The Erdős–Ginzburg–Ziv theorem for dihedral groups

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Abstract

Let $n \geq 23$ be an integer and let $D_{2n}$ be the dihedral group of order $2n$. It is proved that, if $g_1, g_2, \ldots, g_{3n}$ is a sequence of $3n$ elements in $D_{2n}$, then there exist $2n$ distinct indices $i_1, i_2, \ldots, i_{2n}$ such that $g_{i_1}g_{i_2}\cdots g_{i_{2n}} = 1$. This result is a sharpening of the famous Erdős–Ginzburg–Ziv theorem for $G = D_{2n}$.

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Let $G$ be a finite group of order $n$, and let $S = (a_1, \ldots, a_k)$ be a sequence of $k$ elements in $G$ (repetition allowed). We call $S$ a 1-product sequence if $1 = \prod_{i=1}^{k} a_{\tau(i)}$ holds for some permutation $\tau$ of $\{1, \ldots, k\}$. We denote by $\prod(S)$ the product $\prod_{i=1}^{k} a_i$. We call $T = (a_{i_1}, \ldots, a_{i_t})$ a subsequence of $S$ if $1 \leq i_j \leq k$ for each $j$ and $i_j \neq i_t$ when $j \neq t$. Furthermore, if $1 \leq i_1 < \cdots < i_\ell \leq k$, we call $T$ a main subsequence of $S$. Clearly, every subsequence of $S$ can be reordered to form a unique main subsequence of $S$. For example, the subsequence $(a_2, a_1)$ of $S$ can be reordered to a main subsequence $(a_1, a_2)$ of $S$. We denote by $I_T$ the index set $I_T = \{i_1, \ldots, i_\ell\}$ of $T$ and by $ST^{-1}$ the main subsequence obtained by deleting the terms of $T$ from $S$. If $T_1 = (a_{j_1}, \ldots, a_{j_t})$ and $T_2 = (a_{h_1}, \ldots, a_{h_t})$ are two subsequences of $S$, we denote by $T_1 \cap T_2$ the main subsequence $X$ of $S$ such that $I_X = I_{T_1} \cap I_{T_2}$. Let $T_2T_1^{-1}$ be the subsequence obtained by deleting the terms of $T_2 \cap T_1$ from $T_1$. Furthermore, if $T_1$ and $T_2$ are disjoint (i.e. $I_{T_1} \cap I_{T_2} = \emptyset$), we denote by $T_1T_2$ the sequence $(a_{j_1}, \ldots, a_{j_t}, a_{h_1}, \ldots, a_{h_t})$. For each $\ell \in \{1, \ldots, k\}$, we denote by $\sum_{\ell}(S)$ the set consisting of all elements which can be expressed as a product of a subsequence $T$ of $S$ with $|T| = \ell$. In particular,

$$\sum_{\ell}(S) = \{a_{i_1} \cdots a_{i_\ell} \mid 1 \leq i_j \leq k \text{ for each } j, \text{ and } i_j \neq i_t \text{ when } j \neq t\}.$$ 

Set $\sum_{\leq \ell}(S) = \bigcup_{j=1}^{\ell} \sum_j(S)$ and set $\sum(S) = \bigcup_{j=1}^{k} \sum_j(S)$. For each $g \in G$, we denote by $v_g(S)$ the number of times that $g$ occurs in $S$.

Let $D(G)$ be Davenport’s constant of $G$ (i.e. the smallest integer $d$ such that every sequence of $d$ elements in $G$ contains a non-empty 1-product subsequence). We denote by $s(G)$ the smallest integer $t$ such that every sequence
of \( t \) elements in \( G \) contains a 1-product subsequence of length \( n \). In 1961, Erdős, Ginzburg and Ziv [2] proved that \( s(G) \leq 2n - 1 \) for every finite solvable group \( G \), and this result is well known as the Erdős–Ginzburg–Ziv theorem. In 1976, Olson [8] showed that \( s(G) \leq 2n - 1 \) holds for every finite group \( G \). He also conjectured the following stronger result.

**Conjecture 1** ([8]). If \( a_1, \ldots, a_{2n-1} \) is a sequence of \( 2n - 1 \) elements in a finite group \( G \) of order \( n \), then
\[
1 = a_1 a_2 \cdots a_{i_n} \text{ for some } 1 \leq i_1 < i_2 < \cdots < i_n \leq 2n - 1.
\]

Olson [8] pointed out that Conjecture 1 is open even for solvable groups.

Let \( G \) be a finite non-cyclic solvable group of order \( n \). In 1984, Yuster and Peterson [10] proved that \( s(G) \leq 2n - 2 \); in 1988, with the restriction that \( n \geq 600((r - 1)!)^2 \), Yuster [11] proved that \( s(G) \leq 2n - r \); and in 1996, the first author [5] proved that \( s(G) \leq \frac{11}{6} n - 1 \). For some recent related work, we refer the reader to [6].

For a finite abelian group \( G \) of order \( n \), the first author [4] showed that \( s(G) = n - 1 + D(G) \). We note that \( s(G) \geq n - 1 + D(G) \) for any group \( G \) of order \( n \) (see [12]). It is plausible to suggest the following.

**Conjecture 2** ([12]). \( s(G) = n - 1 + D(G) \) holds for every finite group \( G \) of order \( n \).

Zhuang and the first author [12] proved that the equality in Conjecture 2 is true for \( G = D_{2p} \) with prime \( p \geq 4001 \). If Conjecture 2 were true, then it, together with Lemma 4, would imply that \( s(G) \leq 3n/2 \) for any non-cyclic group \( G \) of order \( n \). In this paper we shall confirm Conjecture 2 for the dihedral group \( D_{2n} \) with \( n \geq 23 \).

**Theorem 3.** Let \( n \geq 23 \) be an integer and let \( D_{2n} \) be the dihedral group of order \( 2n \). Then
\[
s(D_{2n}) = |D_{2n}| - 1 + D(D_{2n}) = 3n.
\]

To prove Theorem 3, we need some preliminaries. It is well known that, if \( G \) is the cyclic group of order \( n \), then \( D(G) = n \). Recently, Dimitrov [1] obtained an upper bound of \( D(G) \) for a finite non-abelian \( p \)-group \( G \). In 1977, Olson and White [9] obtained the following result for \( D(G) \) when \( G \) is not cyclic.

**Lemma 4** ([9]). If \( G \) is a finite non-cyclic group of order \( n \), then \( D(G) \leq \lceil \frac{n+1}{2} \rceil \), where \( \lceil x \rceil \) denotes the smallest integer not less than \( x \).

**Lemma 5** ([12]). If \( D_{2n} \) is the dihedral group of order \( 2n \), then \( D(D_{2n}) = n + 1 \).

**Lemma 6** ([4]). Let \( G \) be a finite abelian group of order \( n \), and let \( S \) be a sequence of \( n \) elements in \( G \). Let \( k = \max \{ v_g(S) \mid g \in G \} \) be the maximal value of repetition of an element occurring in \( S \). Then \( 1 \in \sum_{s \leq k}(S) \).

The following technical result is crucial in the proof of Theorem 3.

**Lemma 7.** Let \( G \) be a finite abelian group of order \( n \) and let \( r \geq 2 \) be an integer. Let \( S \) be a sequence of \( n + r - 2 \) elements in \( G \). If \( 1 \notin \sum_{r}(S) \), then \( |\sum_{r-2}(S)| = |\sum_{r}(S)| - r - 1 \).

**Proof.** Since \( G \) is abelian and \( (n - 2) + r = |S| \), we have \( |\sum_{r-2}(S)| = |\sum_{r}(S)| \). So, it suffices to prove that
\[
|\sum_{r}(S)| \geq r - 1.
\]

Set \( k = \max \{ v_g(S) \mid g \in G \} \). Let \( g \in G \) with \( v_g(S) = k \). We multiply every term of \( S \) by \( g^{-1} \) and denote the resulting sequence by \( S' \). Since \( G \) is abelian, we have that
\[
g^{-r} \sum_{r}(S) \overset{\text{def}}{=} g^{-r} \left\{ h : h \in \sum_{r}(S) \right\} = \sum_{r}(S')
\]
and
\[
g^{-n} \sum_{n}(S) \overset{\text{def}}{=} g^{-n} \left\{ h : h \in \sum_{n}(S) \right\} = \sum_{n}(S').
\]
Therefore, \(|\sum_r(S)| = |\sum_r(S')|\) and \(\sum_n(S) = \sum_n(S')\). Hence, \(1 \notin \sum_n(S') = \sum_n(S)\). Then, replacing \(S\) by \(S'\), we may assume that \(g = 1\). Furthermore, by rearranging the subscripts (if necessary), we may assume that
\[
S = (g_1, \ldots, g_{n+r-2-k}, 1, \ldots, 1)
\]
with \(g_i \neq 1\) for every \(i \in \{1, \ldots, n + r - 2 - k\}\).

Set
\[
T = (g_1, \ldots, g_{n+r-2-k}).
\]

Since \(1 \notin \sum_n(S)\), we have \(k \leq n - 1\). We distinguish two cases:

Case 1. \(n - 1 \geq k \geq r - 1\). Then
\[
r - 1 \leq n + r - 2 - k = |T| \leq n - 1.
\]

Let \(W\) be the maximal (in length) \(1\)-product main subsequence of \(T\) (if \(T\) contains no \(1\)-product subsequence, then let \(W\) be the empty sequence). If \(|W| \geq n - k\), then \(W(1, \ldots, 1)\) is a \(1\)-product subsequence of \(S\) with length \(n\), a contradiction. Therefore, \(|W| \leq n - k - 1\) and \(|TW^{-1}| \geq r - 1\). By the choice of \(W\) we infer that \(1 \notin \sum(TW^{-1})\).

Let \(X = (x_1, \ldots, x_{r-1})\) be any \((r - 1)\)-term subsequence of \(TW^{-1}\). Then \(x_1, x_1x_2, \ldots, x_1x_2 \cdots x_{r-1}\) are pairwise distinct. Let \(i \in \{1, \ldots, r - 1\}\). Since \(k \geq r - 1\), we infer that \(x_1 \cdots x_i = x_1 \cdots x_i 1^{r-i} \in \sum_r(X(1, \ldots, 1)) \subseteq \sum_r(S)\).

Therefore, \(|\sum_r(S)| \geq r - 1\).

Case 2. \(k \leq r - 2\). Then \(|T| \geq n\). By using Lemma 6 on \(T\) repeatedly, we can find some disjoint \(1\)-product subsequences \(T_1, \ldots, T_u\) of \(T\) such that \(1 \leq |T_i| \leq k\) for every \(i \in \{1, \ldots, u\}\), and \(|T(T_1 \cdots T_u)^{-1}| \leq n - 1\), where \(u \geq 1\). Therefore,
\[
|T_1| + \cdots + |T_u| = |T| - |T(T_1 \cdots T_u)^{-1}| \geq (n + r - 2 - k) - (n - 1) = r - 1 - k.
\]

This gives that
\[
|T_1| + \cdots + |T_u| + k \geq r - 1. \tag{1}
\]

Let \(W_0\) be the maximal \(1\)-product main subsequence of \(T(T_1 \cdots T_u)^{-1}\) (if \(T(T_1 \cdots T_u)^{-1}\) contains no \(1\)-product subsequence, then let \(W_0\) be the empty sequence). If \(|W_0T_1 \cdots T_u| \geq n - k\), then either \(|W_0T_1 \cdots T_u| \leq n - 1\) and \(W_0T_1 \cdots T_u(1, \ldots, 1)\) is a \(1\)-product subsequence of \(S\) with length \(n\), which is a contradiction, or
\[
|W_0T_1 \cdots T_u| \geq n.
\]

For the latter case, since \(|W_0| \leq |T(T_1 \cdots T_u)^{-1}| \leq n - 1\), there exists \(v \in \{0, 1, \ldots, u - 1\}\) such that
\[
|W_0T_1 \cdots T_v| \leq n - 1 \quad \text{and} \quad |W_0T_1 \cdots T_vT_{v+1}| \geq n \quad \text{(if} \; v = 0, \text{then let} \; W_0T_1 \cdots T_v = W_0)\). It follows from \(|T_{v+1}| \leq k\) that
\[
n - k \leq n - |T_{v+1}| \leq |W_0T_1 \cdots T_v| \leq n - 1.
\]

Therefore, \(W_0T_1 \cdots T_v(1, \ldots, 1)\) is a \(1\)-product subsequence of \(S\) of length \(n\), also a contradiction. So, we may assume that
\[
|W_0T_1 \cdots T_u| \leq n - k - 1.
\]

It follows that
\[
|T(T_1 \cdots T_u)^{-1}W_0^{-1}| = |T| - |W_0T_1 \cdots T_u| \geq r - 1.
\]

By the choice of \(W_0\) we infer that \(1 \notin \sum(T(T_1 \cdots T_u)^{-1}W_0^{-1})\). Let \(X = (x_1, \ldots, x_{r-1})\) be any \((r - 1)\)-term subsequence of \(T(T_1 \cdots T_u)^{-1}W_0^{-1}\). Then \(x_1, x_1x_2, \ldots, x_1x_2 \cdots x_{r-1}\) are pairwise distinct. We show next that
\[
\{x_1, x_1x_2, \ldots, x_1x_2 \cdots x_{r-1}\} \subseteq \sum_r(X(1, \ldots, 1)T_1 \cdots T_u) \subseteq \sum_r(S). \tag{2}
\]
Let \( i \in \{1, \ldots, r - 1\} \). If \( i \geq r - k \), then
\[
x_1x_2 \cdots x_i \in \prod r(X(1, \ldots, 1)) \subseteq \sum r(X(1, \ldots, 1)T_1 \cdots T_u) \subseteq \sum (S).
\]
Now assume \( i \leq r - k - 1 \). By (1) we have \( i \geq 1 \geq r - k - |T_1| - \cdots - |T_u| \), and there is an integer \( m \in \{1, \ldots, u\} \) such that
\[
i \geq r - k - 1 - |T_1| - \cdots - |T_m|
\]
and
\[
i \leq r - k - 1 - |T_1| - \cdots - |T_{m-1}|- (\text{if } m = 1, \text{then let } |T_1| + \cdots + |T_{m-1}| = 0).
\]
Therefore,
\[
r - k \leq i + |T_1| + \cdots + |T_{m-1}| + |T_m| \leq r - 1 - k + |T_m| \leq r - 1.
\]
Thus,
\[
x_1x_2 \cdots x_i = x_1x_2 \cdots x_i \prod (T_1 \cdots T_m) \in \sum r(X(1, \ldots, 1)T_1 \cdots T_m)
\]
\[
\subseteq \sum r(X(1, \ldots, 1)T_1 \cdots T_u) \subseteq \sum (S).
\]
This proves (2) and the lemma follows. \( \square \)

For every positive integer \( n \), we denote by \( \mathbb{Z}_n \) the cyclic group of \( n \) elements. Recall that \( \mathbb{Z}_n \) (as a group) is written multiplicatively in this paper.

**Lemma 8.** Let \( n \geq 8 \), and let \( S \) be a sequence of elements in \( \mathbb{Z}_n \) with \( |S| \geq 2[\log_2 n] \), where \( [x] \) denotes the largest integer not exceeding \( x \). Then (1) there are two disjoint non-empty subsequences \( S_1 \) and \( S_2 \) of \( S \) such that \( \prod (S_1) = \prod (S_2) \) and \( |S_1| = |S_2| \leq [\log_2 n] \); and (2) there are two disjoint subsequences \( C \) and \( D \) of \( S \) such that \( \prod (C) = \prod (D) \) and \( |C| = |D| \geq \frac{|S| - 2[\log_2 n] + 1}{2} \) (for the definition of \( \prod (S) \), see the first paragraph of this paper).

**Proof.** (1) Let \( k = 2[\log_2 n] \), and let \( T \) be a main subsequence of \( S \) with \( |T| = k \). We denote by \( T^{|k/2|} \) the family that consists of all main subsequences of \( T \) of length \( \lfloor k/2 \rfloor \). Then
\[
\prod (T_1) = \prod (T_2) \quad \text{and} \quad |T_1| = |T_2| = \lfloor k/2 \rfloor.
\]
Therefore, there are two distinct main subsequences \( T_1 \) and \( T_2 \) of \( T \) such that
\[
\prod (T_1) = \prod (T_2) \quad \text{and} \quad |T_1| = |T_2| = \lfloor k/2 \rfloor.
\]

Setting \( S_1 = T_1T_2^{-1} = T_1(T_1 \cap T_2)^{-1} \) and \( S_2 = T_2T_1^{-1} = T_2(T_1 \cap T_2)^{-1} \), we get the desired result.

(2) By using (1) repeatedly, we can find some disjoint non-empty subsequences \( U_1, V_1, U_2, V_2, \ldots, U_m, V_m \) of \( S \) such that \( \prod (U_i) = \prod (V_i) \) and \( |U_i| = |V_i| \leq [\log_2 n] \) for every \( i \in \{1, 2, \ldots, m\} \), and such that \( |S(U_1U_2V_2 \cdots U_mV_m)^{-1}| \leq 2[\log_2 n] - 1 \). Now set \( C = U_1U_2 \cdots U_m \) and \( D = V_1V_2 \cdots V_m \). Then \( \prod (C) = \prod (D) \) and \( |C| = |D| \geq \frac{|S| - 2[\log_2 n] + 1}{2} \). \( \square \)

It is well known that the dihedral group \( D_{2n} \) has a unique cyclic subgroup \( H \) of order \( n \). Setting \( N = G \setminus H \) yields that \( N^2 \subseteq H \) and each \( x \in N \) has order 2 in \( D_{2n} \). Let \( S \) be a sequence of elements in \( D_{2n} \). We denote by \( S \cap H \) (respectively \( S \cap N \)) the main subsequence of \( S \) that consists of the terms in \( H \) (respectively \( N \)). The following lemma will be used repeatedly in the proof of Theorem 3.
Lemma 9. Let $S$ be a sequence of $3n$ elements in $D_{2n}$. If one of the following conditions holds, then $S$ contains a 1-product subsequence of length $2n$.

(I) There are two disjoint 1-product subsequences $T_1$ and $T_2$ of $S$ such that $|T_1| = |T_2| = n$.

(II) There are two disjoint subsequences $U$ and $V$ of $S$ satisfying (I) $|U| = |V| \leq n$ and $\prod(U) = \prod(V) \in N$; and (2) $q + w + |U| \geq n$, where $q$ is the maximal non-negative integer such that $(S \cap N)(UV)^{-1}$ has a subsequence of the type $(a_1, a_1) \cdots (a_q, a_q)$, and $w \geq 0$ is the maximal non-negative integer such that $(S \cap H)(UV)^{-1}$ has a subsequence of the type $(b_1, b_1) \cdots (b_w, b_w)$.

Proof. If (I) holds, then $T_1T_2$ is a 1-product subsequence of $S$ of length $2n$.

Suppose that (II) holds. If $|U| = |V| = n$, then $|UV| = 2n$ and $\prod(UV) = \prod(U) \prod(V) = 1$.

Assume that $|U| = |V| < n$ and $q + |U| = q + |V| \geq n$. Setting $k = n - |U| = n - |V|$, then $1 \leq k \leq q$ and $UV(a_1, a_1) \cdots (a_k, a_k)$ is a 1-product sequence of length $2n$.

Now, $q + |U| = q + |V| < n$. Setting $\ell = n - |U| - q = n - |V| - q$, then $1 \leq \ell \leq w$ and $U(b_1, \ldots, b_\ell) V(b_1, \ldots, b_\ell)(a_1, a_1) \cdots (a_q, a_q)$ is a 1-product sequence of length $2n$. So the proof is completed.

The following lemmas will also be used in the proof of Theorem 3.

Lemma 10 ([7], Lemma 2.2). Let $A, B$ be two subsets of a finite group $G$. If $|A| + |B| > |G|$, then $A + B = G$, where $A + B = \{ab | a \in A, b \in B\}$.

Lemma 11 ([3]). Let $n, u$ be integers with $2 \leq u \leq \frac{n}{2} + 2$. Let $S$ be a sequence of $2n - u$ elements in $\mathbb{Z}_n$. If $1 \notin \sum_n(S)$, then there are elements $a, b \in \mathbb{Z}_n$ such that $a(S) \geq v_b(S) \geq n - 2u + 3$ and $ab^{-1}$ generates $\mathbb{Z}_n$.

Proof of Theorem 3. Since $s(D_{2n}) \geq |D_{2n}| + D(D_{2n}) - 1 = 3n$ (the last equality follows from Lemma 4), it suffices to prove that $s(D_{2n}) \leq 3n$. Let $n \geq 3$, and let $S$ be a sequence of $3n$ elements in $D_{2n}$. We have to prove that $S$ contains a 1-product subsequence of length $2n$. Let $H, N, S \cap N$ and $S \cap H$ be defined as prior to Lemma 9.

It is well known that $D_{2n}$ is generated by two elements $x$ and $y$ with ord$(x) = 2$, ord$(y) = n$ and $yx = xy^{-1}$. Then $H = \{1 = y^n, y, y^2, \ldots, y^{n-1}\}$, $N = \{x, xy, \ldots, xy^{n-1}\}$, and $(xy^i)(xy^j) = y^{j-i}$ holds for any two indices $i, j \in \{0, 1, \ldots, n - 1\}$.

By rearranging the subscripts, we may assume that

$$S \cap N = (a_1, a_1)(a_2, a_2) \cdots (a_r, a_r)(c_1, c_2, \ldots, c_u),$$

where $c_1, c_2, \ldots, c_u$ are pairwise distinct, $0 \leq r \leq \frac{|S \cap N|}{2}$ and $2r + u = |S \cap N|$.

Further, assume that

$$S \cap H = (b_1, b_1)(b_2, b_2) \cdots (b_t, b_t)(d_1, d_2, \ldots, d_v),$$

where $d_1, d_2, \ldots, d_v$ are pairwise distinct and $0 \leq t \leq \frac{|S \cap H|}{2}$ with $2t + v = |S \cap H|$.

We shall prove the theorem by showing that at least one of (I) and (II) in Lemma 9 holds for $S$. Set

$$c_i = x y^{m_i}, \quad i = 1, \ldots, u,$$

where $1 \leq m_i \leq n$. Now we distinguish two cases in terms of whether or not $r = 0$.

Case 1. $r = 0$. Then $|S \cap N| = 2r + u = u \leq n$ and $|S \cap H| \geq 2n$. By the Erdős–Ginzburg–Ziv theorem, $S \cap H$ contains a 1-product subsequence $T_1$ with $|T_1| = n$. If $u \leq 1$, then

$$|(S \cap H)T_1^{-1}| = |S \cap H| - n = 3n - |S \cap N| - n = 2n - u \geq 2n - 1,$$

and again by using the Erdős–Ginzburg–Ziv theorem we can find a 1-product subsequence $T_2$ of $(S \cap H)T_1^{-1}$ with $|T_2| = n$ and the theorem follows from Lemma 9(I). Therefore, we may assume that $u \geq 2$. Noting that $c_1c_2, \ldots, c_1c_u$ are pairwise distinct, we have $|\sum 2(c_1, c_2, \ldots, c_u)| \geq u - 1$. We distinguish two subcases.

Subcase 1.1. $|\sum 2(c_1, c_2, \ldots, c_u)| \geq u$. Note that

$$|(S \cap H)T_1^{-1}| = |S \cap H| - n = 3n - |S \cap N| - n = 2n - u.$$
If $1 \in \sum_n((S \cap H)T_1^{-1})$, then the theorem follows from Lemma 9(I) and we are done. Assume that $1 \not\in \sum_n((S \cap H)T_1^{-1})$. It follows from Lemma 7 that $|\sum_{n-2}((S \cap H)T_1^{-1})| \geq n - u + 1$, and therefore,

$$\left|\sum_{n-2}((S \cap H)T_1^{-1}) + \sum_2(c_1, c_2, \ldots, c_u)\right| \geq (n - u + 1) + u > n.$$ 

It follows from Lemma 10 that

$$1 \in H = \sum_{n-2}((S \cap H)T_1^{-1}) + \sum_2(c_1, c_2, \ldots, c_u) \subseteq \sum_n(ST_1^{-1}),$$

and thus the theorem follows again from Lemma 9(I).

Subcase 1.2. $|\sum_2(c_1, c_2, \ldots, c_u)| = u - 1$. Then

$$\{c_1c_2, c_1c_3, \ldots, c_1c_u\} = \sum_2(c_1, c_2, \ldots, c_u) = \{c_2c_1, c_2c_3, \ldots, c_2c_u\}.$$ 

By a straightforward calculation, we have

$$\{c_1c_2, c_1c_3, \ldots, c_1c_u\} = \{y^{m_2-m_1}, y^{m_3-m_1}, \ldots, y^{m_u-m_1}\},$$

and

$$\{c_2c_1, c_2c_3, \ldots, c_2c_u\} = \{y^{m_1-m_2}, y^{m_3-m_2}, \ldots, y^{m_u-m_2}\}.$$ 

Hence,

$$\{y^{m_2-m_1}, y^{m_3-m_1}, \ldots, y^{m_u-m_1}\} = \{y^{m_1-m_2}, y^{m_3-m_2}, \ldots, y^{m_u-m_2}\}.$$ 

In particular,

$$y^{m_2-m_1}y^{m_3-m_1} \ldots y^{m_u-m_1} = y^{m_1-m_2}y^{m_3-m_2} \ldots y^{m_u-m_2}.$$ 

Hence,

$$m_2 - m_1 + m_3 - m_1 + \cdots + m_u - m_1 \equiv m_1 - m_2 + m_3 - m_2 + \cdots + m_u - m_2 \pmod{n}.$$ 

This gives that $u(m_1 - m_2) \equiv 0 \pmod{n}$. Similarly, $u(m_j - m_k) \equiv 0 \pmod{n}$ holds for every pair of $j, k$ with $1 \leq j \neq k \leq u$. Therefore, $y^{m_j - m_k}$ are all in the subgroup $M$ of $H$ with $|M| = \gcd(u, n)$, the greatest common divisor of $u$ and $n$. Therefore,

$$u \geq \gcd(u, n) = |M| \geq |\{1, y^{m_2-m_1}, y^{m_3-m_1}, \ldots, y^{m_u-m_1}\}| = u.$$ 

Hence, $|M| = u, u|n$ and

$$\sum_2(c_1, c_2, \ldots, c_u) = \{y^{m_2-m_1}, y^{m_3-m_1}, \ldots, y^{m_u-m_1}\} = M \setminus \{1\} = \{y^{\frac{u}{n}}, y^{\frac{2u}{n}}, \ldots, y^{(u-1)\frac{u}{n}}\}.$$ 

Thus,

$$\{c_1, c_2, \ldots, c_u\} = \{xy^{m_1}, xy^{m_1+\frac{u}{n}}, xy^{m_1+\frac{2u}{n}}, \ldots, xy^{m_1+(u-1)\frac{u}{n}}\}.$$ 

Suppose first that $u \geq 7$. Then $u - 1 \geq 6$. We may assume that

$$c_2 = xy^{m_1+\frac{2u}{n}}, \quad c_3 = xy^{m_1+\frac{4u}{n}}, \quad c_4 = xy^{m_1+\frac{6u}{n}}, \quad c_5 = xy^{m_1+\frac{8u}{n}}, \quad c_6 = xy^{m_1+\frac{10u}{n}}.$$ 

Then $c_1c_2c_3 = xy^{m_1+\frac{12u}{n}} = c_4c_5c_6$. Set $\ell = \lfloor \frac{u-7}{4} \rfloor$. Let

$$A = (c_1, c_2, c_3, xy^{m_1+\frac{7u}{n}}, \ldots, xy^{m_1+(6+2\ell)\frac{u}{n}})$$

and

$$B = (c_4, c_5, c_6, xy^{m_1+(7+2\ell)\frac{u}{n}}, \ldots, xy^{m_1+(6+4\ell)\frac{u}{n}}).$$
Clearly, \( A \) and \( B \) are two disjoint subsequences of \( S \cap N \) such that \( |A| = |B| = 3 + 2\ell \) and \( \prod(A) = \prod(B) = xy^{m_1 + (2 + \ell)\frac{2}{n}} \in N \).

Recall that \( S \cap H = (b_1, b_1) \cdots (b_1, b_1)(d_1, \ldots, d_v) \), where \( d_1, \ldots, d_v \) are pairwise distinct. Now by Lemma 8 there exist two disjoint subsequences \( C \) and \( D \) of \( (d_1, \ldots, d_v) \) such that \( \prod(C) = \prod(D) \) and \( |C| = |D| \geq \frac{v - 2[\log_2 n] + 1}{2} \) (if \( v \leq 2[\log_2 n] - 1 \), then let \( C \) and \( D \) be the empty sequence and \( \prod(C) = \prod(D) = 1 \)). Therefore, \( \prod(AC) = \prod(BD) = xh \in N \) and \( |AC| = |BD| \), where \( h = y^{m_1 + (2 + \ell)\frac{2}{n}} \). \( \prod(C) \in H \). Note that

\[
\begin{align*}
n \geq & \frac{u}{2} + v > |AC| = |BD| \geq \frac{v - 2[\log_2 n] + 1}{2} + 3 + 2\ell \\
\geq & \frac{v - 2[\log_2 n] + 1}{2} + 3 + 2\left(\frac{u - 7}{4} - \frac{3}{4}\right) \geq \frac{u + v - 2[\log_2 n] - 3}{2}.
\end{align*}
\]

Therefore,

\[
\begin{align*}
t + |AC| = t + |BD| \geq t + \frac{u + v - 2[\log_2 n] - 3}{2} = & \frac{2t + u + v - 2[\log_2 n] - 3}{2} \\
= & \frac{3n - 2[\log_2 n] - 3}{2} \geq n \quad \text{(since } n \geq 23).\text{(since } n \geq 23)\text{.}
\end{align*}
\]

Now the theorem follows from Lemma 9(II) with \( U = AC \) and \( V = BD \). This completes the proof of this subcase with \( u \geq 7 \).

Next suppose that \( 2 \leq u \leq 6 \). If \( 1 \in \sum_n((S \cap H)T^{-1})_1 \), then the theorem follows from Lemma 9(I). Assume that \( 1 \notin \sum_n((S \cap H)T^{-1})_1 \). Since \( (S \cap H)T^{-1}_1 = 2n - u \) and \( 2 \leq u \leq 6 \leq \frac{n}{4} + 2 \), then, by Lemma 11, \( (S \cap H)T^{-1}_1 \) has a subsequence \( (a_1, \ldots, a_r, b_1, \ldots, b_k) \) where \( ab^{-1} \) generates \( H \). It follows from \( n - 2u + 3 \geq \frac{n}{u} = D(H/M) \) that the sequence \( \left(ab^{-1}, \ldots, ab^{-1}\right) \) contains a non-empty subsequence \( T \) such that \( \prod(T) \in M \) and \( |T| \leq \frac{n}{u} \). If \( k = |T| \), then \( a^k b^{-k} = (ab^{-1})^k = \prod(T) \in M \). Note that \( ab^{-1} \) generates \( H \) and \( k < n \); we must have \( a^k b^{-k} \neq 1 \). So, \( a^k b^{-k} \in M \setminus \{1\} = \sum_k(c_1, c_2, \ldots, c_u) \). Without loss of generality, we may assume that \( a^k b^{-k} = c_1 c_2 \). Therefore,

\[
xy^{m_1}a^k = c_1 a^k = c_2 b^k = xy^{m_2}b^k.
\]

Again, without loss of generality, we may assume that

\[
(S \cap H)(c_1, \ldots, a, b, \ldots, b) = (b_1, b_1) \cdots (b_{t-k}, b_{t-k})(d_1, \ldots, d_v).
\]

By Lemma 8 there exist two disjoint subsequences \( C \) and \( D \) of \( (d_1, \ldots, d_{v-1}) \) such that

\[
\prod(C) = \prod(D) \quad \text{and} \quad \left[\frac{v - 1}{2}\right] \geq |C| = |D| \geq \frac{v - 1 - 2[\log_2 n] + 1}{2}.
\]

Therefore,

\[
\prod((c_1, a, \ldots, a)C) = \prod((c_2, b, \ldots, b)D) = xh,
\]

where \( h = y^{m_1}a^k \prod(C) = y^{m_2}b^k \prod(D) \in H \) and \( |(c_1, a, \ldots, a)C| = |(c_2, b, \ldots, b)D| \).

Note that

\[
n \geq 1 + \frac{n}{u} + \left[\frac{v - 1}{2}\right] \geq |(c_1, a, \ldots, a)C| = |(c_2, b, \ldots, b)D| \\
\geq 1 + k + \frac{v - 1 - 2[\log_2 n] + 1}{2} = k + \frac{v - 2[\log_2 n] + 2}{2}.
\]
Therefore,
\[ t - k + \sum_{i=1}^{k} (c_1, a, \ldots, a)C = t - k + \sum_{i=1}^{k} (c_2, b, \ldots, b)D \geq t + \frac{v - 2[\log_2 n] + 2}{2} \]
\[ = 2t + v + u - 2[\log_2 n] + 2 - u = \frac{3n - 2[\log_2 n] + 2 - u}{2} \]
\[ \geq \frac{3n - 2[\log_2 n] - 4}{2} \geq n \quad \text{(since } n \geq 23) \]

Now the theorem follows again from Lemma 9(II) with \( U = (c_1, a, \ldots, a)C \) and \( V = (c_2, b, \ldots, b)D \).

**Case 2.** \( r \geq 1 \). By Lemma 8 there exist two disjoint subsequences \( C \) and \( D \) of \((d_1, \ldots, d_n)\) such that \( \prod(C) = \prod(D) \) and \( |C| = |D| \geq \frac{v - 2[\log_2 n] + 1}{2} \). Let \( \ell = \lfloor \frac{u - 1}{2} \rfloor \). Since
\[ \{c_1c_2, c_3c_4, \ldots, c_{2\ell-1}c_{2\ell}\} \subseteq H, \]
again by Lemma 8 there exist two disjoint subsequences \( A' \) and \( B' \) of \((c_1c_2, c_3c_4, \ldots, c_{2\ell-1}c_{2\ell})\) such that \( \prod(A') = \prod(B') \in H \) and \( |A'| = |B'| \geq \frac{\ell - 2[\log_2 n] + 1}{2} \). Therefore, there exist two disjoint subsequences \( A \) and \( B \) of \((c_1, c_2, \ldots, c_{2\ell})\) such that
\[ \prod(A) = \prod(A') = \prod(B') = \prod(B) \in H \]
and
\[ |A| = |B| = 2|A'| = 2|B'| \geq \ell - 2[\log_2 n] + 1. \]

Now we have
\[ \prod((a_1)AC) = \prod((a_1)BD) = xh \quad \text{for some } h \in H \]
and
\[ |(a_1)AC| = |(a_1)BD| = |D| + |B| + 1 \geq \frac{v - 2[\log_2 n] + 1}{2} + \ell - 2[\log_2 n] + 1 + 1. \]

Thus,
\[ n \geq 1 + \left\lfloor \frac{v}{2} \right\rfloor + \left\lfloor \frac{u - 1}{2} \right\rfloor \geq |(a_1)AC| = |(a_1)BD| \geq \frac{v - 2[\log_2 n] + 1}{2} + \ell - 2[\log_2 n] + 2 \]
\[ \geq \frac{v - 2[\log_2 n] + 1}{2} + \frac{u - 1}{2} - \frac{1}{2} - 2[\log_2 n] + 2 = \frac{u + v - 6[\log_2 n] + 3}{2}. \]

Therefore,
\[ r - 1 + t + |(a_1)AC| = r - 1 + t + |(a_1)BD| \geq r - 1 + t + \frac{u + v - 6[\log_2 n] + 3}{2} \]
\[ = \frac{2r + 2t + u + v - 6[\log_2 n] + 1}{2} = \frac{3n - 6[\log_2 n] + 1}{2} \]
\[ \geq n \quad \text{(since } n \geq 23). \]

The theorem now follows from Lemma 9(II) with \( U = (a_1)AC \) and \( V = (a_1)BD \).  \( \square \)

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