ON SIMPLE CHARACTERIZATIONS OF $k$-TREES*

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Abstract. $k$-trees are a special class of perfect elimination graphs which arise in the study of sparse linear systems. We present four simple characterizations of $k$-trees involving cliques, paths, and separators.

1. Introduction

$k$-trees are a special class of Michigan graphs,¹ that is, graphs $G = (X, E)$, where $X$ is a nonempty finite set of vertices and $E$ is a set of pairs of distinct vertices called edges. Recalling that a clique in $G$ is a nonempty subset of vertices each distinct pair of which is an edge of $G$, $k$-trees are defined recursively as follows. A $k$-tree on $k$ vertices is a graph whose vertex set is a clique on $k$ vertices ($k$-clique); and given any $k$-tree $T_k(n)$ on $n$ vertices, a $k$-tree on $n + 1$ vertices is obtained when the $(n + 1)$st vertex is made adjacent to each vertex of a $k$-clique in $T_k(n)$.

Let $T_k(n) = (X, E)$ be a $k$-tree on $n$ vertices and let $X = \{x_i\}_{i=1}^n$, where $x_i$ is the vertex added to the $k$-tree $T_k(i-1)$ to produce the $k$-tree $T_k(i)$, $i > k$, $\{x_i\}_{i=1}^k$ the “base” clique. Note then that $k$-trees are perfect elimination graphs, that is, graphs $G = (Y, E)$ for which there exists an ordering of the vertex set, say $Y = \{y_i\}_{i=1}^n$, $|Y| = n$, such that in the vertex induced subgraph $G(Y - \{y_i\}_{i=1}^{\ell-1})$ the set

$$C_\ell = \{y_\ell\} \cup \text{Adj}(y_\ell)$$

is a clique. That is, when $\ell = 1$, $y_1$ and its adjacent vertices in $G$ are a

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¹ Generally our graph-theoretic terminology follows [1].
clique, while for \( i > 1 \), \( y_i \) and its adjacent vertices in the subgraph are a clique. For the \( k \)-tree \( T_k(n) = (X, E) \) with \( x_i \) as above, the perfect elimination ordering \( \{y_i\}_{i=1}^n \) is defined simply by \( y_i = x_{n+1-i} \).

Let \( G = (X, E) \) be a graph with \( c(G) \) connected components. Recall that a separator \( S \) of \( G \) is a nonempty subset of \( X \) such that the vertex induced subgraph \( G(X-S) \) has \( c(G(X-S)) > c(G) \). For connected \( G \), \( G(X-S) \) has two or more connected components, say \( C_i = (V_i, E_i) \). The subgraphs \( G(V_i \cup S) \) then are the leaves of \( G \) with respect to \( S \). Similarly, for \( x, y \in X \) with \( xy \notin E \) and \( x \) and \( y \) in the same component of \( G \), an \( x, y \) separator \( S \) is a separator such that \( x \) and \( y \) are in distinct components of \( G(X-S) \). Recall also that any minimal separator is a minimal \( x, y \) separator for some \( x, y \in X \), but a minimal \( x, y \) separator need not be a minimal separator ([4, Fig.1, p. 193]).

Perfect elimination graphs and their role in the algebraic process of symmetric Gaussian elimination in sparse symmetric matrices has been discussed extensively in [4]. Here we apply a portion of the theory developed there to provide a simple characterization of \( k \)-trees.

**Theorem 1.1.** A graph \( G = (X, E) \) is a \( k \)-tree if and only if

- (i) \( G \) is connected,
- (ii) \( G \) has a \( k \)-clique but no \( k+2 \)-clique,
- (iii) every minimal \( x, y \) separator of \( G \) is a \( k \)-clique.

The necessity of (iii) was essentially established in [4, p. 201], however, the presentation given below is clearer. Even in the case \( k = 1 \) (trees), the result does not appear to be well known, although in this case it follows easily from other characterizations of trees (see [1, Theorem 4.1, p. 321]). We have for trees the following:

**Corollary 1.2.** A graph \( G \) is a tree iff \( G \) is connected, without triangles, and every minimal \( x, y \) separator is a single vertex.

For a different approach to the characterization of \( k \)-trees via a generalization of the notion of being "acyclic", see [2]. Our presentation deals only with cliques, paths and separators.
2. Proof of Theorem 1.1

To prove Theorem 1.1, we will borrow some results about perfect elimination graphs as discussed in [4].

Proposition 2.1 ([4, p. 196]). Let \( G = (X, E) \) be connected with a separator \( S \) which is a clique (separation clique) and leaves \( L_i, 1 \leq i \leq n \). If \( S_0 \) is a separator of some \( L_i \), then \( S_0 \) is a separator of \( G \). Furthermore, if \( S_0 \) is a minimal \( x, y \) separator of \( L_i \), then \( S_0 \) is a minimal \( x, y \) separator of \( G \).

Proposition 2.2 ([4, p. 194]). A graph \( G = (X, E) \) is a perfect elimination graph if and only if every minimal \( x, y \) separator is a clique.

We now begin the proof of Theorem 1.1. Let \( G = (X, E) \) be a graph with \( |X| = n \); for any fixed \( k \leq n \), we proceed by induction on \( n \). When \( n = k \) or \( n = k + 1 \), the equivalence of the \( k \)-tree definition and (i)–(iii) is immediate since \( X \) of \( G \) must then be a clique. Assuming the equivalence for graphs with \( k + 1 \leq |X| \leq n - 1 \), we consider a graph with \( |X| = n \).

Necessity. \( G \) is a \( k \)-tree on \( n \) vertices; let \( x_n \) be the vertex added to the \( k \)-tree on \( n - 1 \) vertices in the recursive definition of \( G \). Hence \( G(X - x_n) \) is a \( k \)-tree on \( n - 1 \) vertices. \( G \) is connected, and \( G \) contains a \( k \)-clique but no \( k + 2 \) clique since this is true for \( G(X - x_n) \) and \( |\text{Adj}(x_n)| = k \).

It remains to show that every minimal \( x, y \) separator \( S \) of \( G \) is a \( k \)-clique. Certainly, \( x_n \cup \text{Adj}(x_n) \) must be in the same leaf of \( G \) with respect to \( S \). If \( S = \text{Adj}(x_n) \), \( S \) is a \( k \)-clique. Otherwise, (since \( n \geq k + 2 \)) \( S \) is a minimal \( x, y \) separator of \( G(X - x_n) \), or, if \( x = x_n \), \( S \) is a minimal \( a, y \) separator of \( G(X - x_n) \) for some \( a \in \text{Adj}(x_n) \).

Sufficiency. Let \( G = (X, E) \) with \( |X| = n \) satisfy (i)–(iii) of Theorem 1.1. Then \( G \) is a perfect elimination graph (Proposition 2.2) and has a vertex, say \( x \), such that \( \{x\} \cup \text{Adj}(x) \) is a clique. Certainly, \( |\text{Adj}(x)| \leq k \) since there are not \( k + 2 \) cliques in \( G \). Furthermore, since \( |X| \geq k + 2 \) and \( G \) is connected, \( \text{Adj}(x) \) is a separator, hence \( |\text{Adj}(x)| \geq k \).

So \( |\text{Adj}(x)| = k \) and we finish by showing that \( G(X - x) \) is a \( k \)-tree; then \( G \) is a \( k \)-tree by definition. But certainly \( G(X - x) \) satisfies (i) and (ii); (iii) follows by applying Proposition 2.1 since \( G(X - x) \) is a leaf of \( G \) with respect to \( \text{Adj}(x) \). By the induction hypothesis, \( G(X - x) \) is a \( k \)-tree.
Applying Theorem 1.1 and Proposition 2.1, we have immediately:

Corollary 2.3. Let \( G = (X, E) \) be a \( k \)-tree with separation clique \( S \). Then each leaf of \( G \) with respect to \( S \) is a \( k \)-tree. In particular, for \( |X| \geq k + 1 \), if \( \{x\} \cup \text{Adj}(x) \) is a clique of \( G \), then \( G(X - x) \) is a \( k \)-tree.

Suppose \( S \) is a minimal \( x, y \) separator of a \( k \)-tree; then \( |S| = k \). If \( S \) were not a minimal separator, it must contain properly a minimal separator, say \( S_0 \), which is a minimal \( u, v \) separator for some \( u, v \in X \). But then \( k = |S_0| \leq k - 1 \) so \( S \) itself must be a minimal separator. Hence we have:

Corollary 2.4. For a \( k \)-tree, every minimal \( x, y \) separator is a minimal separator.

3. Other characterizations

In this section we consider some related results about \( k \)-trees. Recall that a graph \( G = (X, E) \) is a tree iff \( G \) is connected and \( |E| = |X| - 1 \). For a \( k \)-tree,

\[
|E| = \frac{1}{2} k(k - 1) + (|X| - k) = k|X| - \frac{1}{2} k(k + 1).
\]

Two characterizations involving (3.1) are presented below.

Proposition 3.1. Let \( G = (X, E) \) be a graph with \( |X| \geq k \) satisfying (ii) of Theorem 1.1 and (iv) every minimal \( x, y \) separator is a clique. Then \( |E| \leq k|X| - \frac{1}{2} k(k + 1) \) with equality holding iff \( G \) is a \( k \)-tree.

Proof. We note that there exists a perfect elimination ordering, say \( X = \{x_i\}_{i=1}^n \) \((|X| = n)\), by Proposition 2.2. Furthermore, we may assume without loss of generality by [4, Corollary 4, p. 198] that the \( k \)-clique \( C \) guaranteed by (ii) is ordered last; i.e., \( C = \{x_i\}_{i=m+1}^n \) with \( m = |X| - k \). Thus \( \{x_j\} \cup \text{Adj}(x_j) \) in \( G(X - \{x_j\}_{j=1}^{i-1}) \) is a clique, \( 1 \leq i \leq m \), and by (ii), \( |\text{Adj}(x_j)| \leq k \). Now such adjacency sets for \( 1 \leq i \leq m \) in their respective reduced subgraphs, count exactly all edges of \( E \) except for those \( G(C) \). Hence
\[ |E| \leq \frac{1}{2} k(i-1) + (|X| - k)k = k|X| - \frac{1}{2} k(k + 1). \]

Clearly the inequality is strict unless \(|\text{Adj}(x_i)| = k, \ 1 \leq i \leq m\), in which case, by definition, \(G\) is a \(k\)-tree.

The following result is an immediate corollary since necessity is clear.

**Theorem 3.2.** \(G = (X, E)\) is a \(k\)-tree if and only if (3.1), and (ii) and (iv) are satisfied.

**Proposition 3.3.** Let \(G = (X, E)\) be a graph with \(|X| \geq k\) satisfying (i) and (iii) of Theorem 1.1. Then \(|E| \geq k|X| - \frac{1}{2} k(k + 1)\).

**Proof.** We sketch the inductive proof, letting \(G\) be a graph with \(|X| \geq k + 1\). Let \(x\) be a vertex such that \(\{x\} \cup \text{Adj}(x)\) is a clique (existence by Proposition 2.2). Then, since \(G\) is connected, either \(X = \{x\} \cup \text{Adj}(x)\) and the inequality is satisfied, or \(\text{Adj}(x)\) is a separation clique with \(|\text{Adj}(x)| \geq k\) by (iii). Using induction on \(G(X-x) = (X', E')\), we have

\[ |E'| \geq k|X'| - \frac{1}{2} k(k + 1). \]

Adding \(|\text{Adj}(x)|\) on the left and \(k\) on the right gives

\[ |E| \geq k|X| - \frac{1}{2} k(k + 1). \]

**Theorem 3.4.** \(G = (X, E)\) is a \(k\)-tree if and only if (i), (3.1) and (iii) are satisfied.

**Proof.** Sufficiency is by induction and follows by observing that (3.1) implies that \(|\text{Adj}(x)| = k\) (where \(\{x\} \cup \text{Adj}(x)\) is a clique) and the inequality of Proposition 3.3 for \(G(X-x)\) is an equality. Hence by induction on \(|X|\), \(G(X-x)\) is a \(k\)-tree, implying \(G\) is a \(k\)-tree.

With a little help from Menger's theorem [1, p. 47], we have:

**Theorem 3.5.** A graph \(G = (X, E)\) is a \(k\)-tree iff (ii), (iv) and (v) for all distinct nonadjacent pairs \(x, y \in X\), there exist exactly \(k\) vertex-adjacent \(k\) paths (except for \(x\) and \(y\)) \(x, y\) paths are satisfied.
Proof. Sufficiency is proved by induction of \(|X|\), the cases \(|X| = k\) and \(|X| = k + 1\) being clear. For \(|X| \geq k + 2\), let \(\{x\} \cup \text{Adj}(x)\) be the clique in \(G\) guaranteed by Proposition 2.2. Since \(G\) is connected by (v), we have that (ii) and (v) imply \(|\text{Adj}(x)| = k\).

Clearly (ii) holds in \(G(X-x)\); (iv) holds by Proposition 2.1. Finally, given any nonadjacent \(u, v\) in \(G(X-x)\), the \(k\) disjoint \(u, v\) paths in \(G\) imply \(k\) disjoint \(u, v\) paths in \(G(X-x)\) since any \(u, v\) path in \(G\) containing \(x\) also contains two vertices of the clique \(\text{Adj}(x)\). Hence (v) holds in \(G(X-x)\), there being no more than \(k\) disjoint \(u, v\) paths in \(G(X-x)\). Thus by induction, \(G(X-x)\) is a \(k\)-tree as is \(G\).

We need only show necessity of (v) which follows from (iii) by Menger's theorem.

As a final remark we note that (iv) above may be replaced by any of several known equivalent conditions. See, for example, [4, p. 194] and [3].

References