JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **168**, 13–27 (1992)

Stability Result for the Inverse Transmissivity Problem

HAMID BELLOUT

Department of Mathematical Sciences, Northern Illinois University, DeKalb, Illinois 60115

Submitted by Avner Friedman

Received February 19, 1991

0. INTRODUCTION

Let Ω be a bounded subdomain of R^n, Γ its boundary, $Q_T = \Omega \times (0, T)$, $\Gamma_T = \Gamma \times (0, T)$ and consider the boundary value problem (in all that follows we will use the standard summation convention)

$$u_t = (a_j u_{x_j})_{x_j} \qquad \text{in } Q_T, \tag{0.1}$$

$$u(x, t) = f(x, t) \qquad \text{on } \Gamma_T, \tag{0.2}$$

$$u(x, 0) = u_0(x) \qquad \text{in } \Omega, \tag{0.3}$$

with $a_j(x) = A_j + K_j \chi_D(x)$; $A_j > 0$, $K_j > 0$, $\forall j = 1, ..., n$, where D is an unknown subdomain of Ω. The inverse problem associated with (0.1)–(0.3) is the determination of D from measurement of the Neumann data

$$a_j u_{\nu_j} = g(x, t) \qquad \text{on } \Gamma_T^1, \tag{0.4}$$

where $\Gamma_T^1 = \Gamma^1 \times (0, T)$ and Γ^1 is an open subset of Γ.

The physical origin of this problem will be discussed in Subsection 0.1.

In this paper we establish a local stability result; more precisely, if D_h is a family of domains such that $D_h \to D$ if $h \to 0$, and if we denote by g_h the data on Γ_T^1 corresponding to D_h, then

$$\liminf_{h \to 0} \int_{\Gamma_T^1} \frac{|g_h - g|}{h}\, dx\, dt > 0$$

$$\text{provided} \quad \liminf_{h \to 0} \frac{1}{h} \operatorname{meas}(D_h \Delta D) > 0. \tag{0.5}$$

409/168/1-2

0022-247X/92 $5.00

The author and A. Friedman [3], established (0.5) for the isotropic (a_j independent of j) elliptic case under the assumption that

$$D_h \subseteq D \qquad (\text{or } D \subseteq D_h)\ \forall h \tag{0.6}$$

Here we will not make assumption (0.6).

0.1. *Physical Origin*

The governing equation of an unsteady flow in a nonhomogeneous confined aquifer can be represented by

$$S(x)\, u_t - (a_j(x)\, u_{x_j})_{x_j} = q(x, t) \qquad \text{in} \quad \Omega \times (0, \infty) \tag{0.7}$$

$$u(x, 0) = g(x) \qquad \text{in } \Omega \tag{0.8}$$

$$u(x, t) = f(x, t) \qquad \text{on} \quad \partial\Omega \times (0, \infty), \tag{0.9}$$

where u, q, S, and a_j represent the piezometric head, the source sink term, the storage coefficient and the transmissivity coefficients, and Ω is an open subset of R^n representing the studied region.

The forward problem (i.e., finding a solution u of problem (0.7)–(0.9) for a given set of data) is used by hydrologists to simulate, for management purposes, the level of aquifers under different use and replenishment conditions (i.e., for different functions q). However, in general, the functions α_j, S are not known and experimental determination can be done (practically) only at finitely many points. It is easy to measure u also in the same experiment. This leads hydrologists to try to determine the functions S and a_j from their values at certain points and some information on the function u for a specific set of data f, g, and q. The type of information available on the function u varies with the situation being considered, and leads to different inverse problems. For instance, if the flow through one of the walls delimiting the region has been monitored for a certain period of time, then the available information is the function $h(x, t)$ defined by

$$a_j(x)\, u_{v_j}(x, t) = h(x, t) \qquad \text{on} \quad \Gamma \times (0, T), \tag{0.10}$$

where Γ is a given hypersurface, part of the boundary of Ω, and v is the outward unitary vector normal to Γ. See [13, 12] for more on this subject.

We will assume $S \equiv 1$, for simplicity, and consider the following inverse problems: given Ω, g, f, q, (0.7)–(0.9), and (0.10), find the function a_j in (0.7).

It is well known that for this kind of problem the a priori assumptions on the type of functions a_j are an essential part of the problem. In the given situation the most natural assumption is that a_j is piecewise constant, i.e., that

$$a_j(x) = a_j^0 + \sum_{i=1}^{I} a_j^i \chi(\Omega_i) \qquad j = 1, ..., n, \tag{0.11}$$

where the a_j^i are positive constants, $\chi(\Omega_i)$ is the characteristic function of the set Ω_i, and $\mathscr{F} = \{\Omega_i : i = 1, ..., I\}$ is a family of (mutually disjoint) open subsets of Ω such that $\bigcup \bar{\Omega}_i \subset \bar{\Omega}$.

We will assume that I and the constants a_j^i are known and that the regions Ω_i are unknown. The inverse problem is thus reduced to finding the family of regions Ω_i.

The above assumption on the structure of the function a is based on the fact that aquifer heterogeneities are due to the presence underground of different materials such as sand, rock, and clay. Also, due to geological processes, those different materials exist in the form of layers and/or lumped bodies. See [4, 11] for more on this subject. The number I of different bodies present in a given area can easily be found from available geological data. The constant transmissivity a_j^i of each homogeneous body needs to be sampled at only one site. On the other hand, the exact shape of the given body would be very hard, if not practically impossible, to find experimentally. The object of the inverse problem is to try to identify the shape of those bodies (i.e., to find Ω_i) in the situations described above.

To the author's best knowledge, the only work that has been done on this problem is the work done by hydrologists, who consider the discrete version of this problem. Once discretized, this problem becomes a problem of identification in a finite dimensional space and different methods have been used to solve this finite dimensional identification problem. On the other hand, mathematicians have considered inverse problems associated with (0.7), but only for continuous functions a_j. See [10, 1] and the references therein.

The relevance of the discretized problem (which is solved numerically) to the actual problem (0.7) hinges on the stability of the inverse problem, i.e., if a small error in h can result in a relatively large error in a_j, then the numerically computed solution using an approximation h^* of h may result in a discretized a^* bearing no connection with the actual a sought.

Discontinuous diffusion coefficients have been considered in inverse problems in the steady state case. See [3, 5, 6] and the references therein.

The inverse problem described in the introduction corresponds then to the case of coefficients a_j given by (0.11); more specifically $I = 2$, with $\Omega_1 = D$ and $d(D, \partial\Omega) > 0$.

1. Notation and Statement of the Main Results

Let Ω be a bounded domain in R^n $(n \geqslant 2)$ with $C^{1,\alpha}$ boundary Γ and let D be a bounded subdomain of Ω, $\bar{D} \subset \Omega$, with $C^{2,\alpha}$ boundary.

Let D_h $(0 < h \leqslant h_0)$ be a family of domains in R^n, such that ∂D_h has the representation

$$\partial D_h : x = x_0 + h\sigma_h(x_0)\, \nu(x_0), \tag{1.12}$$

where x_0 varies on ∂D, $\nu(x_0)$ is the outward normal unit vector to ∂D, $\sigma_h(x_0)$ is continuously differentiable in x_0, and

$$|\sigma_h|_{C^{1+\alpha}} \leqslant B, \tag{1.13}$$

where B is a constant independent of h.

We will assume also that

$$\sigma_h(x) \to \sigma(x) \qquad \text{if} \quad h \to 0, \sigma(x) \not\equiv 0, \tag{1.14}$$

and define the sets S_1, S_2, and $\tilde{S}_\varepsilon$ as

$$S_1 = \overline{\{x \in \partial D \text{ such that } \sigma(x) < 0\}},$$

$$S_2 = \overline{\{x \in \partial D \text{ such that } \sigma(x) > 0\}},$$

$$\tilde{S}_\varepsilon = \{x \in \partial D \text{ such that } d(x, S_1 \cap S_2) \leqslant \varepsilon\}.$$

Let u, respectively, u_h, be the solution of the parabolic diffraction problems,

$$u_t - (a_j u_{x_j})_{x_j} = q(x, t) \qquad \text{in } Q_T \tag{1.15}$$

$$u(x, 0) = u_0(x),\ u(x, t) = f(x, t) \qquad \text{on } \Gamma_T, \tag{1.16}$$

respectively,

$$u_t^h - (a_j^h u_{x_j}^h)_{x_j} = q(x, t) \qquad \text{in } Q_T \tag{1.17}$$

$$u^h(x, 0) = u_0^h(x),\ u^h(x, t) = f(x, t) \qquad \text{in } \Gamma_T, \tag{1.18}$$

where $a_j(x) = A_j + K_j \chi(D)$; $A_j > 0$, $K_j > 0$, $\forall j = 1, ..., n$, respectively, $a_j^h(x) = A_j + K_j \chi(D_h)$; $A_j > 0$, $K_j > 0$, $\forall j = 1, ..., n$.

We will make the following assumptions on the free terms.

(A1) $q \in H^2((0, T)); H^n(\Omega))$, $f \in H^1((0, T); H^3(\Gamma))$, u_0^h, u_0 are in $H^1(\Omega)$.

(A2) $\|u_0^h\|_{H^1}$ is bounded independently of h.

(A3) $\lim_{h \to 0} \int_\Omega (h^{-1}(u_0 - u_0^h))^2\, dx = 0$.

(A4) Compatibility conditions of order 0 are satisfied.

(A5) $u_0(x)$ is not constant.

Under the above assumptions, it is well known (see [8], for instance) that problem (1.15), (1.16), respectively problem (1.17), (1.18), has a unique solution u, respectively, u^h, in $H^1(Q_T)$. Furthermore, from [7, Theorem 7], it follows that u, respectively, u^h, is in $L^2(0, T; H^2(\Omega\backslash D) \cap H^2(D))$, respectively, in $L^2(0, T; H^2(\Omega\backslash D_h) \cap H^2(D_h))$. Also, it results from (A2) that the norm of u^h in the space $L^2(0, T; H^2(\Omega\backslash D_h) \cap H^2(D_h))$ is bounded independently of h.

Set

$$g^h = a_j^h u_{\nu_j}^h, \qquad g = a_j u_{\nu_j} \qquad \text{on } \Gamma_T^1 \tag{1.19}$$

and

$$u^e = u|_{\Omega\backslash D \times (0,T)}, \qquad u^i = u|_{D \times (0,T)}.$$

We will distinguish and treat separately the cases where the set $\{\sigma = 0\}$ has an empty interior or not.

THEOREM 1.1. *Assume that*

(1) $(a_j(u_0)_{x_j})_{x_j} \in C^{0,\alpha}$,

(2) $S_1 \cap S_2$ *is a* C^1 *manifold of dimension* $n-2$ *and*

$$a_j(u_0)_{\nu_j}(x) \neq 0, \qquad \forall x \in \partial D. \tag{1.20}$$

Then for any nonempty open subset Γ^1 *of* $\partial\Omega$

$$\liminf_{h \to 0} \int_0^T \int_{\Gamma^1} \frac{|g^h - g|}{h}\, dx\, dt > 0. \tag{1.21}$$

Remark. Condition 1 is not a real restriction in view of the arbitrariness of the choice (by the observer) of the time $t = 0$ and the fact that $\forall t > 0$, $(a_j u_{x_j})_{x_j}(x, t) \in C^{0,\alpha}(\Omega)$ (even for u_0 only in L^2, see [7]).

THEOREM 1.2. *Assume that* $\sigma = 0$ *on an open subset* Σ *of* ∂D. *Then* (1.21) *holds for any nonempty open subset* Γ^1 *of* Γ.

2. AUXILIARY RESULTS

Set

$$U^h = \frac{u^h - u}{h}; \qquad S^h = D_h \Delta D. \tag{2.22}$$

Taking the difference of Eq. (1.15) and (1.17) we find that U^h is a solution of the parabolic problem

$$U_t^h - (a_j^h U_{x_j}^h)_{x_j} = \frac{1}{h}((a_j^h - a_j)\, u_{x_j})_{x_j} \qquad \text{in } Q_T \tag{2.23}$$

$$U^h(x, 0) = \frac{u_0^h(x) - u_0(x)}{h}, \qquad u(x, t) = 0 \qquad \text{on } \Gamma_T. \tag{2.24}$$

LEMMA 2.1. *There exists $C > 0$ such that $\forall h \geqslant 0$*

$$\int_0^T \int_\Omega (U^h)^2 \, dx\, dt \leqslant C.$$

Proof. Let w^h be the unique solution of the parabolic problem

$$w_t^h(x, t) - (a_j^h w_{x_j}^h(x, t))_{x_j} = U^h(x, T-t) \qquad \text{in } Q_T \tag{2.25}$$

$$w^h(x, 0) = 0, \qquad w^h(x, t) = 0 \qquad \text{on } \Gamma_T. \tag{2.26}$$

By standard results for parabolic equations the problem (2.25), (2.26) has a unique solution w^h in $H^1(Q_T)$. Furthermore, it was proved in [7, Sect. 2] that the following estimate holds for w^h

$$\|w^h\|_{L^2(0, T; H^2(\Omega \backslash D_h))} + \|w^h\|_{L^2(0, T; H^2(D_h))} \leqslant C\, \|U^h\|_{L^2(Q_T)}. \tag{2.27}$$

Multiplying Eq. (2.25), respectively (2.23), by $U^h(x, t)$, respectively $w^h(x, T-t)$, integrating by parts over Q_T and taking the difference we find

$$\begin{aligned} &\int_\Omega U_0^h(x)\, w^h(x, T)\, dx + \int_0^T \int_\Omega (U^h(x, t))^2 \, dx\, dt \\ &= \int_0^T \frac{1}{h} \left(\int_{S^h} K_j (U^h(x, t))_{x_j} (w^h(x, T-t))_{x_j} \right) dx\, dt \\ &= \frac{1}{h} \int_0^h \left(\int_0^T \int_{\partial D_\lambda} (K_j (U^h(x, t))_{x_j} (w^h(x, T-t))_{x_j})\, d\mu\, dt \right) d\lambda, \end{aligned} \tag{2.28}$$

where D_λ is defined by $\partial D_\lambda : x = x_0 + \lambda \sigma_\lambda(x_0)\, v(x_0)$.

Let $\partial D_\lambda = \partial D_{\lambda,1} \cup \partial D_{\lambda,2}$, where $\partial D_{\lambda,1} = \partial D_\lambda \cap (D_h \backslash D)$, and $\partial D_{\lambda,2} = \partial D_\lambda \cap (D \backslash D_h)$. Since $u \in L^2(0, T; H^2(D) \cap H^2(\Omega \backslash D))$, using (2.27) and the traces theorem we have

$$\begin{aligned} &\int_0^T \left| \int_{\partial D_{\lambda,1}} (a_j^h - a_j)(u(x, t))_{x_j} (w^h(x, T-t))_{x_j}\, d\mu \right| dt \\ &\leqslant C \int_0^T \left(\int_{\partial D_{\lambda,1}} |\nabla u(x, t)|^2 \, d\mu \right)^{1/2} \\ &\quad \times \left(\int_{\partial D_{\lambda,1}} |\nabla w^h(x, T-t)|^2 \, d\mu \right)^{1/2} dt \end{aligned}$$

$$\leqslant C\int_0^T \|u(x,t)\|^{1/2}_{H^2(\Omega\setminus D)}\,\|w(x,T-t)\|^{1/2}_{H^2(D_h)}\,dt$$

$$\leqslant C\left(\int_0^T\int_\Omega (U^h)^2\,dx\,dt\right)^{1/2}. \tag{2.29}$$

Similarly,

$$\int_0^T\left|\int_{\partial D_{\lambda,2}}(a_j^h-a_j)(u(x,t))_{x_j}\,(w^h(x,T-t))_{x_j}\,d\mu\right|dt$$

$$\leqslant C\int_0^T \|u(x,t)\|^{1/2}_{H^2(D)}\,\|w^h(x,T-t)\|^{1/2}_{H^2(\Omega\setminus D_h)}\,dt$$

$$\leqslant C\left(\int_0^T\int_\Omega (U^h)^2\,dx\,dt\right)^{1/2}. \tag{2.30}$$

Consequently,

$$\left|\int_0^T\int_{\partial D_\lambda}(K_j(U^h(x,t))_{x_j}\,(w^h(x,T-t))_{x_j})\,d\mu\,dt\right|$$

$$\leqslant C\left(\int_0^T\int_\Omega (U^h)^2\,dx\,dt\right)^{1/2}. \tag{2.31}$$

From (2.28) and the above estimate it follows that

$$\int_0^T\int_\Omega (U^h(x,t))^2\,dx\,dt$$

$$\leqslant -\int_\Omega U_0^h(w)\,w^h(x,T)\,dx$$

$$+C\frac{1}{h}\int_0^h\left(\int_0^T\int_\Omega (U^h(x,t))^2\,dx\,dt\right)^{1/2}d\lambda$$

$$\leqslant C\left(\int_0^T\int_\Omega (U^h(x,t))^2\,dx\,dt\right)^{1/2}$$

$$+C\frac{1}{h}\int_0^h\left(\int_0^T\int_\Omega (U^h(x,t))^2\,dx\,dt\right)^{1/2}d\lambda, \tag{3.32}$$

from which the lemma easily follows.

LEMMA 2.2. *If for a sequence $h\to 0$ we have that $U^h\to U$ weakly in $L^2(Q_T)$, and $\lim_{h\to 0}\int_0^T\int_{\Gamma^1}|g^h-g|/h\,dx\,dt=0$, then $U\equiv 0$ in $(\Omega\setminus D)\times(0,T)$.*

Proof. Letting $h \to 0$ in (2.23) we find that U satisfies in a weak sense the equation

$$U_t - (a_j U_{x_j})_{x_j} = 0 \qquad \text{in} \quad (\Omega \backslash D) \times (0, T).$$

Let $x_0 \in \Gamma^1$, and $r > 0$ be small enough and denote $B(x_0, r) \cap \Omega$ by W. From (2.23) and (2.24) it follows, by standard regularity results for parabolic equations (see [8], for instance), that the sequence U^h is bounded in $C^{1,\alpha}((t_0, \Gamma); C^{2+\alpha}(\bar{W}))$ for some positive α and any positive t_0. Consequently, U is a classical solution of

$$\begin{aligned} U_t - (a_j U_{x_j})_{x_j} &= 0 \qquad \text{in} \quad W \times (0, T) \\ U(x, t) = 0, a_j U_{\nu_j} &= 0 \qquad \text{on} \quad (\partial W \cap \Gamma \times (0, T)). \end{aligned} \tag{2.33}$$

From this we easily deduce that U has a zero of infinite order at the point (x_0, t), $\forall t \in (0, T)$; it then follow by [19, Theorem 7] that $U = 0$ in $W \times (0, T)$.

The lemma is then a direct consequence of the equation

$$U_t - (a_j U_{x_j})_{x_j} = 0 \qquad \text{in} \quad (\Omega \backslash D) \times (0, T)$$

and the unique continuation theorem (see [9]) for parabolic equations.

LEMMA 2.3. *If for a sequence $h \to 0$ we have that $U^h \to U$ weakly in $L^2(Q_T)$ and $\lim_{h \to 0} \int_0^T \int_{\Gamma^1} |g^h - g|/h \, dx \, dt = 0$, then $\forall v \in H^1((0, T); H^2(D))$, $v(x, T) = 0$ we have that*

$$\int_0^T \int_D U(v_t + (a_j v_{x_j}) x_j) \, dx \, dt = \int_0^T \int_{\partial D} \sigma(x) \, b_j(x, t) \, v_{x_j}(x, t) \, d\mu \, dt, \tag{2.34}$$

where $\forall j = 1, ..., n$, $b_j \in L^2(0, T; H^{3/2}(S_1) \cap H^{3/2}(S_2))$.

Proof. Let v be a C^∞ function with compact support in $[0, T) \times \Omega$. Multiplying Eq. (2.23) by v and integrating by parts over Q_T we find

$$\begin{aligned} &\int_0^T \int_{D_h} U^h(v_t + a_j^h v_{x_j x_j}) \, dx \, dt \\ &\qquad + \int_0^T \int_{\Omega \backslash D_h} U^h(v_t + a_j^h v_{x_j x_j}) \, dx \, dt - \int_0^T \int_{\partial D_h} U^h(K_j v_{v_j^e}) \, d\mu \, dt \\ &\quad = -\frac{1}{h} \int_0^T \int_{D \backslash D_h} K_j u^i_{x_j} v_{x_j} \, dt \, dx + \frac{1}{h} \int_0^T \int_{D_h \backslash D} K_j u^e_{x_j} v_j \, dt \, dx.. \end{aligned} \tag{2.35}$$

We first investigate the limit of $\int_0^T \int_{\partial D_h} U^h(K_j v_{v_j^e}) \, d\mu \, dt$. For this purpose we let z^h be a sequence of functions bounded in $C^2((\Omega \backslash D_h) \times [0, T])$ independently of h and such that $z^h = 0$ in a neighborhood of $(\partial \Omega \times [0, T]) \cup$

$(\Omega \times T)$, and that $z^h = 0$ and $(A_j z^h_{v^e_j}) = (K_j v_{v^e_j})$ on $(\partial D_h) \times (0, T)$. Multiplying Eq. (2.23) by z^h and integrating by parts over $(\Omega \backslash D_h) \times (0, T)$ we find that

$$-\int_0^T \int_{\partial D_h} U^h(A_j z^h_{v^e_j})\, d\mu\, dt = \int_0^T \int_{\Omega \backslash D_h} U^h(z^h_t + a^h_j z^h_{x_j x_j})\, dx\, dt$$
$$-\frac{1}{h} \int_0^T \int_{\Omega \backslash D_h} (a^h_j - a_j) u_{x_j} z^h_{x_j}\, dx\, dt. \qquad (2.36)$$

Taking the limit as $h \to 0$ we find, after using Lemma 2.2,

$$\lim_{h \to 0} \int_0^T \int_{\partial D_h} U^h(A_j z^h_{v^e_j})\, d\mu\, dt = \int_0^T \int_{S_1} \sigma K_j u^i_{x_j} z_{x_j}\, d\mu\, dt, \qquad (2.37)$$

where z is the limit of the sequence z^h. Taking $h \to 0$ in (2.35), we obtain after using (2.37) and Lemma 2.1

$$\int_0^T \int_D U(v_t + a_j v_{x_j x_j})\, dx\, dt$$
$$= \int_0^T \int_{S_2} \sigma K_j u^e_{x_j} v_{x_j}\, d\mu\, dt$$
$$+ \int_0^T \int_{S_1} \sigma K_j u^i_{x_j} v_{x_j}\, d\mu\, dt + \int_0^T \int_{S_1} \sigma K_j u^i_{x_j} z_{x_j}\, d\mu\, dt. \qquad (2.38)$$

This last surface integral is a functional of v. Indeed, since S_1 is a C^2 surface there exist p open subsets V_m such that $S_1 \subset \bigcup V_m$ and $S_1 \cap V_m$ can be represented in the form

$$S_1 \cap V_m : x_n = \gamma_m(x_1, ..., x_{n-1}),$$

where γ_m is a C^2 function. After an elementary but lengthy computation we find that

$$\int_0^T \int_{S_1 \cap V_m} \sigma K_j u^i_{x_j} z_{x_j}\, d\mu\, dt$$
$$= \int_0^T \int_{S_1 \cap V_m} \sigma \frac{1 + |\nabla \gamma_m|^2}{\sum_{j=1}^{n-1} A_j \gamma^2_{m,x_j} + A_n}$$
$$\times \left(\sum_{j=1}^{n} K_j u^i_{v^i_j} \right) \left(\sum_{j=1}^{n} K_j v_{v^i_j} \right) d\mu\, dt$$
$$= \int_0^T \int_{S_1 \cap V_m} \sigma \frac{1 + |\nabla \gamma_m|^2}{\sum_{j=1}^{n-1} A_j \gamma^2_{m,x_j} + A_n}$$
$$\times \left(\sum_{j=1}^{n} K_j u^i_{v^e_j} \right) K_j v_{x_j} v^e_j\, d\mu\, dt. \qquad (2.39)$$

For $j = 1, \ldots, n$ we define the functions b_j^e, b_j^i and b_j by

$$b_j^e(x, t) = K_j u^e_{x_j}(x, t)$$

$$b_j^i(x, t) = K_j u^i_{x_j}(x, t) + v_j^e K_j \frac{1 + |\nabla\gamma_m|^2}{\sum_{l=1}^{n-1} A_l \gamma^2_{m, x_l} + A_n} \times \left(\sum_{l=1}^{n} K_l u^i_{v_l^e} \right) \quad \text{for} \quad x \in V_m$$

$$b_j(x, t) = \begin{cases} b_j^e(x, t) & \text{if} \quad x \in S_2 \\ b_j^i(x, t) & \text{if} \quad x \in S_1. \end{cases}$$

This proves the lemma.

3. Proof of Theorem 1.1

We will prove the theorem by exhibiting a function $v \in H^1((0, T); H^2(D))$, $v(x, T) = 0$ for which (2.34) does not hold. For this purpose we will need the following lemma.

Lemma 3.1. *Let $\alpha \in (0, 1)$ be a fixed positive number, $\forall \varepsilon > 0$, $\exists T_2 > 0$ and a function $v_\varepsilon \in H^1((0, T_2); H^2(D))$ such that*

$$v_{\varepsilon, t} + (a_j v_{\varepsilon, x_j}) x_j = 0 \qquad \text{in} \quad D \times (0, T_2) \tag{3.40}$$

$$b_j(x, t)\, v_{\varepsilon, x_j}(x, t) = \frac{\sigma(x)}{\varepsilon} (T_2 - t) \qquad \text{on} \quad (\partial D \backslash \tilde{S}_{4\varepsilon}) \times (0, T_2) \tag{3.41}$$

$$|\nabla v_\varepsilon(x, t)| \leqslant C \frac{T_2 - \tau}{\varepsilon}, \qquad \forall (x, t) \in \partial D \times (\tau, T_2) \tag{3.42}$$

$$v(x, T_2) = 0,$$

where C is independent of ε.

We will first prove Theorem 1.1 using the above lemma, which will be proved later. Setting $v = v_\varepsilon$ in (3.53) and using (3.40) we find

$$\int_0^{T_2} \int_{\partial D} \sigma(x)\, b_j(x, t)\, v_{\varepsilon, x_j}(x, t)\, d\mu\, dt = 0. \tag{3.43}$$

Using (3.41) it follows that

$$\varepsilon^{-1}\int_0^{T_2}\int_{\partial D\setminus\tilde{S}_{2\varepsilon}} \sigma(x)^2\,(T_2-t)\,d\mu\,dt$$
$$+\int_0^{T_2}\int_{\tilde{S}_{2\varepsilon}} \sigma(x)\,b_j(x,t)\,v_{\varepsilon,x_j}(x,t)\,d\mu\,dt=0. \tag{3.44}$$

By standard regularity results (see [7], for example) it follows from assumption 1 of the theorem that

$$u\in C^{0,\alpha}(0,T;C^{2,\alpha}(\bar{D})\cap C^{2,\alpha}(\partial D\cup(\Omega\setminus D))). \tag{3.45}$$

Consequently the functions b_j are bounded and (3.44) yields

$$\varepsilon^{-1}\int_0^{T_2}\int_{\partial D\setminus\tilde{S}_{2\varepsilon}} \sigma(x)^2\,(T_2-t)\,d\mu\,dt$$
$$C\int_0^{T_2}\int_{\tilde{S}_{2\varepsilon}} |\sigma(x)|\,|\nabla v_\varepsilon(x,t)|\,d\mu\,dt$$
$$\leqslant C\int_0^{T_2}\int_{\tilde{S}_{2\varepsilon}} |\sigma(x)|\,C\frac{(T_2-t)}{\varepsilon}\,d\mu\,dt,$$

where (3.42) was used.

Since $|\sigma(x)|\leqslant B\varepsilon$ on $\tilde{S}_{2\varepsilon}$ by (1.13), after integrating in time we find that there exists a new constant C independent of ε such that

$$\int_{\partial D\setminus\tilde{S}_{2\varepsilon}} \sigma(x)^2\,d\mu\leqslant C\varepsilon\cdot\operatorname{meas}\{\tilde{S}_{2\varepsilon}\}$$
$$\leqslant C\varepsilon^2,$$

where we made use of the fact that $\operatorname{meas}\{\tilde{S}_{2\varepsilon}\}\leqslant C\varepsilon$ for some C independent of ε.

Letting ε approach 0 we then find that $\sigma(x)\equiv 0$, which contradicts (1.14).

Proof of Lemma 3.1. From assumption 2 of the theorem it follows that $\forall\varepsilon>0$, $\exists\psi_\varepsilon^l(x)\in C^1(\partial D)$, $l=1,2$, such that

$$\psi_\varepsilon^l(x)=\begin{cases}1 & \text{if } x\in(S_l\setminus\tilde{S}_{2\varepsilon})\\ 0 & \text{if } x\in\tilde{S}_\varepsilon\cup(\partial D\setminus S_l)\end{cases}$$
$$0\leqslant\psi_\varepsilon^l\leqslant 1,\qquad |\nabla\psi_\varepsilon^l|_{C^s(\partial D)}\leqslant C\varepsilon^{-s}, \tag{3.46}$$

where C is independent of ε.

Set

$$b_{\varepsilon,j}(x,t)=\psi_\varepsilon^1 b_j^e(x,t)+\psi_\varepsilon^2 b_j^i(x,t)+v_j^e(1-\psi_\varepsilon^1-\psi_\varepsilon^2) \qquad j=1,\dots,n \tag{3.47}$$
$$B_{\varepsilon,j}(x,t)=\varepsilon b_{\varepsilon,j}(x,T_2-t), \qquad j=1,\dots,n,$$

where T_2 is a small positive number to be specified later.

From (1.20) and the continuity (in t) of $\nabla u^e(x,t)$, $\nabla u^i(x,t)$ (see (3.45)) we deduce that there exists $T_0>0$ such that

$$K_j u^e_{x_j}(x,t)\,v_j^e\neq 0, \qquad K_j u^i_{x_j}(x,t)\,v_j^e\neq 0, \qquad \forall (x,t)\in\partial D\times(0,T_0). \tag{3.48}$$

We will prove the lemma here, assuming that $K_j u^e_{x_j}(x,t)\,v_j^e\geqslant 0$ and $K_j u^i_{x_j}(x,t)\,v_j^e\geqslant 0$; the proof for the case $K_j u^e_{x_j}(x,t)\,v_j^e\leqslant 0$ and/or $K_j u^i_{x_j}(x,t)\,v_j^e\leqslant 0$ is similar. It then follows from (3.48) that there exists $T_0>0$ and $\delta>0$ such that

$$K_j u^e_{x_j}(x,t)\,v_j^e\geqslant\delta, \qquad K_j u^i_{x_j}(x,t)\,v_j^e\geqslant\delta, \qquad \forall (x,t)\in\partial D\times(0,T_0), \tag{3.49}$$

and consequently

$$b_{\varepsilon,j}(x,t)\,v_j^e\geqslant\delta', \qquad \forall (x,t)\in\partial D\times(0,T_0). \tag{3.50}$$

For fixed $\varepsilon>0$ let z be the solution of the parabolic problem

$$z_t-(a_j z_{x_j})_{x_j}=0 \qquad \text{in} \quad D\times(0,T_1), \tag{3.51}$$
$$z(x,0)=0, \qquad \text{in } D, \tag{3.52}$$
$$B_{\varepsilon,j}(x,t)\,z_{x_j}(x,t)=\sigma(x)\cdot t \qquad \text{on} \quad \partial D\times(0,T_1). \tag{3.53}$$

The existence, uniqueness, and regularity of a solution z to the above problem result from (3.50) and [8, p. 322, Theorem 5.4]. Furthermore, z satisfies [8, p. 322, estimate (5.14)].

$$|z|^{\alpha+2}_{D\times(0,\tau)}\leqslant C_1\,|\sigma(x)t|^{\alpha+1}_{\partial D\times(0,\tau)}, \tag{3.54}$$

where $\alpha\in(0,1)$ and the norms $|\cdot|$ are as defined in [8].

A careful examination of the proof of (3.54) (see [8, pp. 324–328], for example) reveals that $\exists\tau_\varepsilon>0$ such that, independently of T_2,

$$C_1\leqslant C_2\,\frac{C_3}{\sum_{j=1}^n |B_{\varepsilon,j}|^0_{\partial D\times(0,\tau)}}\leqslant 2\,\frac{C_3}{\sum_{j=1}^n |B_{\varepsilon,j}|^0_{\partial D\times(0,\tau)}}, \qquad \forall\tau\leqslant\tau_\varepsilon \tag{3.55}$$

provided τ_ε is small enough. Here C_3 depends only D and a_j and hence is independent of ε.

From the definition of $B_{\varepsilon,j}$ it follows using the smoothness of u and (3.49), that

$$\sum_{j=1}^{n} |B_{\varepsilon,j}|^{\alpha+1}_{\partial D\times(0,\tau)} \leqslant \varepsilon^{-\alpha}\cdot C_4 \tag{3.56}$$

$$C_5 \leqslant \sum_{j=1}^{n} |B_{\varepsilon,j}|^{0}_{\partial D\times(0,\tau)} \leqslant \varepsilon\cdot C_6 \tag{3.57}$$

$$C_7 \leqslant \frac{|\sum_{j=1}^{n} B_{\varepsilon,j} v_j^e|^0_{\partial D\times(0,\tau)}}{\sum_{j=1}^{n} |B_{\varepsilon,j}|^0_{\partial D\times(0,\tau)}} \leqslant C_8, \tag{3.58}$$

where the constants $C_4, ..., C_8$ are independent of ε and τ provided $\tau\in(0, T_0/2)$, where T_0 is defined in (3.49).

This allows for a choice of τ_ε which is independent of T_2.

Choose T_2 in (3.47) such that $T_2\leqslant\tau_\varepsilon$, and (3.54) yields

$$|z|^{\alpha+2}_{D\times(0,\tau)} \leqslant \varepsilon^{-1}\cdot C_9 \tau^{1/2-\alpha/2}, \qquad \forall\tau\leqslant T_2 m$$

where C_9 is independent of τ and ε.

Thus

$$|\nabla z|^{0}_{D\times(0,\tau)} \leqslant \varepsilon^{-1}\cdot C_{10}\tau, \qquad \forall\tau\leqslant T_2. \tag{3.59}$$

Set $v_\varepsilon(x, t) = z(x, T_2 - t)$. This function satisfies all the requirement of the lemma.

4. Proof of Theorem 1.2

Let $x_0\in\Sigma$, $W = B(x_0, r)$, where r is sufficiently small. For any $v\in C_0^2(W\times(0, T))$,

$$\int_0^T \int_{D\cap W} U(v_t + (a_j v_{x_j})_{x_j})\, dx\, dt$$
$$= \int_0^T \int_{(\partial D)\cap W} \sigma(x)\, b_j(x, t)\, v_{x_j}(x, t)\, d\mu\, dt = 0, \tag{4.60}$$

by the previous lemma, since $\sigma = 0$ on Σ.

Setting $a_j^*(x) = A_j + K_j$, $\forall x\in\Omega$, and using that $U(x, t) = 0$, $\forall x\in\Omega\backslash D$, it follows from the previous equality that

$$\int_0^T \int_W U(v_t + (a_j^* v_{x_j})_{x_j})\, dx\, dt = 0, \qquad \forall v\in C_0^2(W\times(0, T)). \tag{4.61}$$

Thus, U satisfies

$$U_t - a_j^* U_{x_j x_j} = 0 \qquad \text{in} \quad W \times (0, T). \tag{4.62}$$

It then follows from Lemma 2.2 and the unique continuation theorem that $U = 0$ in $W \times (0, T)$. Similarly, we deduce from (2.34) that $U = 0$ in $D \times (0, T)$.

Using this fact, (2.38) becomes

$$\int_0^T \int_{(\partial D) \cap W} \sigma(x)\, b_j(x, t)\, v_{x_j}(x, t)\, d\mu\, dt = 0, \qquad \forall v \in C_0^2(\Omega \times (0, T)). \tag{4.63}$$

We will now prove Theorem 1.2. We consider two cases:

Case 1. $\sigma \not\equiv 0$ on S_1.

Let $\xi_\varepsilon(x) \in C^1(\bar{D})$ be such that

$$\xi_\varepsilon = 1 \qquad \text{on} \quad \partial D \cap \{\sigma < -\varepsilon\} \qquad \xi_\varepsilon = 0 \text{ on } \partial D \cap \{\sigma \geqslant 0\} \tag{4.64}$$

$$|\nabla \xi_\varepsilon| \leqslant C/\varepsilon \qquad \text{in } \bar{D}, \tag{4.65}$$

and $\psi(t) \in C_0^1((0, T))$.

Taking $v = \psi \xi_\varepsilon u^i$ in (4.63) and using (2.39), (4.64), and (4.65) we obtain

$$\begin{aligned} -\int_0^T \int_{S_1 \cap V_m \cap \{\sigma > -\varepsilon\}} & \psi\sigma \frac{1 + |\nabla \gamma_m|^2}{\sum_{j=1}^{n-1} A_j \gamma_{m,x_j}^2 + A_n} \\ & \times \left(\sum_{j=1}^n K_j u^i_{v_j^e} \right) \left(\sum_{j=1}^n K_j u^i_{v_j^e} \right) d\mu\, dt \\ & - \int_0^T \int_{S_1 \cap \{\sigma > -\varepsilon\}} \sigma K_j u^i_{x_j} u^i_{x_j}\, d\mu\, dt \\ & \leqslant C \operatorname{meas}\{0 > -\sigma > -\varepsilon\} \to 0 \qquad \text{if} \quad \varepsilon \to 0. \end{aligned}$$

Hence, $\nabla u^i = 0$ on $(\{\sigma < 0\} \cap S_1) \times (0, T)$ and by the unique continuation theorem (see [9]) $u^i = u^e = \text{const.}$ in Q_T, which contradicts the assumption (A5).

Case 2. Assume now that $\sigma \geqslant 0$ then $\sigma \equiv 0$ on S_1 and (4.63) becomes

$$\int_0^T \int_{S_2} \sigma K_j u^e_{x_j} v_{x_j}\, d\mu\, dt = 0, \qquad \forall v \in C_0^2(\Omega \times (0, T)). \tag{4.66}$$

Let $\psi(t)$ be a smooth nonnegative function with compact support in $(0, T)$. By standard regularity results (see [8], for example) u^e is regular on

$\partial D \times (0, T)$, then the function ψu^e is $C_0^1((0, T); \bar{D})$. Setting $v = \psi u^e$ in (4.66) yields

$$\int_0^T \int_{S_2} \psi \sigma K_j u^e_{x_j} u^r_{x_j} \, d\mu \, dt = 0.$$

Hence, $\nabla u^e = 0$ on $(\{\sigma > 0\} \cap S_1) \times (0, T)$ and by the unique continuation theorem (see [9]) $u^i = u^e = \text{const.}$ in Q_T, which contradicts the assumption (A5).

REFERENCES

1. G. ALESSANDRINI AND S. VESSELA, Error estimates in an identification problem for a parabolic equation, *Boll. Un. Mat. Ital. C (6)* **4** (1985), 87–111.
2. M. S. AGRANOVICH AND M. I. VISHIK, Elliptic problems with a parameter and parabolic problems of general type, *Russian Math. Surveys* **19**, No 3 (1964), 53–158.
3. H. BELLOUT AND A. FRIEDMAN, Identification problems in potential theory, *Arch. Rational Mech. Anal.* **101** (1988), 143–160.
4. G. E. FOGG, Ground water flow and sand bdy interconnectedness in a thick, multiple-aquifer system, *Water Resour. Res.* **22** (1986), 679–694.
5. A. FRIEDMAN AND M. VOGELIUS, Identification of small inhomogeneities of extreme conductivity by boundary measurements: A continuous dependence results, preprint.
6. V. ISAKOV, On uniqueness of recovery of a discontinuous conductivity coefficient, *Comm. Pure Appl. Math.* **41** (1988), 841–853.
7. O. A. LADYZHENSKAYA, V. JA. RIVKIND, AND N. N. URALCEVA, The classical solvability of diffraction problems, *Proc. Steklov Inst. Math.* **92** (1966), 132–166.
8. O. A. LADYZHENSKAYA, V. A. SOLONIKOV, AND N. N. URALCEVA, Linear and quasilinear equations of parabolic type, *in* "Translations of Mathematical Monographs," Vol. 2.3, Amer. Math. Soc., Providence, RI, 1968.
9. E. M. LANDIS AND A. O. OLEINIK, Generalized analyticity and some related properties of solutions of elliptic and parabolic equations, *Russian Math. Surveys* **29**, No. 2 (1974), 195–212.
10. C. D. PAGANI, Determining a coefficient of a parabolic equation, preprint.
11. N. Z. SUN AND W. W.-G. YEH, Identification of parameter structure in ground water inverse problem, *Water Resour. Res.* **21** (1985), 869–883.
12. S. YAKOWITZ AND L. DUCKSTEIN, Instability in aquifer identification: Theory and case studies, *Water Resour. Res.* **16** (1980), 1045–1064.
13. W. W.-G. YEH, Review of parameter identification procedures in ground water hydrology: The inverse pioblem, *Water Resour. Res.* **22** (1986), 95–108.