Singular Two Point Boundary Value Problems for Second Order Differential Systems

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1. INTRODUCTION

Consider the following singular two point boundary value problem (SBVP) on \([a, b]\):

\[
\begin{aligned}
A(t, x, x') x'' &= f(t, x, x'), \\
(x(a) - P x'(a)) &= 0, \\
x(b) + Q x'(b) &= 0,
\end{aligned}
\]  \hspace{1cm} (1.1a)

\[
\hspace{1cm} (1.1b)
\]

where \(x(t): [a, b] \to \mathbb{R}^n\), \(A(t, x, y) \in C([a, b] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^{n \times n})\) is symmetric and positive definite on \([a, b] \times \mathbb{R}^n \times \mathbb{R}^n\), \(\text{rank}\ A(b, -Q y, y) < n\) for \(y \in \mathbb{R}^n\). \(P\) and \(Q\) are \(n \times n\) constant matrices, \(f(t, x, y) \in C([a, b] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)\).

In recent years, the scalar problems (i.e., \(n = 1\)) of singular differential equations (1.1a), with various kinds of boundary conditions have been widely studied (see, for example, [1–6] and references therein). For \(n > 1\), Li Y ong [7] has proven an existence theorem for (1.1a) with Dirichlet boundary conditions by employing the geometric method posed in [8].

The investigations of the solvability to the problem (1.1) for \(n > 1\) are rare; \(\text{SBVP}\ (1.1)\) with Robin boundary conditions (1.1b) has not been seen yet. The goal of this paper is to give some existence results for \(\text{SBVP}\ (1.1)\)
by exploiting some known results [9, 10] on nonsingular differential equations. Hence we develop the theory of singular boundary value problems, and provide a method to investigate the singular differential systems.

The paper is organized as follows. In Section 2, we extend some known results for later use. In Section 3, we give the main results of the paper, and apply those results to some examples of semilinear boundary value problems.

We impose the following assumptions on \( f(t, x, y) \) and \( A(t, x, y) \):

(H1) There are constants \( k > 0 \) and \( \alpha \geq 0 \) such that, for \( t \in [a, b] \), \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^n \),

\[
\|f(t, x, y)\| \leq k(1 + \|y\|^\alpha).
\]

(H2) Let \( \lambda_i(t, x, y) \) be the \( i \)th eigenvalue of \( A(t, x, y) \), \( i = 1, 2, \ldots, n \), \( D = [a, b] \times \mathbb{R}^n \); there is a function \( \lambda_0(t) \in C([a, b], \mathbb{R}) \), \( \lambda_0(t) > 0 \) for \( t \in [a, b] \) and \( \int_a^b (ds/\lambda_0(s)) < +\infty \), such that \( \lambda_i(t, x, y) \geq \lambda_0(t) \) for \( (t, x) \in D \) and \( y \in \mathbb{R}^n \).

A function \( x(t); [a, b] \to \mathbb{R}^n \) is said to be a solution of \( \text{SBVP} \) (1.1) if \( x(t) \) belongs to \( C^2([a, b], \mathbb{R}^n) \cap C^1([a, b], \mathbb{R}) \) (under the sense of Lemma 2.4 in this paper), and \( x(t) \) satisfies equation (1.1a) with boundary conditions (1.1b).

Remark 1.1. For Robin type boundary conditions (1.1b), it is obvious that if \( x(t) \) is a solution to the \( \text{SBVP} \) (1.1), then the derivative of \( x(t) \) at \( t = b \) must exist; hence the condition "\( \int_a^b (ds/\lambda_0(s)) < +\infty \)" in (H2) is needed. To see this, let us consider a scalar equation: \( \lambda_0(t)x'' = x' \), \( t \in [a, b] \), where \( A(t, x, y) = \lambda_0(t) \) and \( f(t, x, y) = y \). (H1) is satisfied with \( k = 1 \) and \( \alpha = 1 \). The first order derivatives of the solution to the problem are \( x'(t) = c \cdot e^{\int_a^t (ds/\lambda_0(s))}, c \in \mathbb{R} \). If \( \lim_{t \to b^-} \int_a^t (ds/\lambda_0(s)) = +\infty \) and \( c \neq 0 \), then the derivative \( x'(t) \) of any nontrivial solution to the problem will go to infinity exponentially as \( t \to b^- \) (we refer to [7] for this treatment about Dirichlet boundary conditions).

Remark 1.2. If we consider the nonhomogeneous boundary conditions

\[
x(a) - Px'(a) = \alpha, \quad x(b) + Qx'(b) = \beta.
\]

\( \alpha, \beta \in \mathbb{R}^n \), we can change them to homogeneous conditions (1.1b) by a linear substitution \( x = y + g(t) \), where \( g(t) \) is a certain linear vector-valued function (see Lemma 2.1 below).
2. PRELIMINARIES

In this section, we give some preliminary results for later use. We start with the following Green's function expression for SBVP (1.1).

**Lemma 2.1.** Suppose that \(A(t, x, y)\) satisfies (H2); \(f(t, x, y)\) is continuous on \([a, b] \times \mathbb{R}^n \times \mathbb{R}^n\); both \(P\) and \(Q\) are symmetric and positive semidefinite. Let \(\bar{x}(t) \in C^1([a, b], \mathbb{R})\) and

\[
y(t) = \int_a^b G(t, s) A^{-1}(s, \bar{x}(s), \bar{x}'(s)) f(s, \bar{x}(s), \bar{x}'(s)) \, ds + g(t);
\]

then \(y(t) \in C^2([a, b], \mathbb{R}^n) \cap C^1([a, b], \mathbb{R}^n)\), and satisfies

\[
\begin{align*}
\begin{cases}
A(t, \bar{x}, \bar{x}') y'' = f(t, \bar{x}, \bar{x}'), \\
y(a) - Py(a) = \alpha, & y(b) + Qy'(b) = \beta,
\end{cases}
\end{align*}
\]

(2.1)

where

\[
G(t, s) = \begin{cases}
((t - b)E - Q)( (b - a)E + P + Q)^{-1} [(s - a)E + P], & a \leq s \leq t \leq b, \\
((t - a)E + P)( (b - a)E + P + Q)^{-1} [(s - b)E - Q], & a \leq t \leq s \leq b
\end{cases}
\]

(2.2)

(Green's matrix), \(E\) denotes the unit matrix on \(\mathbb{R}^{n \times n}\), and \(g(t)\) is the unique solution to the following problem:

\[
\begin{align*}
\begin{cases}
x'' = 0, \\
x(a) - Px'(a) = \alpha, & x(b) + Qx'(b) = \beta.
\end{cases}
\end{align*}
\]

(2.3)

**Proof.** Let \(\|G\| = \max_{a \leq t, s \leq b} \|G(t, s)\|, \|f\| = \max_{a \leq s \leq b} \|f(s, \bar{x}(s), \bar{x}'(s))\|\); then

\[
\begin{align*}
\int_a^b \|G(t, s)\| \|A^{-1}(s, \bar{x}(s), \bar{x}'(s)) f(s, \bar{x}(s), \bar{x}'(s))\| ds \\
\leq \|G\| \|f\| \int_a^b \frac{1}{\lambda_0(s)} \, ds < +\infty, & t \in [a, b];
\end{align*}
\]

thus \(y(t)\) is well defined in \([a, b]\).
It is easy to see that \( y(t) \in C^2([a, b], R^n) \cap C^1([a, b], R^n) \).

Now, we go to prove that \( y'(t) = \int_a^b G_i(t, s)A^{-1}(s, \bar{x}(s), \bar{x}'(s))f(s, \bar{x}(s), \bar{x}'(s))ds + \bar{g}(t) \), where

\[
G_i(t, s) = \begin{cases} 
((b-a)E + P + Q)^{-1}[(s-a)E + P], & a \leq s \leq t \leq b, \\
((b-a)E + P + Q)^{-1}[(s-b)E - Q], & a \leq t \leq s \leq b.
\end{cases}
\] (2.4)

Since \( y(t) = \int_a^b G_i(t, s)A^{-1}f ds + \int_a^b G(t, s)A^{-1}f ds + g(t) \), by (2.2) we have

\[
y'(t) = \int_a^b G_i(t, s)A^{-1}f ds + g'(t)
\]

\[
= \int_a^b G_i(t, s)A^{-1}f ds + g'(t)
\]

\[
= \int_a^b G_i(t, s)A^{-1}f ds + g'(t)
\]

\[
= I + \int_a^b G_i(t, s)A^{-1}f ds + g'(t).
\] (2.5)

In the following, we are trying to prove \( I = 0 \). Since

\[
[((b-a)E + P + Q)^{-1}[(t-a)E + P]
\]

\[
= \int_a^b [-(b-a)E + P - Q + (t-a)E + P]dP
\]

\[
= \int_a^b [(b-a)E + P + Q]^{-1}[(t-a)E + P]dP
\]

\[
= \int_a^b [(b-a)E + P + Q]^{-1}[(t-a)E + P]dP
\]

\[
= \int_a^b [(t-a)E + P][((b-a)E + P + Q)^{-1}[(t-b)E - Q],
\]

so by (2.5), \( I = 0 \) and \( y'(t) = \int_a^b G_i(t, s)A^{-1}f ds + g'(t) \).

From (2.4), (2.3), we have \( y''(t) = A^{-1}(t, \bar{x}(t), \bar{x}'(t))f(t, \bar{x}(t), \bar{x}'(t)) \) for \( t \in [a, b] \), and \( y(t) \) satisfies the boundary conditions in (2.1). Finally, by a simple calculation, we get that \( g(t) = c_1 + c_2(t-a) \), where \( c_1 = \alpha + P \cdot c_2, c_2 = \frac{(b-a)E + P}{Q}^{-1}(\beta - \alpha) \).
Let $\lambda \in [0, 1]$, $S(\lambda)$ be the set of $C^2([a, b], R^n) \cap C^1([a, b], R^n)$ functions $x(t)$ which satisfies the following problem:

$$
\begin{align*}
A(t, x, x') x'' &= \lambda f(t, x, x'), \\
x(a) - Px'(a) &= \lambda a, \\
x(b) + Qr(b) &= \lambda \beta.
\end{align*}
$$

(2.6a) (2.6b)

By Leray–Schauder topological degree, we present the following continuation theorem, which allows one to pass from the solvability of the nonsingular problem ($\lambda = 0$) to the singular problem with $\lambda = 1$. The related result on nonsingular differential systems can be found in [9].

**Lemma 2.2.** Suppose that all conditions in Lemma 2.1 hold, and moreover, there is a constant $B > 0$ independent of $\lambda$ such that, for any given $x(t) \in S(\lambda)$ and all $t \in [a, b]$, $\|x(t)\| \leq B$ and $\|x'(t)\| \leq B$ hold; then $S(1)$ is nonempty.

**Proof.** We define an operator $F$ on $C^1([a, b], R^n)$: $\forall x(t) \in C^1([a, b], R^n)$.

$$
F(x)(t) = \int_a^b G(t, s) A^{-1}(s, x(s), x'(s)) f(s, x(s), x'(s)) \, ds + g(t),
$$

where $G(t, s)$ and $g(t)$ are given by (2.2), (2.3) respectively. From Lemma 2.1, the operator $F$ is well defined. Moreover, the continuity of $F$ on $C^1([a, b], R^n)$ directly follows from (H1) and (H2). If $x(t) \in C^1([a, b], R^n)$ such that $x = F(x)$, then $x(t) \in S(1)$.

Let $(x_n(t))$ be a bounded sequence on $C^1([a, b], R^n)$; then from the assumptions of the lemma, we have

(a) $(F(x_n))$ is uniformly bounded.

(b) $(F'(x_n))$ is uniformly bounded; therefore $(F(x_n))$ is equicontinuous on $C([a, b], R^n)$.

(c) $F(x_n)(t) \in C^2([a, b], R^n)$; thus for $t_1, t_2 \in [a, b]$,

$$
\begin{align*}
\|F'(x_n)(t_2) - F'(x_n)(t_1)\| &= \\left\| \int_{t_1}^{t_2} F''(x_n)(s) \, ds \right\| \\
&\leq \int_{t_1}^{t_2} \|F''(x_n)(s)\| \, ds \\
&= \int_{t_1}^{t_2} \|A^{-1}(s, x_n(s), x'_n(s)) f(s, x_n(s), x'_n(s))\| \, ds \\
&\leq \int_{t_1}^{t_2} \frac{\|f(s, x_n(s), x'_n(s))\|}{\lambda_0(s)} \, ds.
\end{align*}
$$

(2.7)
Since \( \{x_n(t)\} \) is bounded on \( C^1([a, b], R^n) \), there is a \( M > 0 \) (independent of \( n \)) such that 
\[
\|f(s, x_n(s), x'_n(s))\| \leq M, \text{ } s \in [a, b].
\]
From (2.7),
\[
\|F'(x_n)(t_2) - F'(x_n)(t_1)\| \leq M \left| \int_{t_1}^{t_2} \frac{1}{\lambda_0(s)} \, ds \right|
\]
thus \( \{F'(x_n)\} \) is equicontinuous on \( C([a, b], R^n) \). We now deduce from the Arzela–Ascoli theorem that \( (F(x_n)) \) is a compact set in \( C^1([a, b], R^n) \), and therefore the operator \( F \) is absolutely continuous.

Consider the operator \( I - \lambda F, \lambda \in [0, 1] \), where \( I \) denotes the identity on \( C^1([a, b], R^n) \). Define:
\[
\Omega = \left\{ x \in C^1([a, b], R^n); \quad \|x\|_1 < r_0 \right\},
\]
where \( \|\cdot\|_1 \) is the norm of \( C^1([a, b], R^n) \). Take \( r_0 > 0 \) sufficiently large such that if \( \|x\|_1 = r_0 \), then either \( \|x\| > B \) or \( \|x'\| > B \). Since \( \Omega \) is open and bounded, and for \( \lambda \in [0, 1] \), the operator \( I - \lambda F \) does not vanish on \( \partial \Omega \) (otherwise contradicts the choice of \( r_0 \)), by Leray–Schauder theory, the degree \( d[I - \lambda F, \Omega, 0] \) exists. By invariance of degree under homotopy,
\[
d[I - F, \Omega, 0] = d[I, \Omega, 0] = 1.
\]
Thus there is a \( x(t) \in \Omega \) such that \( x = F(x) \), or \( x(t) \in S(1) \). This ends the proof.

Lemma 2.2 gives the existence of a solution to \textbf{SBVP} (1.1) if \( S(\lambda) \) is bounded in the \( C^1 \) norm. So as to get a \textit{a priori} bound, we need the following result (see [10] for a related result on the nonsingular case).

\textbf{Lemma 2.3.} Let (H1), (H2) be satisfied and \( \mu_0 \) be a positive constant and \( \mu_0 < b - a \). Suppose further that there is a \( r_0 > 0 \) such that \( \|x(t)\| \leq r_0 \) for any \( x(t) \) satisfying equation (2.6a), \( t \in [a, b] \); and the exponential \( \sigma \) in (H1) satisfies
\[
0 \leq \sigma < \min \left\{ 1 + \frac{1}{k\lambda_0(2r_0/\mu_0 + k\lambda_0)}, \quad 2 \right\}
\]
where \( \lambda_0 = \int_a^b (ds/(\lambda_0(s))) \). Then, there is a constant \( M > 0 \), only dependent on \( k \), \( \lambda_0 \) and \( r_0 \), such that \( \|x(t)\| \leq M, t \in [a, b] \).

\textbf{Proof.} Let \( t_0 \in [a, a + \mu_0] \) such that \( \|x'(t_0)\| = \max_{t \in [a, a + \mu_0]} \|x'(t)\| \). Denote \( q = \|x'(t_0)\| \), and choose a \( \delta \) so that \( |\delta| \leq \mu_0/2 \) and \( t_0 + \delta \in [a, a + \mu_0] \).
Since \( x(t_0 + \delta) - x(t_0) = \int_{t_0}^{t_0+\delta} x'(s) \, ds \), we have:

\[
x(t_0 + \delta) - x(t_0) - \delta x'(t_0) = \int_{t_0}^{t_0+\delta} x'(s) \, ds - \int_{t_0}^{t_0+\delta} x'(t_0) \, ds
\]

\[
= \int_{t_0}^{t_0+\delta} x''(\tau) \, d\tau \, ds = \int_{t_0}^{t_0+\delta} x''(\tau) \, ds \, d\tau
\]

\[
= \int_{t_0}^{t_0+\delta} x''(\tau)(t_0 + \delta - \tau) \, d\tau.
\]

Let \( S = (\tau - t_0)/\delta \); then

\[
x(t_0 + \delta) - x(t_0) = \delta x'(t_0) + \int_{0}^{1} x''(t_0 + s\delta)(1 - s) \delta^2 \, ds
\]

\[
= \delta x'(t_0) + \int_{0}^{1} \lambda A^{-1}(t_0 + s\delta, x, x') \times f(t_0 + s\delta, x, x')(1 - s) \delta^2 \, ds; \quad (2.9)
\]

thus \( |\delta||x'(t_0)|| \leq 2r_0 + |\delta|^2 \int_{0}^{1} \frac{\lambda k(1 + q^\alpha)}{\lambda_0(t_0 + s\delta)}(1 - s) \, ds. \quad (2.10) \)

For any \( s \) between \( t_0 \) and \( t_0 + \delta \), \( 1/(\lambda_0(s)) \) is bounded; thus we denote

\[
k_0 = \max_{s \in [a, a + \mu_0]} \left\{ \frac{1}{\lambda_0(s)} \right\}.
\]

Note that \( 0 \leq \lambda \leq 1 \); from (2.10), we have

\[
|\delta|q \leq 2r_0 + |\delta|^2 k_0 k(1 + q^\alpha) \int_{0}^{1} (1 - s) \, ds = 2r_0 + \frac{1}{2}|\delta|^2 k_0 k(1 + q^\alpha).
\]

(2.11)

Let \( \varphi(s) = k_0 k(1 + s^\alpha) \). Since \( \sigma < 2 \), we have \( s^2/(\varphi(s)) \to \infty \) as \( s \to \infty \).

Choose \( q^* > 0 \) such that \( s^2/(\varphi(s)) > 4r_0 \) for \( s > q^* \), and let \( M_0 = \max(q^*, 8r_0/\mu_0) \). Now, we go to prove that \( \|x'(t)\| \leq M_0 \) for any \( t \in [a, a + \mu_0] \). To do this, it is sufficient to show that \( q > q^* \) implies \( q < 8r_0/\mu_0 \).
Note that \( q^2 / \varphi(q) > 4r_0 \) for \( q > q^* \). If \( 4r_0/q \leq \frac{1}{2} \mu_0 \), choose \( \delta \), so that \( |\delta| = 4r_0/q \) (recall that \( |\delta| \leq \frac{1}{2} \mu_0 \)). Then by (2.11), we get
\[
q \leq \frac{2r_0}{|\delta|} + \frac{|\delta|}{2} \cdot \varphi(q) < \frac{2r_0}{|\delta|} + \frac{|\delta|}{2} \cdot \frac{q^2}{4r_0} = \frac{2r_0}{4r_0} \cdot \frac{q}{2} + \frac{q}{4r_0} = \frac{q}{2} + \frac{q}{2} = q,
\]
a contradiction. So it must be \( 4r_0/q > \frac{1}{2} \mu_0 \), which implies \( q < 8r_0/\mu_0 \).

When \( t \in [a + \mu_0, b) \), \( t - \mu_0 \in [a, b) \), similar to (2.9), we have
\[
x(t - \mu_0) = x(t) - \mu_0 x'(t) + \int_0^1 x''(t + s\mu_0)(1 - s) \mu_0^2 ds, \quad \text{or}
\]
\[
\mu_0 x'(t) = x(t) - x(t - \mu_0) + \mu_0^2 \int_0^1 x''(t - s\mu_0)(1 - s) ds
\]
\[
= x(t) - x(t - \mu_0) + \mu_0 \int_{t - \mu_0}^t x'(s) \left(1 - \frac{t - s}{\mu_0}\right) ds;
\]
thus
\[
\|x'(t)\| \leq \frac{2r_0}{\mu_0} + \int_{t - \mu_0}^t \frac{k(1 + \|x'(s)\|^\sigma)}{\lambda_0(s)} ds \\
\leq \frac{2r_0}{\mu_0} + k\lambda_0 + k \int_a^t \frac{\|x'(s)\|^\sigma}{\lambda_0(s)} ds. \quad (2.12)
\]

Let
\[
F(t) = \frac{2r_0}{\mu_0} + k\lambda_0 + k \int_a^t \frac{\|x'(s)\|^\sigma}{\lambda_0(s)} ds.
\]

We conclude from (2.12) that if \( 0 \leq \sigma < 1 \), then
\[
F(t) \leq \left[(1 - \sigma)k\lambda_0 + \left(\frac{2r_0}{\mu_0} + k\lambda_0\right)^{1-\sigma}\right]^{1/(1-\sigma)} \overset{\text{def}}{=} M_1;
\]
if \( \sigma = 1 \), then
\[
F(t) \leq \left(\frac{2r_0}{\mu_0} + k\lambda_0\right)e^{k\lambda_0} \overset{\text{def}}{=} M_2;
\]
and if

\[ 0 \leq \sigma < \left( 1 + \frac{1}{k\lambda_0 \left( \frac{2r_0}{\mu_0} + k\lambda_0 \right)} \right), \]

then

\[ F(t) \leq \frac{1}{\left[ 1 - (\sigma - 1)k\lambda_0 k_1^{\sigma - 1} \right]^{1/(\sigma - 1)}} = M_3, \]

where \( k_1 = (2r_0/\mu_0) + k\lambda_0 \). If \( k_1 > 1 \), then \( 1 - (\sigma - 1)k\lambda_0 k_1^{\sigma - 1} > 1 - (\sigma - 1)k\lambda_0 k_1 > 0 \); otherwise \( k_1 \leq 1 \); then \( k\lambda_0 < 1 \). Since \( \sigma < 2 \), we get \((\sigma - 1)k\lambda_0 k_1^{\sigma - 1} < 1 \). So \( M_3 \) is well defined. Finally we choose \( M = \max(M_0, M_1, M_2, M_3) \) to complete the proof.

**Lemma 2.4.** If the assumptions in Lemma 2.3 are satisfied, then \( \lim_{t \to b} x(t) \) exists.

**Proof.** Let \( y(t) = (t - a)x'(t) \), \( t \in [a, b] \); then \( y'(t) = x'(t) + (t - a)x''(t) \). Assume \( t_1, t_2 \in [a, b], t_1 \leq t_2 \); then

\[ y(t_2) - y(t_1) = x(t_2) - x(t_1) + \int_{t_1}^{t_2} (s - a)x''(s) \, ds. \]

Thus,

\[ \|y(t_2) - y(t_1)\| \leq \|x(t_2) - x(t_1)\| + \int_{t_1}^{t_2} (s - a)\|x''(s)\| \, ds. \tag{2.13} \]

\[ \forall \varepsilon > 0, \text{ since } x(t) \text{ is continuous in } t, \text{ there is a } \delta_1 > 0, \text{ such that for } |t_2 - t_1| < \delta_2, \]

\[ \|x(t_2) - x(t_1)\| < \frac{\varepsilon}{2}. \tag{2.14} \]

From Lemma 2.3, \( \exists M > 0 \) so that \( \|x'(t)\| \leq M \) for \( t \in [a, b] \); thus

\[ \int_{t_1}^{t_2} (s - a)\|x''(s)\| \, ds \leq (b - a)k\int_{t_1}^{t_2} \frac{1}{\lambda_0(s)} \left( 1 + \|x'(s)\|^{\sigma} \right) \, ds \]

\[ \leq k(b - a)(1 + M^{\sigma}) \int_{t_1}^{t_2} \frac{1}{\lambda_0(s)} \, ds. \]
So, \( \exists \delta_2 > 0 \), if \( |t_2 - t_1| < \delta_2 \) then
\[
k(b - a)(1 + M^\sigma) \int_{t_1}^{t_2} \frac{ds}{\lambda_0(s)} < \frac{\varepsilon}{2}.
\]

Hence if we choose \( \delta = \min\{\delta_1, \delta_2\} \), then from (2.13), we have \( \|y(t_2) - y(t_1)\| < \varepsilon \) for \( |t_2 - t_1| < \delta \). This shows the uniform continuity of \( y(t) \) on \([a, b]\); therefore \( \lim_{t \to b^-} y(t) \) exists. Thus, we get
\[
\lim_{t \to b^-} x'(t) = \lim_{t \to b^-} \frac{(t - a)x'(t)}{t - a} = \frac{1}{b - a} \lim_{t \to b^-} y(t)
\]
This ends the proof.

3. MAIN RESULTS

Let \( r(t, x): [a, b] \times \mathbb{R}^n \to \mathbb{R} \) be of class \( C^2 \), \( r(t, x) \to +\infty \) uniformly in \( t \in [a, b] \) as \( \|x\| \to \infty \). Again, we denote as \( S(\lambda) \), the set of \( C^2([a, b], \mathbb{R}^n) \cap C^1([a, b], \mathbb{R}^n) \) functions which satisfy
\[
\begin{cases}
A(t, x, x')x'' = \lambda f(t, x, x'), \\
x(a) - Px'(a) = 0, \quad x(b) + Qx'(b) = 0.
\end{cases}
\]

(3.1a)
(3.1b)

For a \( x(t) \in S(\lambda) \), let \( m(t) = r(t, x(t)) \) for \( t \in [a, b] \).

**Theorem 3.1.** Suppose (H1), (H2) hold, and in addition,

(H31) There are constants \( r^* > 0, k_0 > 0 \) such that for \( t \in [a, b] \), \( r(t, x) > r^*, \quad x' \in \mathbb{R}^n \),
\[
r''(t, x) = U(t, x, x') + \frac{\partial r}{\partial x} \cdot A^{-1}(t, x, x')f(t, x, x') \geq -k_0,
\]

(3.2)

where
\[
U(t, x, x') = \frac{\partial^2 r}{\partial t^2} + 2 \frac{\partial}{\partial x} \left( \frac{\partial r}{\partial t} \right) x' + \frac{\partial^2 r}{\partial x^2} x' \geq 0.
\]

(3.3)

(H312) The boundary conditions (3.1b) imply \( m(a) \leq cm'(a) \) and \( m'(b) \leq 0 \) for a certain \( c \geq 0 \).
Then, there is a $r_0 > 0$ such that, if $\sigma$ in (H1) satisfies (2.8), i.e.,

$$0 \leq \sigma < \min \left\{ 1 + \frac{1}{k \lambda_0 (2r_0/\mu_0 + k \lambda_0)^2} \right\}$$

for a certain $0 < \mu_0 < b - a$, then SBVP (1.1) has at least one solution.

**Proof.** Choose $\kappa$, an integer such that $e^{(1/\kappa)/(b - a)} \leq 2$, $\kappa \geq c$. Let $L_0$ be so large that $L_0 \geq 2r^*$ and

$$\sqrt{\frac{L_0}{2 \kappa^2 k_0}} \arctg \sqrt{\frac{L_0}{2 \kappa^2 k_0}} > b - a.$$

(3.4)

Our next step is to prove that for any $x(t) \in S(\lambda)$, the corresponding function $m(t)$ satisfies $m(t) \leq L_0$ for all $t \in [a, b]$. Suppose to the contrary that the set $\{t \in [a, b]; m(t) > L_0\}$ is nonempty for a certain $x(t) \in S(\lambda)$. Let $t_1 \in [a, b]$ such that $m(t_1) = \max_{a \leq t \leq b} m(t)$; then $m(t_1) > L_0$.

If $t_1 = a$, by (H3.12), $m(a) \leq cm'(a) \leq 0$ (since $c \geq 0$, $m'(a) = m'(t_1) \leq 0$), a contradiction. Thus $a < t_1 \leq b$.

Define $t_0$: if for any $t \in [a, t_1]$, $m(t) > L_0/2$, then let $t_0 = a$; otherwise, let

$$t_0 = \sup \left\{ t \in [a, t_1]; \frac{m'(t)}{\kappa} \geq \frac{m(t)}{\kappa} \right\}.$$

In order to show that the above definition for $t_0$ is reasonable, we should prove that when $t_0 \neq a$, the set $\{t \in [a, t_1]; m'(t) \geq (m(t))/\kappa\}$ is nonempty. If not, then $m'(t) < (m(t))/\kappa$ for $t \in [a, t_1]$. Thus, according to the definition of $t_1$, we can choose a $t_2$, $a < t_2 < t_1$, so that $m(t_2) = L_0/2$.

$m(t) > L_0/2$ for $t \in (t_2, t_1)$. Integrating from $t_2$ to $t_1$ in both sides of inequality $m'(t) < (m(t))/\kappa$, we get

$$m(t_1) \leq m(t_2) e^{(1/\kappa)(t_1 - t_2)} = \frac{L_0}{2} e^{(1/\kappa)(t_1 - t_2)} \leq \frac{L_0}{2} e^{(1/\kappa)(b - a)} \leq L_0;$$

(3.5)

this contradicts $m(t_1) > L_0$. Hence $t_0$ is well defined.

We now point out that $m'(t_0) \geq (m(t_0))/\kappa$. In fact, if $t_0 = a$, then $m(a) > L_0/2$; hence $c > 0$ ($c = 0$ implies $m(a) \leq cm'(a) = 0$), by (H3.12), $m'(a) \geq (1/c)m(a) \geq (m(t))/\kappa$ (since $\kappa \geq c$). If $a < t_0 < t_1$ ($t_0 \neq t_1$, since $m'(t_1) = 0$), then by the definition of $t_0$, $m'(t_0) \geq (m(t))/\kappa$. On the other hand, similar to the analysis for (3.5), we have $m(t) \geq L_0/2$ for $t \in [t_0, t_1]$.

Set $y(t) = (m'(t))/(m(t))$ for $t \in [t_0, t_1]$. Since $L_0 \geq 2r^*$, by (H3.11), $m'(t) \geq -k_0$. Thus

$$y'(t) = \frac{m''(t)m(t) - (m'(t))^2}{m^2(t)} \geq -\frac{k_0}{m(t)} - y^2(t) \geq -\frac{2k_0}{L_0} - y^2,$$
or by integration,
\[ \int_{y(t_0)}^{y(t_1)} \frac{ds}{(2k_0/L_0) + s^2} \geq -(t_1 - t_0). \]

It follows that
\[
\sqrt{\frac{L_0}{2k_0} \arctg{\frac{L_0}{2k_0} y(t_1)}} - \sqrt{\frac{L_0}{2k_0} \arctg{\frac{L_0}{2k_0} y(t_0)}} \geq -(t_1 - t_0). 
\]

(3.6)

Since \( a < t_1 \leq b \), by (H 3.11) and the definition of \( t_1 \), \( m'(t) = 0 \), this implies \( y(t_1) = 0 \). Note that \( y(t_0) = (m'(t_0))/(m(t_0)) \geq 1/\kappa \); from (3.6), we have
\[
\sqrt{\frac{L_0}{2k_0} \arctg{\frac{L_0}{2k_0} y(t_1)}} \leq \sqrt{\frac{L_0}{2k_0} \arctg{\frac{L_0}{2k_0} y(t_0)}} \leq t_1 - t_0 \leq b - a. 
\]

Hence there is a contradiction (recall the choice of \( L_0 \) in (3.4)). This shows that for any \( x(t) \in S(\lambda), m(t) \leq L_0 \) for all \( t \in [a, b] \), or equivalently, \((t, x(t)) \in ((t, x); r(t, x) \leq L_0)\) for all \( x(t) \in S(\lambda) \) and \( t \in [a, b] \). Since \( L_0 \) is independent of \( \lambda \), and the set \(((t, x); r(t, x) \leq L_0)\) is bounded and \( \lambda \)-independence, we have, for \( \lambda \in [0, 1] \) and any solution \( x(t) \) to the problem (3.1), that there is a \( r_0 > 0 \) independent of \( \lambda \), such that \( \|x(t)\| \leq r_0 \) for \( t \in [a, b] \). Now, all the conditions in Lemma 2.3 are satisfied and thus there is a \( M > 0 \) only dependent on \( k, \lambda_0 \), and \( r_0 \) such that \( \|x'(t)\| \leq M \) for \( t \in [a, b] \). The existence of a solution to \( \text{SBVP} (1.1) \) then follows from Lemma 2.2. The proof is completed.

Assume for every \( t \in (a, b), \{x \in \mathbb{R}^n; r(t, x) < 0\} \) is nonempty; then we define:
\[
D_0 = \{(t, x); t \in (a, b), r(t, x) < 0\}. 
\]

(3.7)

By the properties of function \( r(t, x) \), \( D_0 \) is bounded in \( \mathbb{R}^{n+1} \). Here is the question: how does one ensure the existence of a solution \( x(t) \) satisfying \((t, x(t)) \in D_0 \) for \( t \in [a, b] \)? With a modification from the proof of Theorem 3.1, we have the following theorem to give the answer.

**Theorem 3.2.** Assume that all assumptions in Theorem 3.1 hold with \( r^* = k_0 = 0 \) and the region \( D_0 \) defined by (3.7); then \( \text{SBVP} (1.1) \) has at least one solution \( x(t) \) which satisfies \((t, x(t)) \in D_0 \) for \( t \in [a, b] \).

**Proof.** Let \( x(t) \) satisfy (3.1), \( m(t) = r(t, x(t)), t \in [a, b] \). If \( \{t \in [a, b]; m(t) > 0\} \) is nonempty, there is a \( t_1 \in [a, b] \) such that \( m(t_1) = \)
max_{a \leq t \leq b}(m(t)) > 0. Assume \( t_1 = a, \) since \( m(a) \leq cm'(a), \) \( c > 0 \) and \( m'(a) \leq 0; \) then \( m(a) \leq 0 \) contradicts \( m(t_1) = m(a) > 0; \) thus \( a < t_1 \leq b. \)

Choose a \( t_0 \) by this way; if \( m(t) > 0 \) for all \( t \in [a, t_1], \) then let \( t_0 = a; \) otherwise let \( t_0 = \sup\{t \in [a, t_1]; m(t) = 0\}. \) Clearly, \( t_0 \) is well defined. We now claim that there is a \( t_2, t_0 < t_2 < t_1 \) such that \( m'(t_2) > 0. \) If not, for all \( t \in [t_0, t_1], \) \( m'(t) \leq 0, \) we have \( m(t_2) \leq m(t_0). \) If \( t_0 = a, \) then we have \( m(t_2) \leq m(a); \) it is impossible since we have proven that \( t_2 > a. \) If \( t_0 > a, \)

then by the definition of \( t_0, m(t_0) = 0; \) thus \( m(t_2) \leq 0. \) This is also impossible since we have assumed that \( m(t_1) > 0. \) Thus such a \( t_2 \) exists.

Let \( y(t) = (m'(t))/m(t) \) for \( t \in [t_2, t_1], \) \( y(t) \) is well defined since \( m(t) > 0 \) on \([t_2, t_1]. \) By (H3.11) with \( k_0 = 0, m''(t) \geq 0; \) thus

\[
y'(t) = \frac{m''(t)m(t) - (m'(t))^2}{m^2(t)} = \frac{m''(t)}{m(t)} - y^2 \geq -y^2.
\]

Consider the initial value problem

\[
\begin{cases}
z' = -z^2, \\
z(t_2) = y(t_2).
\end{cases}
\]

The solution of the problem is

\[
z(t) = \frac{y(t_2)}{1 + y(t_2)(t - t_2)} (t > t_2).
\]

By the comparison theorem, \( y(t) \geq z(t), \) or equivalently, \( m'(t) \geq z(t)m(t) \) for \( t \in [t_2, t_1]. \) Since on \([t_2, t_1], \) \( z(t) > 0 \) and \( m(t) > 0; \) thus \( m'(t) > 0 \) on the whole interval and specifically, \( m'(t_1) > 0. \) On the other hand, if \( a < t_2 < b, m(t) \) achieves its maximum at \( t_2; \) thus \( m'(t_2) = 0, \) a contradiction. Otherwise \( t_1 = b; \) then by (H3.12), \( m'(b) \leq 0 \) also leads to a contradiction. This proves that the set \( \{t \in [a, b]; m(t) > 0\} \) is empty, thus for any solution \( x(t) \) to (3.1), we have \((t, x(t)) \in \overline{D}_0 \) for all \( t \in [a, b]. \) Furthermore, by the boundedness of \( D_0, \) there is a \( r_0 > 0 \) such that \( \|x(t)\| \leq r_0, \)

\( t \in [a, b]. \) The rest part of the proof is similar to that of Theorem 3.1; we omit the details.

The following theorem gives another set of assumptions which also leads us to the same conclusion. Especially, the assumption \( "m(a) \leq c \cdot m'(a)" \)

in (H3.12) will be replaced by the following (H3.33).

**Theorem 3.3.** Suppose (H1),(H2) hold, and \( D_0 \) is defined by (3.7). In addition, assume

...
(H 3.31) For \( t \in [a, b] \), \( r(t, x) \geq 0 \), \( x' \in \mathbb{R}^n \),

\[
r'(t, x) = U(t, x, x') + \frac{\partial r}{\partial x} \cdot \mathcal{A}^{-1}(t, x, x') f(t, x, x') \geq 0, \tag{3.8}
\]

\[
U(t, x, x') = \frac{\partial^2 r}{\partial t^2} + 2 \frac{\partial}{\partial x} \left( \frac{\partial r}{\partial t} \right) \cdot x' + \frac{\partial^2 r}{\partial x^2} x' \geq 0. \tag{3.9}
\]

(H 3.32) For any solution \( x(t) \to (3.1) \) and \( \lambda \in [0, 1] \), if the set \( \{ t \in [a, b]; r(t, x(t)) > 0 \} \) is nonempty, then \( r(t, x(t)) \neq \text{const.} \) on \( [a, b] \).

(H 3.33) The boundary conditions (3.1b) imply \( m'(a) \geq 0 \) and \( m'(b) \leq 0 \).

Then, there is a \( r_0 > 0 \) such that, if \( \sigma \) in (H 1) satisfies (2.8) for a certain \( 0 < \mu_0 < b - a \), \textbf{SBVP} (1.1) has at least one solution.

Proof. Let \( m(t) = r(t, x(t)), \ t \in [a, b] \); if \( \{ t \in [a, b]; m(t) > 0 \} \) is nonempty, then there is a \( t_1 \in [a, b] \) such that \( m(t_1) = \max_{a \leq t \leq b} m(t) > 0 \). There are three cases under consideration: (a) \( t_1 = a \); (b) \( a < t_1 < b \); and (c) \( t_1 = b \). We only prove case (a).

Since \( t_1 = a \) and \( m(t) \neq \text{const.} \), there must be a \( t_0 > a \) so that \( m'(t_0) < 0 \) and \( m(t) > 0 \) for \( t \in [a, t_0] \). To see this, let us define \( t^* \) as if \( m(t) > 0 \) for \( t \in [a, b] \), then \( t^* = b \); otherwise \( t^* = \inf \{ t; m(t) = 0 \} \). Obviously, \( t^* > a \) and we therefore can choose a \( t_0 \in [a, t^*] \) such that \( m'(t_0) < 0 \) and \( m(t) > 0 \) on \( [a, t_0] \).

Let \( y(t) = \langle m'(t) \rangle/\langle m(t) \rangle, t \in [a, t_0] \); then \( y'(t) \geq -y^2(t) \). Since the solution of the problem

\[
\begin{cases}
z' = -z^2, \\
z(t_0) = y(t_0)
\end{cases}
\]

is

\[
z(t) = \frac{y(t_0)}{1 + y(t_0)(t - t_0)}, \quad t < t_0,
\]

by the comparison theorem, \( y(t) \leq z(t) \) or equivalently: \( m'(t) \leq z(t)m(t) \) for \( t \in [a, t_0] \). Then, for \( t = a \), we get

\[
m'(a) \leq z(a)m(a) = \frac{y(t_0)}{1 + y(t_0)(a - t_0)} \cdot m(a) < 0. \tag{3.10}
\]

Since \( t_1 = a \), by the definition of \( t_1 \), \( m'(a) \leq 0 \). However, by (H 3.33), \( m'(a) \geq 0 \); thus \( m'(a) = 0 \), a contradiction to (3.10). This shows that the set \( \{ t \in [a, b]; m(t) > 0 \} \) is empty. We refer to the proofs of Theorem 3.1 and Theorem 3.2 for the rest part of the proof, since they are similar.
Corollary 3.4. Suppose that all the assumptions in Theorem 3.3 are satisfied except (H 3.32), and (3.8) is modified by:

\[ r''(t, x) = U(t, x, x') + \frac{\partial r}{\partial x} \cdot A^{-1}(t, x, x') f(t, x, x') > 0. \] (3.11)

Then the conclusion of Theorem 3.3 still holds.

Proof. We only need to show that (3.1) implies (H 3.32). Let \( x(t) \) be a solution to the problem (3.1) for \( \lambda \in [0, 1] \), such that \( r(t, x(t)) \equiv \text{const.} > 0 \) on \([a, b] \); clearly \( r''(t, x(t)) = 0 \). This is contrary to (3.11).

As an example, let us consider the following singular semilinear boundary value problem

\[
\begin{align*}
A(t)x'' = f(t, x), \\
x(a) - Px'(a) = 0, \\
x(b) + Qx'(b) = 0,
\end{align*}
\] (3.12)

where \( A(t) \in C([a, b], R^{n \times n}) \) is symmetric and positive definite in \( t \in [a, b] \), \( \text{rank} \ A(b) < n \). There is a real function \( \lambda_{\beta}(t) \in C([a, b], R), \lambda_{\beta}(t) > 0 \) for \( t \in [a, b] \), \( \int_{a}^{b} (ds/\lambda_{\beta}(s)) < +\infty \), so that \( \lambda_{\beta}(t) \geq \lambda(t), t \in [a, b], i = 1, 2, \ldots, n \), where \( \lambda(t) \) is the \( i \)-th eigenvalue of \( A(t) \).

Theorem 3.5. Assume that

1. Either the matrix \( P \) is positive definite or \( P = 0; Q \) is positive semidefinite.
2. There are \( r_0 > 0, k_0 > 0 \) such that, for \( t \in [a, b], \|x\| > r_0 \),

\[ x \cdot A^{-1}(t)f(t, x) \geq -k_0. \] (3.13)

Then the problem (3.12) has at least one solution.

Proof. Let \( r(t, x) = \frac{1}{2}\|x\|^2 \); then by (3.13),

\[ r'' = U(t, x, x') + \frac{\partial r}{\partial x} \cdot A^{-1}(t)f(t, x) = \|x\|^2 + x \cdot A^{-1}f \geq -k_0, \]

where \( U(t, x, x') = \|x\|^2 \geq 0 \).

If \( P \) is positive definite, there is a \( \alpha > 0 \) so that, for any \( \beta \in R^n \), \( \beta \cdot P \beta \geq \alpha \|\beta\|^2 \). Let \( x(t) \) be a solution to (3.1); then at \( t = a \), \( x(a) = Px'(a) \). Thus from \( m(t) = r(t, x(t)) = \frac{1}{2}\|x(t)\|^2 \), we have

\[ m(a) = \frac{1}{2}\|x(q)\|^2 = \frac{1}{2}\|Px'(a)\|^2 \leq \frac{1}{2}\|P\|^2 \|x'(a)\|^2 \] (3.14)

Furthermore, \( r'(t, x(t)) = x'(t) \cdot x(t) = x'(t) \cdot Px'(t) \); hence

\[ m'(a) = r'(a, x(a)) = x'(a) \cdot Px'(a) \geq \alpha \|x'(a)\|^2 \]
By (3.14) and (3.15), we have

\[ m(a) \leq \frac{1}{2\alpha} \|P\|^2 \|x'(a)\|^2 \leq \frac{\|P\|^2}{2\alpha} m'(a). \quad (3.15) \]

At \( t = b \), \( x(b) = -Qx'(b) \). Since \( Q \) is positive semidefinite,

\[ m'(b) = x'(b) \cdot x(b) = -x'(b) \cdot Qx'(b) \leq 0. \]

Thus when \( P \) is positive definite, we take \( c = \|P\|^2 / 2\alpha > 0 \), and when \( P = 0 \), choose \( c = 0 \). Note that, with the semilinear case of nonlinearity \( f(t, x) \), (2.8) is naturally satisfied with \( \sigma = 0 \); thus the conclusion of our theorem immediately follows from applying Theorem 3.1.

**THEOREM 3.6.** Assume that

1. Both \( P \) and \( Q \) are positive semidefinite.
2. There is a \( r_0 > 0 \) such that, for \( t \in [a, b] \), \( \|x\| > r_0 \).

\[ x \cdot A^{-1}(t)f(t, x) > 0. \quad (3.16) \]

Then the problem (3.12) has at least one solution \( x(t) \), \( \|x(t)\| \leq r_0 \) for \( t \in [a, b] \).

**Proof.** Let \( r(t, x) = \frac{1}{2}\|x\|^2 - \frac{1}{2}r_0^2 \); by (3.16),

\[ r'' = U(t, x, x') + \frac{\partial}{\partial x} \cdot A^{-1}f = \|x'\|^2 + x \cdot A^{-1}(t)f(t, x) > 0, \]

where \( U(t, x, x') = \|x'\|^2 \geq 0 \), \( x' \in \mathbb{R}^n \).

For \( x(t) \), a solution to (3.1), \( r'(t, x(t)) = x'(t) \cdot x(t) \); hence

\[ m'(a) = r'(a, x(a)) = x'(a) \cdot x(a) = x'(a) \cdot Px'(a) \geq 0. \]

Similarly, \( m'(b) \leq 0 \). Then we apply Corollary 3.4 to get the desired conclusion.

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REFERENCES


