_t y_t) + 2b(y_t x_t \delta - y_t x_t \delta_t) + 2b(y_t x_t \delta - y_t x_t \delta_t)] \]
\[ + [a(y_y y_y y_t - y_y y_t y_t) + 2b(y_t x_t \delta - y_t x_t \delta_t) + 2b(y_t x_t \delta - y_t x_t \delta_t)] \]
\begin{align*}
+ c(x_i x_s \delta - x_i^2 \delta_s) \delta^{-1} (\delta^{-2} u_s) \\
+ [a(y_s y_s - y_s y_s) + b(x_i y_s - y_s x_i) + c(x_i x_s - x_i x_s)] \\
+ [a(y_s y_s \delta - y_s^2 \delta_s) + b(y_s x_s \delta - y_s x_i \delta)] \\
+ c(x_i x_s \delta - x_i^2 \delta_s) \delta^{-1}] (\delta^{-2} u_t).
\end{align*}

Finally, one can show that the terms in the expression for \( L'u \) of order \( O(\delta^4) \) are given in the original \((x, y)\) coordinates by \(-\delta^{-1} \nabla \cdot \left[ \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right] \nabla u\), where \( \nabla \) is the gradient operator.

**ACKNOWLEDGMENTS**

The author wishes to thank Professor Morris Marx for calling his attention to the theorem of G. T. Whyburn.

**REFERENCES**
