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A unified stabilized method for Stokes' and Darcy's equations

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Abstract

We use the lowest possible approximation order, piecewise linear, continuous velocities and piecewise constant pressures to compute solutions to Stokes equation and Darcy's equation, applying an edge stabilization term to avoid locking. We prove that the formulation satisfies the discrete *inf-sup* condition, we prove an optimal a priori error estimate for both problems. The formulation is then extended to the coupled case using a Nitsche-type weak formulation allowing for different meshes in the two subdomains. Finally, we present some numerical examples verifying the theoretical predictions and showing the flexibility of the coupled approach.

Keywords: Stokes' equation; Darcy's equation; Stabilized methods; Finite element; Interior penalty method; Nitsche's method; Domain decomposition; Inf–sup condition

1. Introduction

Our aim in this paper is to present a unified treatment of Stokes' equation and Darcy's equation. In infiltration problems, like the ones encountered in groundwater flow or bioflows, one is interested in solving a problem where the flow in one part of the domain is governed by Stokes' equation and in the other by Darcy's. It is then convenient to work with a method that may treat both equations in the same manner and also yield the same convergence orders in both cases. Since the flow field in such cases often is used as data for a system of transport equations from an engineering point of view one wishes to have continuous velocities for which one has control of the error in the incompressibility. In the Darcy case the mixed form is often advocated since it is more robust with respect to variations in the material data. It was pointed out in [13] that the construction of a finite element method which is uniformly well behaved with respect to both the Darcy case and the Stokes case is not trivial, in fact some popular LBB-satisfying velocity pressure pairs such as the non-conforming Crouzeix-Raviart element or the piecewise quadratics for the velocities together with piecewise constants for the pressure fail to converge at all in the Darcy limit. Other methods such as the standard Taylor–Hood element of the minielement converges but with a loss of convergence order and without convergence of the divergence of the velocities. The authors then propose an LBB-stable, H(div) conforming element that gives optimal order estimates in both regimes. However, the construction of this element is rather complicated and moreover geometry dependent and therefore less attractive for engineering purposes. We will propose a very simple and efficient

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method that allows for optimal error estimates in the energy norm in both regimes. Assuring the convergence of the divergence of the velocity field. We will consider equations of the following form:

$$
\begin{aligned}
A(u) + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega,
\end{aligned} \tag{1}
$$

where Ω is an open subset of $\mathbb{R}^d$, A is some selfadjoint positive definite operator, u denotes the velocity vector, p the pressure and $f \in [L^2(\Omega)]^d$. For the choice of A we focus on two cases of importance in fluid dynamics

- $A(u) := Iu$ corresponding to Darcy's equation.
- $A(u) := -2\mu\nabla \cdot \boldsymbol{\varepsilon}(u)$, where $\boldsymbol{\varepsilon}(u)$ is the symmetric part of the velocity gradient, corresponding to Stokes equation.

For simplicity we assume Dirichlet conditions on the boundary, that is, $u = 0$ on $\partial\Omega$ for Stokes and $u \cdot n = 0$ on $\partial\Omega$ for Darcy. Moreover, our results immediately carry over to the Brinkman model, where $A(u) := (\mu/\kappa)u - 2\mu\nabla \cdot \boldsymbol{\varepsilon}(u)$. It is well known that the computation of solutions to such systems require that some care is taken in the choice of approximating spaces in order to make the discrete problem well posed. In particular, the naive choice of piecewise linear finite elements for both the velocities and the pressure or piecewise linear finite elements for the velocities and piecewise constants for the pressure results in ill posed discretizations. The solution is either to enrich the velocity space, using higher order interpolation or local so called, bubble functions, or to stabilize the method using a Galerkin/least-squares formulation. A vast number of discretizations and stabilizations for the Stokes equation are proposed in the literature, see, e.g., [3,10,11,6,1,4,12]. For stabilized finite element methods treating the case of Darcy flow we refer to [14] and references therein. To remain competitive with the approach where Darcy's equation is treated as an elliptic Poisson's equation we wish to keep down the number of degrees of freedom as much as possible. The method for the Stokes system which is in some sense minimal would be to use piecewise constant (discontinuous) approximation for the pressures and piecewise linear (continuous) approximation for the velocities. This however results in a much too rich pressure space and the only velocity that can satisfy the incompressibility constraint is $u \equiv 0$. Indeed the discrete divergence operator becomes injective instead of surjective, a phenomenon known as "locking". The key to "unlock" the problem is to add a consistent stabilizing term to the formulation. We propose to add a symmetric stabilization term penalizing the jumps over the element edges of the piecewise constant pressures. This stabilization was first introduced in the context of Stokes equation in [11] in a global form and then considered in a local form in [12]. Comparisons with other stabilized methods for the Stokes equations were carried out in [16].

The main difference between Stokes and Darcy's equations, from the point of view of analysis, is that in Stokes the velocities are $[H^1(\Omega)]^d$ whereas in the case of Darcy they are only in $H_{\text{div}}(\Omega)$. This loss of regularity must be accounted for in the analysis, and this is the main reason why the stabilized mixed P_1/P_0 is an ideal candidate for the problem: since the incompressibility condition is tested with constants we obtain $H_{\text{div}}(\Omega)$ stability without additional least-squares stabilization.

In this paper we apply this mixed stabilized method to Stokes' equations and Darcy's equations in a unified manner and prove optimal a priori estimates in the energy norm applying to both systems. We also propose a Nitsche type weak coupling for Stokes and Darcy which can handle non-matching meshes on the separating interface. This can be convenient since the two media can have different material properties and therefore need different meshes. Finally, we show some numerical examples of showing the performance of the method on the separate problems and on a coupled example. Only the case of global stabilization is accounted for, but our results generalize to the local form analyzed in [12] and extend it to include Darcy flow. It is interesting to note that if the local version of the stabilization is applied the method enjoys local mass conservation on the macro element scale. For some recent results on the theoretical and numerical aspects on the coupling of the Stokes and the Darcy equation we refer to [5].

2. Finite element formulation

In order to formulate our finite element method we first introduce the weak formulation of problem (1). We introduce the Hilbert spaces

$$W^D = \{v \in H_{\text{div}}(\Omega) : v \cdot n = 0 \text{ on } \partial\Omega\},$$

$$W^S = \{v \in [H_0^1(\Omega)]^d\},$$

and

$$L_0^2 = \left\{ q \in L^2(\Omega) : \int_\Omega q \, \mathrm{d}x = 0 \right\},$$

with Ω some open subset of R^d. We denote the product space $W^X \times L_0^2$ by $\mathscr{W}^X$ where X is chosen to D or S depending on the choice of equation and define the following norm on $\mathscr{W}^X$:

$$\|(u, p)\|^2_{\mathscr{W}^X} = \|u\|^2_{l,\Omega} + \|\nabla \cdot u\|^2_{0,\Omega} + \|p\|^2_{0,\Omega},$$

with $l = 0$ for Darcy and $l = 1$ for Stokes. Let $a(u, v)$ be the bilinear form corresponding to the weak formulation of $A(u)$ and consider the bilinear form

$$B[(u, p), (v, q)] = a(u, v) - (p, \nabla \cdot v)_{0,\Omega} + (q, \nabla \cdot u)_{0,\Omega}. \tag{2}$$

The weak formulation of (1) now takes the form, find $(u, p) \in \mathscr{W}^X$ such that

$$B[(u, p), (v, q)] = (f, v)_{0,\Omega} \quad \forall (v, q) \in \mathscr{W}^X. \tag{3}$$

Let $\mathscr{T}_h$ be a conforming, shape regular triangulation of Ω. We introduce the two classical finite element spaces of piecewise linears and piecewise constants

$$V_h^0 = \{v : v|_K \in P_1(K); v \in C^0(\Omega); v|_{\partial\Omega} \equiv 0\},$$

$$V_h = \{v : v|_K \in P_1(K); v \in C^0(\Omega)\},$$

$$Q_h = \left\{ q : q|_K \in P_0(K); \int_\Omega q \, \mathrm{d}x = 0 \right\}.$$

The velocity field will be sought in $W_h^S = [V_h^0]^d$ for Stokes and in $W_h^D = \{v \in [V_h]^d : v \cdot n = 0 \text{ on } \partial\Omega\}$ for Darcy's equation and the pressure field in Q_h. In analogy with the notation above we denote the discrete counterpart of $\mathscr{W}^X$, $W_h^X \times Q_h$, by $\mathscr{W}_h^X$. We introduce the following bilinear form on which we will base our finite element method:

$$B_h[(u, p), (v, q)] = a(u, v) - (p, \nabla \cdot v)_{0,\Omega} + (q, \nabla \cdot u)_{0,\Omega} + J(p, q), \tag{4}$$

where

$$J(p, q) = \delta \sum_K \int_{\partial K \setminus \partial\Omega} h_{\partial K} [p][q] \, \mathrm{d}s,$$

with $[\cdot]$ denoting the jump over the element edge (taken on interior edges only). We propose the following finite element formulation: find $(u_h, p_h) \in \mathscr{W}_h^X$ such that

$$B_h[(u_h, p_h), (v_h, q_h)] = (f, v_h)_{0,\Omega} \quad \forall (v_h, q_h) \in \mathscr{W}_h^X. \tag{5}$$

This finite element formulation is simply the standard Galerkin formulation with the penalizing term $J(p, q)$ added. In the following we will assume that the pressure is in $H^1(\Omega)$: then the penalizing term is consistent and we have the following

Lemma 1. *If* (u, p) *is a weak solution to* (1) *with* $(u, p) \in W^X \times H^1(\Omega) \cap L_0^2$ *then*

$$B_h[(u - u_h, p - p_h), (v_h, q_h)] = 0 \quad \forall (v_h, q_h) \in \mathscr{W}_h^X.$$

Proof. Immediate by noting that if $p \in H^1(\Omega)$ then the trace of p is well defined and hence $J(p, q_h) = 0$ for all $q_h \in Q_h$. $\square$

3. Stability

Since it is a well known fact that the above choice of finite element spaces results in an ill posed discrete problem if used in a standard Galerkin method, the crucial point is to show that our stabilization operator $J(p, q)$ introduces sufficient coupling between the degrees of freedom in the pressure field such that an *inf-sup* condition is satisfied. In the analysis, we will use the following norm:

$$|||(u, p)|||^2 := \|(u, p)\|^2_{\mathscr{W}^X} + J(p, p).$$

Note that the triple norm contains the L^2-norm of $\nabla \cdot u$; this term is superfluous for Stokes since we already control the H^1-norm of the velocities, but of vital importance for Darcy. In fact, the control of the divergence is what allows us to prove optimal error estimates in the energy norm for sufficiently regular solutions. The main result of this section is the following theorem, assuring the wellposedness of our discretization.

Theorem 2. *The finite element formulation* (5) *satisfies the following inf-sup condition*

$$\gamma|||(u_h, p_h)||| \leqslant \sup_{(v_h, q_h) \in \mathscr{W}^X_h} \frac{B_h[(u_h, p_h), (v_h, q_h)]}{|||(v_h, q_h)|||} \quad \forall (u_h, p_h) \in \mathscr{W}^X_h.$$

Proof. Taking first $(v_h, q_h) = (u_h, p_h)$ we obtain

$$B_h[(u_h, p_h), (u_h, p_h)] \geqslant C_a \|u_h\|^2_{l,\Omega} + J(p_h, p_h), \tag{6}$$

where, using Korn's inequality,

$$2\mu\|\varepsilon(v)\|^2_{L_2(\Omega)} \geqslant C_{\mathrm{K}} \|v\|^2_{1,\Omega} \quad \forall v \in [H^1_0]^d,$$

we have set

$$C_a = \begin{cases} 1 & \text{for } l = 0, \\ C_{\mathrm{K}} & \text{for } l = 1. \end{cases}$$

As a consequence of the surjectivity of the divergence operator there exists a function $v_p \in [H^1_0(\Omega)]^d$ such that $\nabla \cdot v_p = p_h$ and

$$\|v_p\|_{1,\Omega} \leqslant c\|p_h\|_{0,\Omega}. \tag{7}$$

Let $\pi_h v_p$ denote the Clément projection of v_p onto $[V^0_h]^d$. By the stability of the projection we have $\|\pi_h v_p\|_{1,\Omega} \leqslant \tilde{c}\|p_h\|_{0,\Omega}$. We now take $(v_h, q_h) = (\pi_h v_p, 0)$ and add $0 = \|p_h\|^2 - (p_h, \nabla \cdot v_p)_{0,\Omega}$ to obtain

$$B_h[(u_h, p_h), (\pi_h v_p, 0)] = a(u_h, \pi_h v_p) + \|p_h\|^2 + (p_h, \nabla \cdot (\pi_h v_p - v_p))_{0,\Omega}.$$

We integrate the third term by parts on each triangle K

$$B_h[(u_h, p_h), (\pi_h v_p, 0)] = a(u_h, \pi_h v_p) + \|p_h\|^2_{0,\Omega} + \sum_K \frac{1}{2} \int_{\partial K} [p_h](\pi_h v_p - v_p) \cdot n \, \mathrm{d}s.$$

Applying now Cauchy-Schwarz inequality followed by the arithmetic–geometric inequality in the first and last term and using the stability estimate on $\pi_h v_p$ we obtain

$$\begin{aligned} B_h[(u_h, p_h), (\pi_h v_p, 0)] \geqslant & -\frac{C_b}{\alpha}\|u_h\|^2_{l,\Omega} - (1 - \tilde{c}\alpha)\|p_h\|^2_{0,\Omega} - \frac{1}{\alpha} J(p_h, p_h) \\ & - \alpha \sum_K \int_{\partial K} h^{-1} |(\pi_h v_p - v_p) \cdot n|^2 \, \mathrm{d}s, \end{aligned}$$

where

$$C_b = \begin{cases} 1 & \text{for } l = 0, \\ 2\mu & \text{for } l = 1. \end{cases}$$

To conclude we need the following trace inequality, cf. [17]:

$$\|u \cdot n\|_{0,\partial K}^2 \leqslant C\|u\|_{0,K}(h^{-1}\|u\|_{0,K} + \|u\|_{1,K}) \quad \forall u \in [H^1(K)]^d \tag{8}$$

from which we deduce

$$\|(\pi_h v_p - v_p) \cdot n\|_{0,\partial K}^2 \leqslant Ch\|v_p\|_{1,K}^2.$$

Taking into account (7) we may write

$$\sum_K \int_{\partial K} h^{-1}|\pi_h v_p - v_p|^2 \,\mathrm{d}s \leqslant c_{\mathrm{t}}\|p\|_{0,\Omega}^2$$

which leads to

$$\begin{aligned} B_h[(u_h, p_h), (\pi_h v_p, 0)] \geqslant & -\frac{C_b}{\alpha}\|u_h\|_{1,\Omega}^2 + (1 - (\tilde{c} + c_{\mathrm{t}})\alpha)\|p_h\|_{0,\Omega}^2 \\ & - \frac{1}{\alpha}J(p_h, p_h). \end{aligned} \tag{9}$$

The control of $\|\nabla \cdot u_h\|_{0,\Omega}^2$ is obtained by choosing $(v_h, q_h) = (0, \nabla \cdot u_h)$.

$$\begin{aligned} B_h[(u_h, p_h), (0, \nabla \cdot u_h)] &= \|\nabla \cdot u_h\|_{0,\Omega}^2 + J(p_h, \nabla \cdot u_h) \\ &\geqslant (1 - C\alpha)\|\nabla \cdot u_h\|_{0,\Omega}^2 - \frac{1}{\alpha}J(p_h, p_h), \end{aligned} \tag{10}$$

where we used that $\|h^{1/2}\nabla \cdot u_h\|_{\partial K}^2 \leqslant C\|\nabla \cdot u_h\|_K^2$ by a scaling argument if $\nabla \cdot u_h$ is elementwise constant. Finally, we take $(v_h, q_h) = (\beta u_h + \pi_h v_p, \beta p_h + \nabla \cdot u_h)$, with

$$\beta \geqslant (1 - (\tilde{c} + c_{\mathrm{t}})\alpha) + \alpha^{-1}\left(\frac{C_b}{C_a} + 2\right),$$

which yields by (6), (9), (10)

$$B_h[(u_h, p_h), (v_h, q_h)] \geqslant (1 - (\tilde{c} + c_{\mathrm{t}})\alpha)|||(u_h, p_h)|||^2.$$

The claim now follows by taking α sufficiently small and noting that $\exists C$ such that $|||(u_h, p_h)||| \geqslant C|||(v_h, q_h)|||$. $\square$

4. Error analysis

4.1. A priori estimates

First of all we note that applying the trace inequality (8) we easily derive the following approximation property for couples of functions $(u, p) \in [H^2(\Omega)]^d \times H^1(\Omega)$:

$$|||(u - \pi_h u, p - \pi_h p)||| \leqslant ch(\|u\|_{2,\Omega} + \|p\|_{1,\Omega}). \tag{11}$$

Proposition 3. *Assume that the solution* (u, p) *to problem* (1) *resides in* $[H^2(\Omega)]^d \times H^1(\Omega) \cap L_0^2(\Omega)$; *then the finite element solution* (5) *satisfies the error estimate*

$$|||(u - u_h, p - p_h)||| \leqslant ch(\|u\|_{2,\Omega} + \|p\|_{1,\Omega}).$$

Proof. In view of (11) we only need to show the inequality for $|||(u_h - \pi_h u, p_h - \pi_h p)|||$. By Theorem 2 and using Galerkin orthogonality we obtain, with the notation $\eta_h = u_h - \pi_h u$ and $\zeta_h = p_h - \pi_h p$,

$$
\begin{aligned}
|||(\eta_h, \zeta_h)||| &\leqslant \frac{1}{\gamma} \sup_{(v_h,q_h)\in \mathscr{W}_h^X} \frac{B_h[(\eta_h, \zeta_h), (v_h, q_h)]}{|||(v_h, q_h)|||} \\
&\leqslant \frac{1}{\gamma} \sup_{(v_h,q_h)\in \mathscr{W}_h^X} \frac{B_h[(u - \pi_h u, p - \pi_h p), (v_h, q_h)]}{|||(v_h, q_h)|||}.
\end{aligned}
$$

It remains to use interpolation estimates to bound the terms on the right-hand side. The result follows from standard interpolation theory and (8). We have

$$
\begin{aligned}
& a(u - \pi_h u, v_h) \leqslant ch\|u\|_{2,\Omega} |||(v_h, q_h)|||, \\
& \quad -(p - \pi_h p, \nabla \cdot v_h)_{0,K} = 0, \\
& \quad (q_h, \nabla \cdot (u - \pi_h u))_{0,K} \leqslant ch\|u\|_{2,K} |||(v_h, q_h)|||, \\
& \quad J(p - \pi_h p, q_h) \leqslant ch\|p\|_{1,\Omega} |||(v_h, q_h)|||.
\end{aligned}
$$

Using the Aubin-Nitsche duality argument we prove the following $L_2(\Omega)$-estimate for the velocities for the case of Stokes problem when the mesh parameter is smaller than the viscosity. □

Proposition 4. *For Stokes problem we have, if the problem is regularizing,*

$$
\|u - u_h\|_{0,\Omega} \leqslant ch^2(\|u\|_{2,\Omega} + \|p\|_{1,\Omega}).
$$

Proof. Let $(\varphi, r) \in \mathscr{W}^X$ be the solution of the dual equation

$$
B[(v, q), (\varphi, r)] = (\psi, v)_{0,\Omega} \quad \forall (v, q) \in \mathscr{W}^X, \tag{12}
$$

and we assume that this dual solution enjoys the additional regularity

$$
\|\varphi\|_{2,\Omega} + \|r\|_{1,\Omega} \leqslant c\|\psi\|_{0,\Omega}. \tag{13}
$$

Choosing $v = u - u_h$, $q = 0$ and $\psi = u - u_h$, we may write

$$
\|u - u_h\|_{0,\Omega}^2 = a(u - u_h, \varphi) + (\nabla \cdot (u - u_h), r)_{0,\Omega}
$$

and proceed using Galerkin orthogonality to obtain

$$
\begin{aligned}
\|u - u_h\|_{0,\Omega}^2 &= a(u - u_h, \varphi - \pi_h \varphi) + (\nabla \cdot (u - u_h), (r - \pi_h r))_{0,\Omega} \\
&\quad + (\nabla \cdot \pi_h \varphi, p - p_h)_{0,\Omega} + J(p - p_h, \pi_h r) \\
&\leqslant \|u - u_h\|_{1,\Omega} \|\varphi - \pi_h \varphi\|_{1,\Omega} + \|\nabla \cdot (u - u_h)\|_{0,\Omega} \|r - \pi_h r\|_{0,\Omega} \\
&\quad + |\pi_h \varphi - \varphi|_{1,\Omega} \|p - p_h\|_{0,\Omega} \\
&\quad + J(p - p_h, p - p_h)^{1/2} J(r - \pi_h r, r - \pi_h r)^{1/2}.
\end{aligned}
$$

As a consequence of proposition 3 and the regularity hypothesis (13) we may conclude, keeping in mind that

$$
\|\nabla \cdot (u - u_h)\|_{0,\Omega} \leqslant |||(u - u_h, 0)|||
$$

and using the interpolation result

$$
J(r - \pi_h r, r - \pi_h r)^{1/2} \leqslant h\|r\|_{1,\Omega},
$$

that

$$
\|u - u_h\|_{0,\Omega}^2 \leqslant ch^2\|\varphi\|_{2,\Omega} + ch^2\|r\|_{1,\Omega} + ch^2\|r\|_{1,\Omega} \leqslant ch^2\|u - u_h\|_{0,\Omega}. \qquad \square
$$

5. The coupled problem

The aim of this paper is to propose a unified approach to Stokes' and Darcy's equation in order to solve problems where the flow in one part of the domain $\Omega_1 := \Omega_S$ is approximated by the former system of equations and in another part $\Omega_2 := \Omega_D$ by the latter. To be able to handle completely independent triangulations of the different domains, we apply a Nitsche-type method of weak coupling between the domains. We will also split the viscous stress vector $2\mu\,\boldsymbol{\varepsilon}\cdot n$ into a scalar normal stress $\sigma_n = 2\mu\, n\cdot(\boldsymbol{\varepsilon}\cdot n)$ and a tangential stress vector $\sigma_t = 2\mu\,\boldsymbol{\varepsilon}\cdot n - \sigma_n\, n$ on the interface $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$, where $n := n_1$ is the outer unit normal to Ω_1. We note in particular that

$$(2\mu\boldsymbol{\varepsilon}\cdot n)\cdot v = (\sigma_t + \sigma_n n)\cdot v = \sigma_n v\cdot n + \sigma_t\cdot v. \tag{14}$$

Denoting by $(u|_{\Omega_i}, p|_{\Omega_i}) = (u_i, p_i), \quad i = 1, 2$, we consider the following conditions on Γ:

$$\begin{aligned} -\sigma_n(u_1) + p_1 = p_2, \quad \sigma_t(u_1) = 0 \quad &\text{(force balance)},\\ n\cdot u_1 = n\cdot u_2 \quad &\text{(continuity of normal velocity)}. \end{aligned} \tag{15}$$

We remark that the "full slip" condition in the tangential direction is not physically realistic. Beavers–Joseph (Robin-like) conditions like

$$\sigma_t\cdot t_j = -k(u_1 - u_2)\cdot t_j,$$

with k a stiffness parameter and t_j, $j=1,\dots,d$ denoting the tangential vectors on the interface, can easily be incorporated into the bilinear form in the standard way, but for ease of presentation we choose the simpler form in (15). An alternative implementation of this condition along the lines of [9] is described in Section 5.2.

In the following we will write $\tilde{u} = (u_1, u_2) \in V_1 \times V_2$ with the continuous spaces:

$$V_1 = \{v \in [H^1(\Omega_i)]^d : v|_{\partial\Omega\cap\partial\Omega_i} = 0\},$$

$$V_2 = \{v \in H_{\mathrm{div}}(\Omega_i) : v\cdot n|_{\partial\Omega\cap\partial\Omega_i} = 0\}.$$

To formulate our method, we suppose that we have regular finite element partitionings $\mathscr{T}_h^i$ of the subdomains Ω_i into shape regular simplexes. We shall consider one-sided mortaring using the trace mesh

$$\mathscr{G}_h = \{E : E = K\cap\Gamma,\ K \in \mathscr{T}_h^2\}. \tag{16}$$

We seek the approximation $\tilde{u}_h = (u_{1,h}, u_{2,h}) \in V^h = V_1^h \times V_2^h$ and $\tilde{p}_h = (p_{1,h}, p_{2,h}) \in Q^h = Q_1^h \times Q_2^h$, where

$$V_i^h = \{v_i \in V_i : v_i|_K \text{ is linear for all } K \in \mathscr{T}_h^i\},$$

$$Q_i^h = \{q_i \in Q_i : q_i|_K \text{ is constant for all } K \in \mathscr{T}_h^i\}.$$

On the interface we will use the notation $[\tilde{v}] = v_1 - v_2$ and we denote the diameter of $E \in \mathscr{G}_h$ by h_E. A variant of the method of Nitsche [15,2] can now be formulated as follows: find $\tilde{u}_h \in V^h$ such that

$$a_h(\tilde{u}_h, \tilde{v}) + b_h(\tilde{p}_h, \tilde{v}) + b_h(\tilde{q}, \tilde{u}_h) + J(\tilde{p}_h, \tilde{q}) = f_h(\tilde{v}) \tag{17}$$

for all $\tilde{v} \in V^h$ and $\tilde{q} \in Q^h$, with

$$a_h(\tilde{w}, \tilde{v}) := a(w_1, v_1) + a(w_2, v_2) + \gamma_0 \sum_{E\in\mathscr{G}_h} h_E^{-1} \int_E [\tilde{w}\cdot n]\,[\tilde{v}\cdot n]\,\mathrm{d}s, \tag{18}$$

$$b_h(\tilde{p}, \tilde{v}) := -\sum_i (p_i, \nabla\cdot v_i)_{\Omega_i} + \int_\Gamma p_2[\tilde{v}\cdot n]\,\mathrm{d}s, \tag{19}$$

$$J(\tilde{p}, \tilde{q}) = \sum_i \delta_i \sum_{K\in\mathscr{T}_h^i} \int_{\partial K\setminus\Gamma} h_{\partial K}[p_i][q_i]\,\mathrm{d}s,$$

and

$$f_h(\tilde{v}) := \sum_{i=1}^{2} (f, v_i)_{\Omega_i}, \tag{20}$$

with γ_0 sufficiently large (see below). The method is clearly consistent in the sense that it holds for the exact solution, and we also have stability as sketched in the following section.

5.1. Stability of the coupled formulation

To prove stability for the coupled problem we follow [7], we consider the simple case of two subdomains only. The main difficulty compared with the uncoupled analysis is the fact that the global pressure $\tilde{p}$ has mean zero, but the pressures in the two subdomains taken separately do not. We decompose the pressure in each subdomain into a constant part $\bar{p}$ and a part with mean value zero p^0, $\tilde{p} = \bar{p} + p^0$. It follows that $\|\tilde{p}\|_0^2 = \|p^0\|_0^2 + \|\bar{p}\|_0^2$ and that

$$\bar{p}_1|\Omega_1| + \bar{p}_2|\Omega_2| = 0. \tag{21}$$

Hence we may write the L_2 norm as

$$\|\bar{p}\|_0^2 = \bar{p}_1^2|\Omega_1| + \bar{p}_2^2|\Omega_2| = \bar{p}_2^2\left(\frac{|\Omega_2|^2}{|\Omega_1|} + |\Omega_2|\right).$$

For the velocities and p^0 we may use the same argument as in the previous section with some slight modifications to take into account the Nitsche-type coupling. We outline this first part in the following proposition.

Proposition 5. *The coupled formulation* (17) *satisfies the inf-sup condition of theorem* 2 *for with the triple norm given by*

$$|||(\tilde{u}, p^0)|||_C^2 = \sum_{i=1}^{2} |||(u_i, p_i^0)|||_{\Omega_i}^2 + \sum_{E \in \mathcal{G}_h} h_E^{-1} \int_E [\tilde{u} \cdot n]^2 \, \mathrm{d}s.$$

Proof. We will only point out how to modify theorem 2 to account for the coupled case.

(1) Note that when testing with $(\tilde{u}_h, \tilde{p}_h)$ we obtain the additional stabilizing term $\gamma_0 \sum_{E \in \mathcal{G}_h} h_E^{-1} \int_E [\tilde{u}_h \cdot n]^2 \, \mathrm{d}s$.
(2) To control the pressure we choose $v_{i,p} \in H_0^1(\Omega_i)$ such that $\nabla \cdot v_{i,p} = p_{i,h}^0$ in the two domains separately. This way the coupling terms do not interfere, since $[\pi_h \tilde{v}_p \cdot n] = (v_{1,p} - v_{2,p}) \cdot n = 0$. Furthermore, $b_h(\bar{p}, \pi_h v_p - v_p) = 0$ since $\bar{p}$ is piecewise constant on each subdomain and $v_p \in [H_0^1(\Omega_1 \cup \Omega_2)]^2$.
(3) When choosing $(\tilde{v}_h, \tilde{q}_h) = (0, \nabla \cdot \tilde{u}_h)$ the one-sided mortaring produces a term

$$\int_\Gamma \nabla \cdot u_{2,h} \, [\tilde{u}_h \cdot n] \, \mathrm{d}s$$

to control this term we use Cauchy-Schwarz inequality, the arithmetic–geometric inequality followed by a scaling argument to obtain

$$\langle \nabla \cdot u_{2,h}, [\tilde{u}_h \cdot n] \rangle_\Gamma \leqslant \frac{C\,c}{2} \|\nabla \cdot u_{2,h}\|_{\Omega_2^\Gamma}^2 + \frac{1}{2c} \sum_{E \in \mathcal{G}_h} h_E^{-1} \int_E [\tilde{u}_h \cdot n]^2 \, \mathrm{d}s,$$

where Ω_2^Γ denotes the union of the triangles in Ω_2 neighbouring to the boundary Γ. The second term on the right-hand side is controlled by the additional stabilizing term from (1), choosing γ_0 sufficiently large, and the proof is complete. □

To conclude we need a similar result for the constant part $\bar{p}$. To this end we will construct functions $v_{\bar{p}_i} \in V^h$, $i = 1, 2$, such that

$$\int_\Gamma v_{\bar{p}_i,1} \cdot n_1 \,\mathrm{d}s = -\int_\Gamma v_{\bar{p}_i,2} \cdot n_2 \,\mathrm{d}s = \bar{p}_i |\Omega_i|, \tag{22}$$

where n_1 and n_2 denotes the normal taken from domain Ω_1 and Ω_2, respectively. Moreover let v_{p_i} satisfy

$$\int_\Gamma [v_{\bar{p}_i}] \cdot n \,\mathrm{d}s = 0, \tag{23}$$

$$v_{\bar{p}_i}|_{\partial\Omega\setminus\Gamma} = 0, \tag{24}$$

$$|||(v_{\bar{p}_i}, 0)|||_h \leqslant C \|\bar{p}_i\|_{0,\Omega_i}. \tag{25}$$

We first assume that this function exists and prove the inf-sup condition for $\bar{p}$, then we show how to construct v_i.

Lemma 6. *We have that*

$$2\|\bar{p}\|_{0,\Omega} \leqslant C \sup_{(v_h,q_h)\in V^h\times Q^h} \frac{b(\bar{p}, v_h)}{|||(v_h, q_h)|||}.$$

Proof. We choose $(v_h, q_h) = (v_{\bar{p}_1} - v_{\bar{p}_2}, 0)$ and note that by the definition of $v_{\bar{p}_i}$ we have

$$\int_\Gamma [v_h \cdot n] \,\mathrm{d}s = 0$$

and hence

$$\int_\Gamma \bar{p}_2 [v_h \cdot n] \,\mathrm{d}s = \bar{p}_2 \int_\Gamma [v_h \cdot n] \,\mathrm{d}s = 0$$

leading to

$$b_h(\bar{p}, v_{\bar{p}_1} - v_{\bar{p}_2}) := -\sum_i (\bar{p}_i, \nabla \cdot v_i)_{\Omega_i} + \int_\Gamma \bar{p}_2 [v_h \cdot n] \,\mathrm{d}s = -\sum_i (\bar{p}_i, \nabla \cdot v_i)_{\Omega_i}.$$

We now integrate by parts in the remaining term and use (21) to obtain

$$\begin{aligned} -\sum_i (\bar{p}_i, \nabla \cdot (v_{\bar{p}_1} - v_{\bar{p}_2}))_{\Omega_i} &= \int_\Gamma \bar{p}_1 (v_{\bar{p}_1} - v_{\bar{p}_2}) \cdot n_1 \,\mathrm{d}s + \int_\Gamma \bar{p}_2 (v_{\bar{p}_1} - v_{\bar{p}_2}) \cdot n_2 \,\mathrm{d}s \\ &= \bar{p}_1^2 |\Omega_1| - \bar{p}_1 \bar{p}_2 (|\Omega_1| + |\Omega_2|) + \bar{p}_2^2 |\Omega_2| \\ &= 2\bar{p}_2^2 |\Omega_2| \left(\frac{|\Omega_2|}{|\Omega_1|} + 1 \right) = 2\|\bar{p}\|_{0,\Omega}^2. \end{aligned}$$

We may now conclude using (21) and (25)

$$C \frac{b(\bar{p}, v_h)}{|||(v_h, 0)|||} \geqslant \frac{b(\bar{p}, v_h)}{\|\bar{p}\|_{0,\Omega}} \geqslant \frac{2\bar{p}_2^2 |\Omega_2| \, (|\Omega_2|/|\Omega_1| + 1)}{\|\bar{p}\|_{0,\Omega}} = 2\|\bar{p}\|_{0,\Omega}. \qquad \square$$

To include the control of $\|\bar{p}\|$ given by the result of Lemma 6 in the inf-sup condition we show stability of the form $b(q_h, v_h)$ for the complete pressure.

Lemma 7. *There holds*

$$\|\tilde{p}\|_{0,\Omega} \leqslant C \sup_{(v_h,q_h)\in V^h\times Q^h} \frac{b(\tilde{p}, v_h)}{|||(v_h, q_h)|||} - J(\tilde{p}, \tilde{p}).$$

Proof. We choose $(v_h, q_h)=(\pi_h\tilde{v}_p+\lambda(v_{\bar{p}_1}-v_{\bar{p}_2}), 0)$ where $\tilde{v}_p$ is the function from point two in the proof of proposition 5 and $v_{\bar{p}_1}-v_{\bar{p}_2}$ is the function from Lemma 6. Following Theorem 2 and Lemma 6 we may write

$$\begin{aligned} b(\tilde{p}, \pi_h\tilde{v}_p+\lambda(v_{\bar{p}_1}-v_{\bar{p}_2})) &= \|p^0\|^2_{0,\Omega}+b(p^0+\bar{p}, \pi_h\tilde{v}_p-\tilde{v}_p+\lambda(v_{\bar{p}_1}-v_{\bar{p}_2}))\\ &= \|p^0\|^2_{0,\Omega}+b(p^0, \pi_h\tilde{v}_p-\tilde{v}_p)+b(p^0+\bar{p}, \lambda(v_{\bar{p}_1}-v_{\bar{p}_2}))\\ &\geqslant \|p^0\|^2_{0,\Omega}-cJ(\tilde{p},\tilde{p})\|p^0\|_{0,\Omega}+b(p^0, \lambda(v_{\bar{p}_1}+v_{\bar{p}_2}))+2\lambda\|\bar{p}\|^2_{0,\Omega}. \end{aligned}$$

Recalling the stabilizing term of point 1 in the proof of Proposition 5 we have that

$$b(p^0, \lambda(v_{\bar{p}_1}+v_{\bar{p}_2}))\geqslant -\lambda c\|p^0\|_{0,\Omega}|||(v_{\bar{p}_1}+v_{\bar{p}_2},0)|||\geqslant -\lambda c\|p^0\|_{0,\Omega}\|\bar{p}\|_{0,\Omega}$$

and we may conclude using Young's inequality and choosing $\lambda=1/(c)$

$$\begin{aligned} b(\tilde{p}, \pi_h\tilde{v}_p+\lambda(v_{\bar{p}_1}+v_{\bar{p}_2})) &\geqslant \frac{1}{2}\|p^0\|^2_{0,\Omega}-cJ(\tilde{p},\tilde{p})\|p^0\|_{0,\Omega}+2\lambda(1-\frac{\lambda c}{2})\|\bar{p}\|^2_{0,\Omega}\\ &\geqslant \frac{1}{2}\|p^0\|^2_{0,\Omega}-cJ(p^0,p^0)\|p^0\|_{0,\Omega}+\frac{1}{c}\|\bar{p}\|^2_{0,\Omega}. \end{aligned}$$

The claim now follows since $|||(v_{\bar{p}_1}-v_{\bar{p}_2},0)|||\leqslant c\|\tilde{p}\|_{0,\Omega}$. □

It remains to prove that we can construct the function $v_{\bar{p}_1}+v_{\bar{p}_2}$. We will for simplicity assume that the interface is plane. We denote the leftmost node in the domain Ω_i by x_i^L and the rightmost node by x_i^R, the nodes closest to the interface midpoint we denote by x_i^M, we assume that there exists some constant c_{nc} such that $|x_1^W-x_2^W|\leqslant c_{nc}h$, $W=L,R,M$ and that the distance between the endpoints and the midpoint is of order $O(|\Gamma|)$. Let w_1 denote the continuous function which is affine on the intervals $[x_1^L, x_1^M]$ and $[x_1^M, x_1^R]$ and is zero for $x\leqslant x_1^L$, for $x\geqslant x_1^R$ and for $x\in\partial\Omega_1\backslash\Gamma$ and for which $\int_\Gamma w_1\,\mathrm{d}s=1$. We define w_2 in the same way. We now use the harmonic extension R_iw_i to define w in the whole domain as

$$w=R_iw_i \quad \text{in } \Omega_i.$$

By the properties of the harmonic extension we have

$$\|w\|_{1,\Omega_1\cup\Omega_2}\leqslant c\sum_{i=1}^{2}\|w_i\|_{1/2,\Gamma}\leqslant\frac{C}{|\Gamma|}$$

and by the bound on the distance $|x_1^W-x_2^W|$, $W=L,R,M$ one readily computes that

$$\int_\Gamma\frac{\gamma_0}{h}[w]^2\,\mathrm{d}s \leqslant C\frac{c_{nc}}{|\Gamma|}$$

and it follows that the function defined by

$$v_{\bar{p}_i}=(0,w)\bar{p}_i|\Omega_i|$$

enjoys the required properties.

We thus have stability and consistency, and optimal convergence follows using the same techniques of proof as previously noting that

$$\gamma_0\sum_{E\in\mathscr{G}_h}h_E^{-1}\int_E[(\tilde{u}-\pi_h\tilde{u})\cdot n]^2\,\mathrm{d}s\leqslant Ch^2\sum_{i=1}^{2}\|u_i\|^2_{2,\Omega_i}$$

by the trace inequality (8).

5.2. Implementation of a Beavers–Joseph law

We finally consider a non-standard implementation of a Beavers–Joseph law governing the tangential stress vector on Γ,

$$[u] = -kP_t\,\sigma \cdot n, \quad P_t := I - n \otimes n, \tag{26}$$

where k is a material parameter (usually assumed to be proportional to the square root of the porosity, cf. [8]) and $\otimes$ denotes the exterior product. In [9] it was noted that for vanishing values of k, using non-matching meshes, there may be stability problems if the standard method of implementation (e.g., [8]) is used. Thus, we follow [9] and propose the following consistent penalty method: find $\tilde{u}_h \in V^h$ such that

$$a_h^*(\tilde{u}_h, \tilde{v}) + b_h(\tilde{p}_h, \tilde{v}) + b_h(\tilde{q}, \tilde{u}_h) + J(\tilde{p}_h, \tilde{q}) = f_h(\tilde{v}) \tag{27}$$

for all $\tilde{v} \in V^h$ and $\tilde{q} \in Q^h$, with $b_h(\cdot,\cdot)$, $J(\cdot,\cdot)$, and $f_h(\cdot)$ as above, and

$$\begin{aligned} a_h^*(\tilde{w}, \tilde{v}) := {} & a_h(\tilde{w}, \tilde{v}) - ([\tilde{w}] + kP_t\sigma(\tilde{w}_1)\cdot n, P_t\sigma(\tilde{v}_1)\cdot n)_\Gamma - (P_t\sigma(\tilde{w}_1)\cdot n, [\tilde{v}] + kP_t\sigma(\tilde{v}_1)\cdot n)_\Gamma \\ & + (kP_t\sigma(\tilde{w}_1)\cdot n, P_t\sigma(\tilde{v}_1)\cdot n)_\Gamma + (S_h([\tilde{w}] + kP_t\sigma(\tilde{w}_1)\cdot n), [\tilde{v}] + kP_t\sigma(\tilde{v}_1)\cdot n)_\Gamma. \end{aligned} \tag{28}$$

Here S_h is a matrix which depends on the interface conditions of the problem, the local mesh size, and a penalty parameter γ_S which has to be large enough for the method to be stable. More precisely, on an element K with diameter h_K,

$$S_h|_K = \left(\frac{h_K}{\gamma_S} I + kP_t\right)^{-1}.$$

Here γ_S can be chosen freely, but for stability it must be related to the constant C_I in the following trace inequality:

$$\|h_K^{1/2}\sigma(v_1)\|^2_{L_2(\partial K)} \leqslant C_I \|\sigma(v_1)\|^2_{L_2(K)} \quad \forall v_1 \in V_1^h.$$

For details, see [9].

The rationale behind the method (27) is as follows: the term

$$-(P_t\sigma(\tilde{w}_1)\cdot n, [\tilde{v}])_\Gamma \tag{29}$$

is a consistency term emanating from integration by parts,

$$-([\tilde{w}] + kP_t\sigma(\tilde{w}_1)\cdot n, P_t\sigma(\tilde{v}_1)\cdot n)_\Gamma,$$

which is consistent by (26), is artificially added to symmetrize the problem, and

$$(S_h([\tilde{w}] + kP_t\sigma(\tilde{w}_1)\cdot n), [\tilde{v}] + kP_t\sigma(\tilde{v}_1)\cdot n)_\Gamma,$$

also consistent by (26), is added to make the discrete problem coercive. We now note that (29) is in fact *not* consistent because there is no tangential force emanating from the Darcy side. From this point of view, the Beavers–Joseph-Saffman law

$$u_1 = -kP_t\sigma \cdot n \tag{30}$$

is more natural. This law is implemented in the formulation (27) simply by replacing $[\tilde{w}]$ by $\tilde{w}_1$ and $[\tilde{v}]$ by $\tilde{v}_1$ in $a^*(\cdot,\cdot)$. For the discrete problem to be well posed this point does not matter, but as will be shown numerically the relation (26) seems less stable in the case of Darcy-Stokes couplings.

6. Numerical results

6.1. Convergence study for Darcy flow

The first numerical example, taken from [14], is a study of convergence rates for Darcy flow. The domain under consideration is the unit square with a given exact pressure solution $p = \sin 2\pi x \sin 2\pi y$. The exact velocity field is

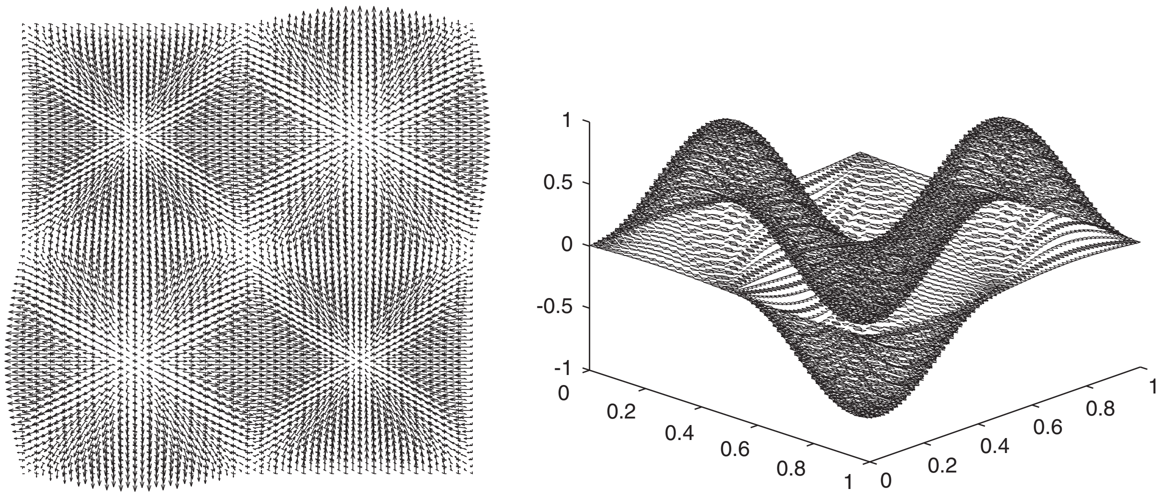

Fig. 1. Approximate velocity field and elevation of the pressure on the final mesh in a sequence.

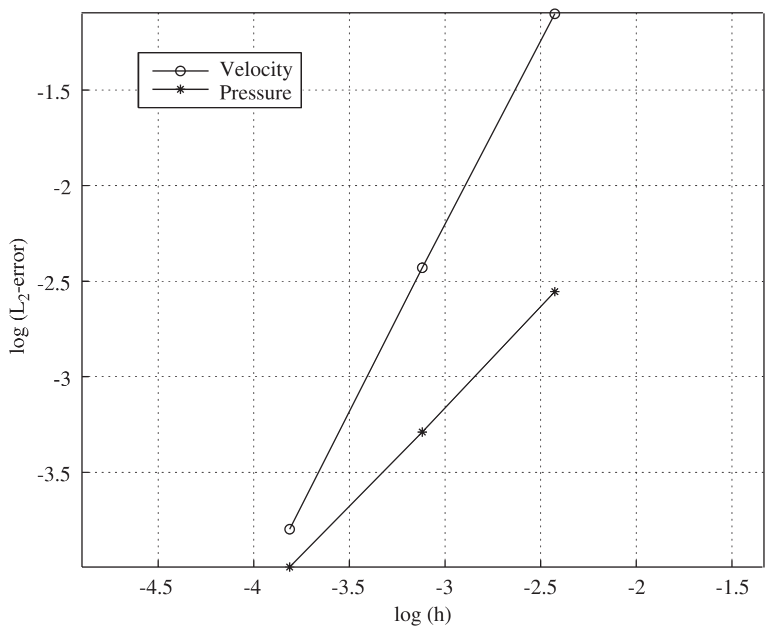

Fig. 2. L_2-norm convergence of the velocity and of the pressure for Darcy.

then computed from Darcy's law to give boundary conditions and a source term for the divergence. In order to create a unique pressure field we also impose zero mean pressure. We set $\delta = 10$.

In Fig. 1, we show the approximate velocities and pressures on the final mesh in a sequence. In Fig. 2, we show the convergence of the method in the L_2-norm, which yields second order accuracy for the velocities and first order for the pressure.

6.2. Convergence study for Stokes flow

Again, we consider the unit square with exact flow solution (from [16]) given by $u = (20xy^3, 5x^4 - 5y^4)$ and $p = 60x^2y - 20y^3 + C$. Choosing $\delta = 1/10$ and imposing zero mean pressure ($C = -5$), we obtain the optimal convergence shown in Fig. 3.

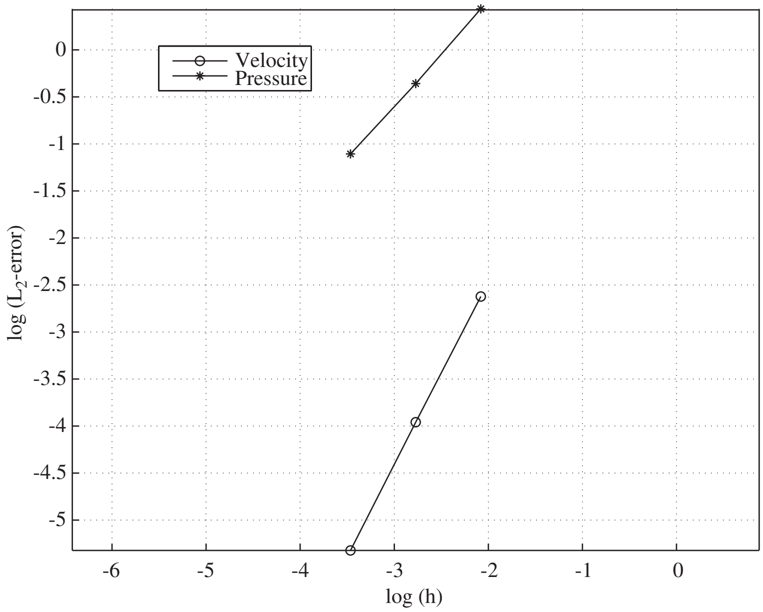

Fig. 3. L_2-norm convergence of the velocity and of the pressure for Stokes.

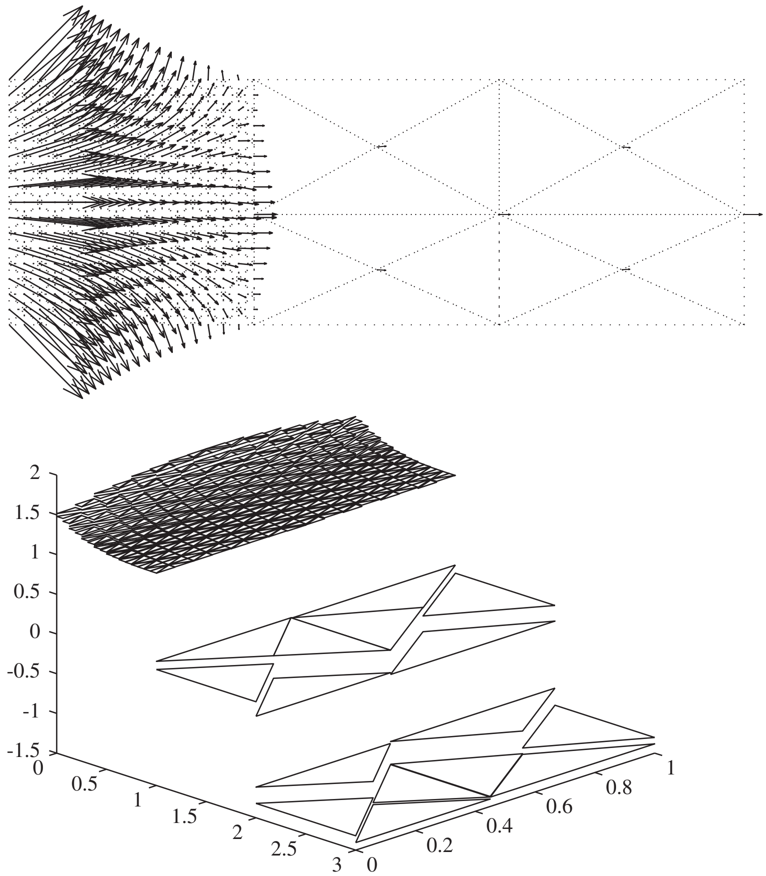

Fig. 4. Velocity and pressure solutions for the coupled problem.

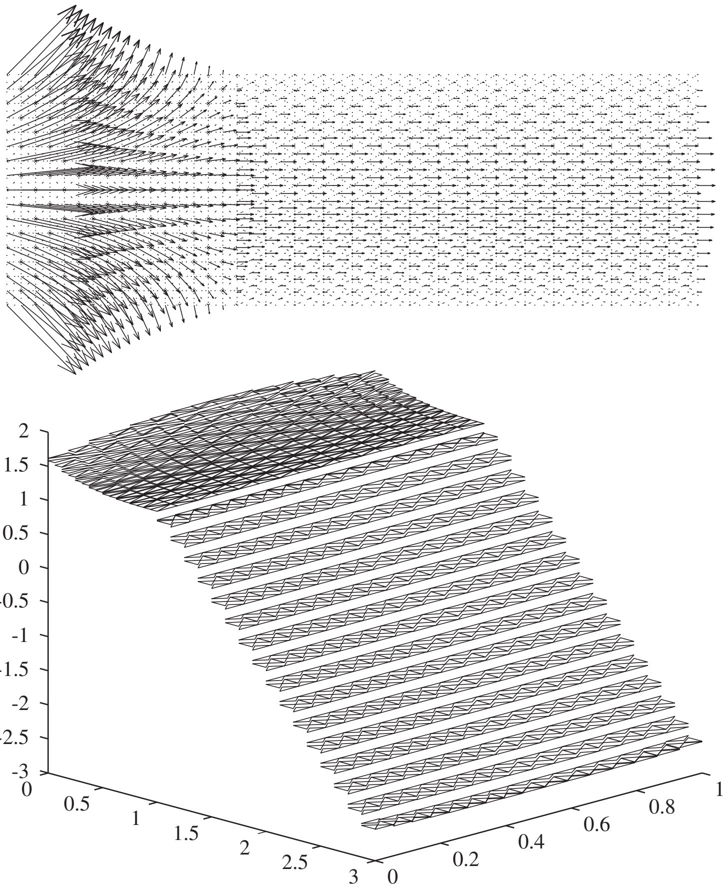

Fig. 5. Velocity and pressure solutions for the coupled problem.

6.3. Coupling of Stokes and Darcy in the normal direction

We consider an artificial example: in a domain $(0, 3) \times (0, 1)$ the flow is governed by Darcy on $(0, 1) \times (0, 1)$ and by Stokes on $(1, 3) \times (0, 1)$. The velocity solution for Darcy is given by the exact pressure solution

$$p = (1 - x)y(1 - y) - x + x^2 - x^3/3 + C_1,$$

i.e.,

$$u = (1 - 2x + x^2 + y - y^2, -1 + x + 2y - 2xy),$$

which is divergence free and has a parabolic profile at $x = 1$. We prescribe $u \cdot n$ at $y = \pm 1$ and $x = 0$. For Stokes, we prescribe $u = 0$ at $y = \pm 1$ and $u = (y(1 - y), 0)$ at $x = 2$, corresponding to Poiseuille flow. Here we have used $A = -\mu \Delta u$ instead of $A = -2\mu \nabla \cdot \boldsymbol{\varepsilon}(u)$ in the Stokes domain to obtain the usual Poiseuille linear pressure increase also at in- and outflow. Note that this does not affect the coupling terms at $x = 1$.

In Fig. 4, we show the effect of a coarse triangulation on one side; note that the solution on the interface is not parabolic due to the poor resolution on the Stokes domain. In Fig. 5, we give the corresponding solution using a finer resolution for the Stokes part. Note that the meshes still do not match across the interface.

For the convergence check we use the same example and note that the pressure from the Darcy problem is constant at $x = 1$. Thus, we have $p = -2x + C_2$ in the Poiseuille flow and continuity of the pressure across the interface. Imposing

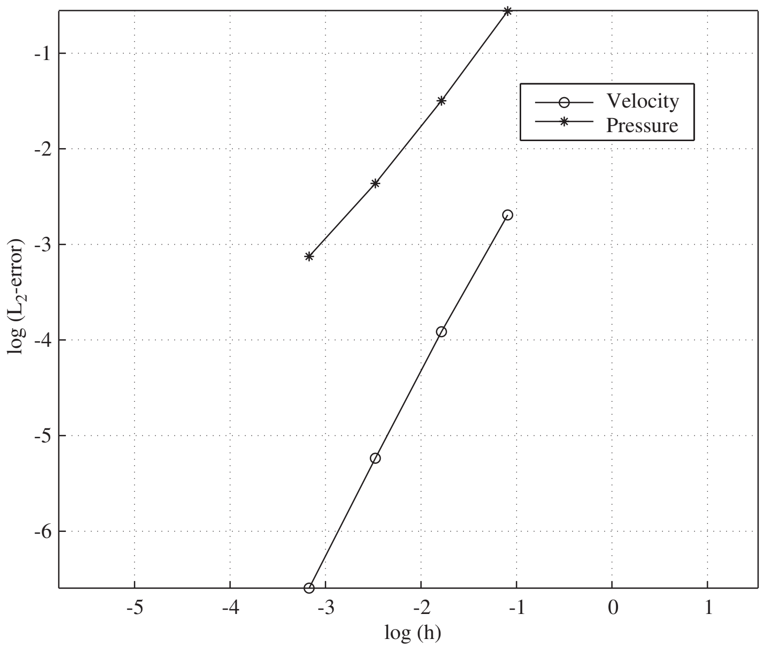

Fig. 6. L_2-norm convergence of the velocity and of the pressure for the coupled problem.

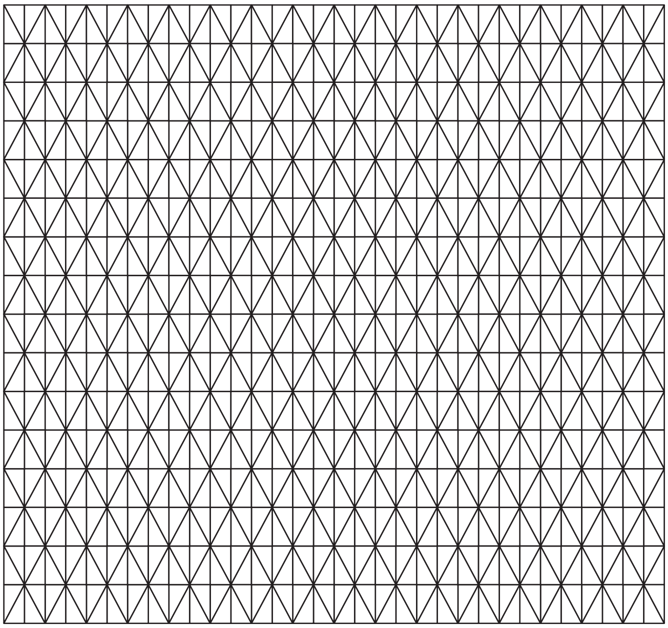

Fig. 7. Mesh used for the tangential coupling example.

mean pressure zero, these conditions yield $C_1 = 29/18$, $C_2 = 59/18$. The convergence of the pressure and the velocity in L_2, on a sequence of unfitted meshes (one of which is shown in Fig. 5) is given in Fig. 6, showing first order and second order convergence, respectively.

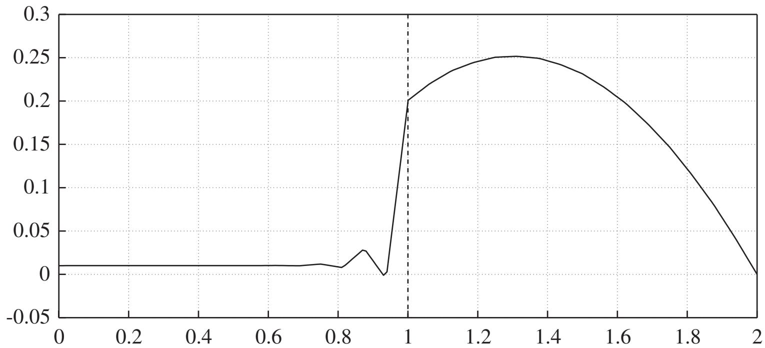

Fig. 8. Velocity profile using the interface law (26).

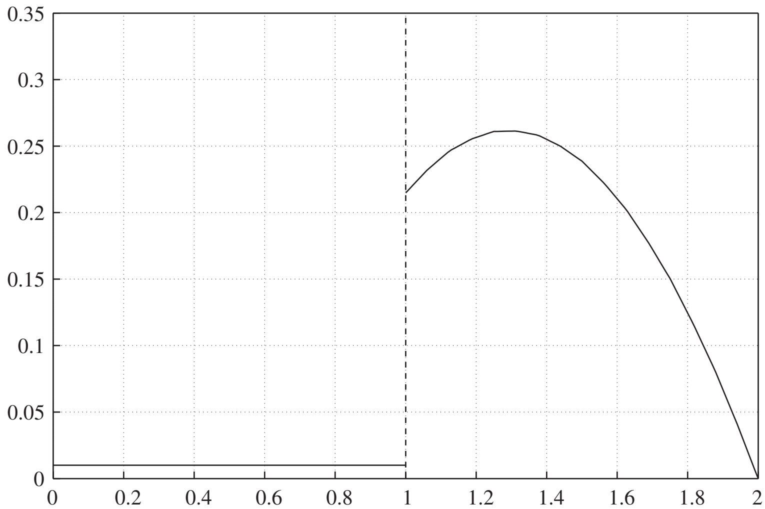

Fig. 9. Velocity profile using the interface law (30).

6.4. Tangential coupling of Stokes and Darcy

We give an example close to the one presented by Gartling et al. [8]. Here we solve a Darcy problem of the type

$$cu - \nabla p = f, \quad \nabla \cdot u = 0,$$

with c a constant. Data for this problem are as follows: on the domain $\Omega = (0, 2) \times (0, 2)$ with $\Omega_1 = (1, 2) \times (0, 2)$ and $\Omega_2 = \Omega \backslash \Omega_1$, we set $f = (0, 100)$, $\mu = 10$, and $c = 10^4$. Boundary conditions are $u \cdot n = 0$ at $x = 0$ and $u = 0$ at $x = 2$. The computational mesh used is shown in Fig. 7. In Fig. 8, we give the velocity profile at $y = 1$ for a Beavers–Joseph condition (26) with $k = 0$ which corresponds to continuity in tangential velocity. Note the oscillations at the interface which are consistent with the results of [8]. In contrast, if we use condition (30), with non-zero k, we used $k = 2/(3\sqrt{c})$, we obtain the profile shown in Fig. 9. This solution is completely stable, and it might be argued that it is unreasonable to enforce tangential velocities for Darcy in any case (unlike the coupling between a Brinkman model and Stokes).

7. Conclusion

We have applied the mixed P_1/P_0 stabilized finite element method allowing the use of piecewise linear approximation for the velocities and piecewise constant approximation for the pressures to Stokes and Darcy's equation. This formulation is a natural generalization of the Brezzi–Pitkäranta penalization [3], but remains consistent for sufficiently smooth exact solutions. We have proved optimal a priori estimates for both problems indicating that this method might be a suitable candidate for problems where one wishes to compute flows where (Navier-) Stokes and Darcy's equations are coupled. Moreover we discussed the coupling of the two systems using a Nitsche-type method. Some numerical results were reported showing good agreement with the theoretical predictions.

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