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Special transformations in algebraically closed valued fields

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ABSTRACT

We present two of the three major steps in the construction of motivic integration, that is, a homomorphism between Grothendieck semigroups that are associated with a first-order theory of algebraically closed valued fields, in the fundamental work of Hrushovski and Kazhdan (2006) [8]. We limit our attention to a simple major subclass of V-minimal theories of the form $\mathrm{ACVF}_S(0,0)$, that is, the theory of algebraically closed valued fields of pure characteristic 0 expanded by a (VF, Γ)-generated substructure S in the language $\mathcal{L}_{\mathrm{RV}}$. The main advantage of this subclass is the presence of syntax. It enables us to simplify the arguments with many different technical details while following the major steps of the Hrushovski–Kazhdan theory.

1. Introduction

The theory of motivic integration in valued fields has been progressing rapidly since its first introduction by Kontsevich. Early developments by Denef and Loeser et al. have yielded many important results in many directions. The reader is referred to [7] for an excellent introduction to the construction of motivic measure.

There have been different approaches to motivic integration. The comprehensive study in Cluckers–Loeser [4] has successfully united some major ones on a general foundation. Their construction may be applied in general to the field of formal Laurent series over a field of characteristic 0 but heavily relies on the Cell Decomposition Theorem of Denef–Pas [6,12]. We note that cell decomposition is also achieved in other cases, for example, in certain finite extensions of p-adic fields [13] and in henselian fields with respect to a first-order language that is equipped with, instead of an angular component, a collection of residue multiplicative structures [3]. On the other hand, the Hrushovski–Kazhdan integration theory [8] is a major development that does not require the presence of an angular component map and hence is of great foundational importance. Its basic objects of study are models of the so-called V-minimal theories, for example, the theory of algebraically closed valued fields of pure characteristic 0 and the theories of its rigid analytic expansions [10,11]. The method of the Hrushovski–Kazhdan integration theory is based on a fine analysis of definable subsets up to definable bijections in a first-order language $\mathcal{L}_{\mathrm{RV}}$ for valued fields. Of course the method of the Cluckers–Loeser approach [4] is similar, but the "up to definable bijections" point of view is not so much stressed. In fact both approaches are rooted in the Cohen–Denef analysis of definable sets that leads to cell decomposition [5,6].

The language $\mathcal{L}_{\mathrm{RV}}$ has two sorts: the VF-sort and the RV-sort. One of the main features of $\mathcal{L}_{\mathrm{RV}}$ is that the residue field and the value group are wrapped together in one sort RV. Let (K, val) be a valued field and $\mathcal{O}$, $\mathcal{M}$, $\overline{\mathrm{K}}$ the corresponding valuation ring, its maximal ideal, and the residue field. Let $\mathrm{RV}(K) = K^\times/(1+\mathcal{M})$ and $\mathrm{rv} : K^\times \longrightarrow \mathrm{RV}(K)$ the quotient map. Note that, for each $a \in K$, val is constant on the subset $a + a\mathcal{M}$ and hence there is a naturally induced map vrv from $\mathrm{RV}(K)$ onto

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the value group Γ. The situation is illustrated in the following commutative diagram

$$
\begin{tikzcd}
\mathcal{O} \smallsetminus \mathcal{M} \arrow[r, hook] \arrow[d, two heads, "\text{quotient}"'] & K^{\times} \arrow[d, two heads, "\text{rv}"] \arrow[dr, two heads, "\text{val}"] & \\
\overline{\mathrm{K}}^{\times} \arrow[r, hook] & \mathrm{RV}(K) \arrow[r, two heads, "\text{vrv}"] & \Gamma
\end{tikzcd}
$$

where the bottom sequence is exact.

Let $\mathrm{VF}_*[\cdot]$ and $\mathrm{RV}[*, \cdot]$ be two categories of definable sets that are respectively associated with the VF-sort and the RV-sort. In $\mathrm{VF}_*[\cdot]$, the objects are definable subsets of products of the form $\mathrm{VF}^n \times \mathrm{RV}^m$ and the morphisms are definable functions. On the other hand, for technical reasons (particularly for keeping track of dimensions), $\mathrm{RV}[*, \cdot]$ is formulated in a somewhat complicated way (see Section 4). The main construction of the Hrushovski–Kazhdan theory is a canonical homomorphism from the Grothendieck semigroup $\mathbf{K}_+ \mathrm{VF}_*[\cdot]$ to the Grothendieck semigroup $\mathbf{K}_+ \mathrm{RV}[*, \cdot]$ modulo a semigroup congruence relation $\mathrm{I_{sp}}$ on the latter. In fact, it turns out to be an isomorphism. This construction has three main steps.

- *Step* 1. First we define a lifting map $\mathbb{L}$ from the set of the objects in $\mathrm{RV}[*, \cdot]$ into the set of the objects in $\mathrm{VF}_*[\cdot]$; see Definition 4.18. Next we single out a subclass of the isomorphisms in $\mathrm{VF}_*[\cdot]$, which are called special bijections; see Definition 5.1. Then we show that for any object A in $\mathrm{VF}_*[\cdot]$ there is a special bijection T on A and an object $\mathbf{U}$ in $\mathrm{RV}[*, \cdot]$ such that $T(A)$ is isomorphic to $\mathbb{L}(\mathbf{U})$. This implies that $\mathbb{L}$ hits every isomorphism class of $\mathrm{VF}_*[\cdot]$. Of course, for this result alone we do not have to limit our means to special bijections. However, in Step 3 below, special bijections become an essential ingredient in computing the congruence relation $\mathrm{I_{sp}}$.
- *Step* 2. For any two isomorphic objects $\mathbf{U}_1, \mathbf{U}_2$ in $\mathrm{RV}[*, \cdot]$, their lifts $\mathbb{L}(\mathbf{U}_1), \mathbb{L}(\mathbf{U}_2)$ in $\mathrm{VF}_*[\cdot]$ are isomorphic as well. This shows that $\mathbb{L}$ induces a semigroup homomorphism from $\mathbf{K}_+ \mathrm{RV}[*, \cdot]$ into $\mathbf{K}_+ \mathrm{VF}_*[\cdot]$, which is also denoted by $\mathbb{L}$.
- *Step* 3. A number of classical properties of integration can already be (perhaps only partially) verified for the inversion of the homomorphism $\mathbb{L}$ and hence, morally, this third step is not necessary. To facilitate computation in future applications, however, it seems much more satisfying to have a precise description of the semigroup congruence relation induced by it. The basic notion used in the description is that of a blowup of an object in $\mathrm{RV}[*, \cdot]$. We then show that, for any objects $\mathbf{U}_1, \mathbf{U}_2$ in $\mathrm{RV}[*, \cdot]$, there are isomorphic iterated blowups $\mathbf{U}_1^\sharp, \mathbf{U}_2^\sharp$ of $\mathbf{U}_1, \mathbf{U}_2$ if and only if $\mathbb{L}(\mathbf{U}_1), \mathbb{L}(\mathbf{U}_2)$ are isomorphic. The "if" direction essentially contains a form of Fubini's Theorem and is the most technically involved part of the construction.

The inverse of $\mathbb{L}$ thus obtained is a Grothencieck homomorphism. If the Jacobian transformation preserves integrals, that is, the change of variables formula holds, then it may be called a motivic integration. When the Grothendieck semigroups are formally groupified this integration is recast as a ring homomorphism.

In this paper we give a presentation of the first two steps. The sections are organized as follows. Throughout we shall follow the terminology and notation of [16]. For the reader's convenience some key definitions and notational conventions are recalled in Section 2, where new ones are introduced as well. To delineate the basic geography of definable subsets, many structural properties concerning the three sorts VF, RV, and Γ are needed. These are discussed in Sections 3 and 8. In Section 4 we first discuss various notions of dimension, mainly VF-dimension and RV-dimension, and then describe the relevant categories of definable subsets and the formulation of their Grothendieck semigroups. The fundamental lifting map $\mathbb{L}$ between VF-categories and RV-categories is also introduced here. The central topic of Section 5 is RV-pullbacks and special bijections on them. Corollary 5.6 corresponds to Step 1 above. In Section 6 we describe the "descent" technique and use it to obtain a general quantifier elimination result for henselian fields.

Section 7 is devoted to showing Step 2 above. The notion of a $\vec{\gamma}$-polynomial is introduced here, which generalizes the relation between a polynomial with coefficients in the valuation ring and its projection into the residue field. This leads to Lemma 7.2, a generalized form of the multivariate version of Hensel's lemma. Note that in order to apply Lemma 7.2 to a given definable subset we need to find suitable polynomials with a simple common residue root. This is investigated in Lemma 7.4, which does not hold when the substructure in question contains an excessive amount of parameters in the RV-sort. This is the reason why motivic integration is constructed only when parameters are taken from a (VF, Γ)-generated substructure.

For finer categories of definable subsets that can handle the Jacobian transformation, a notion of the Jacobian is needed. This is provided in Section 9. Then in Section 10 we define these finer categories and explain how to carry out Step 1 and Step 2 for them.

While we do follow the broad outline of [8], there are significant technical differences. To begin with, our construction is specialized for $\mathrm{ACVF}_S(0, 0)$, that is the theory of algebraically closed valued fields of pure characteristic 0, formulated in the language $\mathcal{L}_{\mathrm{RV}}$ and expanded by a substructure S, where S is generated by elements in the field sort and the (imaginary) value group sort. For this simple major subclass of V-minimal theories we are able to work with syntax. Very often, in order to grasp the geometrical content of a definable subset A, it is a very fruitful exercise to analyze the logical structure of a typical formula that defines A, especially when quantifier elimination is available. Consequently, in the context of this paper, syntactical analysis affords tremendous simplifications of many lemmas in [8]. It also gives rise to technical tools that are especially powerful for $\mathrm{ACVF}_S(0, 0)$, the most important of which is Theorem 5.5.

Step 3 of the construction of motivic integration will be presented in a sequel.

2. Preliminaries

Throughout this paper we shall use the terminology and notation introduced in [16]. For the reader's convenience, we recall a few key definitions here.

Definition 2.1. The language $\mathcal{L}_{\mathrm{RV}}$ has the following sorts and symbols:

(1) a VF-sort, which uses the language of rings $\mathcal{L}_{\mathrm{R}} = \{0, 1, +, -, \times\}$;
(2) an RV-sort, which uses
 (a) the group language $\{1, \times\}$,
 (b) two constant symbols 0 and ∞,
 (c) a unary predicate $\overline{\mathrm{K}}^{\times}$,
 (d) a binary function $+ : \overline{\mathrm{K}}^2 \longrightarrow \overline{\mathrm{K}}$ and a unary function $- : \overline{\mathrm{K}} \longrightarrow \overline{\mathrm{K}}$, where $\overline{\mathrm{K}} = \overline{\mathrm{K}}^{\times} \cup \{0\}$,
 (e) a binary relation $\leq$;
(3) a function symbol rv from the VF-sort into the RV-sort.

The two sorts without the zero elements are respectively denoted by $\mathrm{VF}^{\times}$ and RV; $\mathrm{RV} \smallsetminus \{\infty\}$ is denoted by $\mathrm{RV}^{\times}$; and $\mathrm{RV} \cup \{0\}$ is denoted by RV_0.

Definition 2.2. *The theory* ACVF *of algebraically closed valued fields in* $\mathcal{L}_{\mathrm{RV}}$ states the following:

(1) $(\mathrm{VF}, 0, 1, +, -, \times)$ is an algebraically close field;
(2) $(\mathrm{RV}^{\times}, 1, \times)$ is a divisible abelian group, where multiplication $\times$ is augmented by $t \times 0 = 0$ for all $t \in \overline{\mathrm{K}}$ and $t \times \infty = \infty$ for all $t \in \mathrm{RV}_0$;
(3) $(\overline{\mathrm{K}}, 0, 1, +, -, \times)$ is an algebraically closed field;
(4) the relation $\leq$ is a preordering on RV with ∞ the top element and $\overline{\mathrm{K}}^{\times}$ the equivalence class of 1;
(5) the quotient $\mathrm{RV} / \overline{\mathrm{K}}^{\times}$, denoted as $\Gamma \cup \{\infty\}$, is a divisible ordered abelian group with a top element, where the ordering and the group operation are induced by $\leq$ and $\times$, respectively, and the quotient map $\mathrm{RV} \longrightarrow \Gamma \cup \{\infty\}$ is denoted as vrv;
(6) the function $\mathrm{rv} : \mathrm{VF}^{\times} \longrightarrow \mathrm{RV}^{\times}$ is a surjective group homomorphism augmented by $\mathrm{rv}(0) = \infty$ such that the composite function

$$\mathrm{val} = \mathrm{vrv} \circ \mathrm{rv} : \mathrm{VF} \longrightarrow \Gamma \cup \{\infty\}$$

is a valuation with the valuation ring $\mathcal{O} = \mathrm{rv}^{-1}(\mathrm{RV}^{\geq 1})$ and its maximal ideal $\mathcal{M} = \mathrm{rv}^{-1}(\mathrm{RV}^{>1})$, where

$$\mathrm{RV}^{\geq 1} = \{x \in \mathrm{RV} : 1 \leq x\}, \quad \mathrm{RV}^{>1} = \{x \in \mathrm{RV} : 1 < x\}.$$

Semantically we shall treat Γ as an imaginary sort and write RV_{Γ} for $\mathrm{RV} \cup \Gamma$. However, syntactically any reference to Γ may be eliminated in the usual way and we shall still work with $\mathcal{L}_{\mathrm{RV}}$-formulas.

Theorem 2.3 (*[16, Theorem 3.10]*)**.** *The theory* ACVF *admits quantifier elimination.*

Since a VF-sort literal can be equivalently expressed as an RV-sort literal, we may assume that an $\mathcal{L}_{\mathrm{RV}}$-formula contains no VF-sort literals at all. In particular, we may assume that every VF-sort polynomial $F(\vec{X})$ in a formula ϕ occurs in the form $\mathrm{rv}(F(\vec{X}))$. This understanding sometimes makes the discussion more streamlined. We say that $F(\vec{X})$ is an *occurring polynomial* of ϕ.

Definition 2.4. Let $\vec{X}$ be VF-sort variables and $\vec{Y}$ be RV-sort variables.

A $\overline{\mathrm{K}}$-*term* is an $\mathcal{L}_{\mathrm{RV}}$-term of the form $\sum_{i=1}^{k}(\mathrm{rv}(F_i(\vec{X})) \cdot r_i \cdot \vec{Y}^{\vec{n}_i})$ with $k > 1$, where $F_i(\vec{X})$ is a polynomial with coefficients in VF and $r_i \in \mathrm{RV}$. An RV-*literal* is an $\mathcal{L}_{\mathrm{RV}}$-formula of the form

$$\mathrm{rv}(F(\vec{X})) \cdot \vec{Y}^{\vec{m}} \cdot T(\vec{X}, \vec{Y}) \,\square\, \mathrm{rv}(G(\vec{X})) \cdot r \cdot \vec{Y}^{\vec{l}} \cdot S(\vec{X}, \vec{Y}),$$

where $F(\vec{X}), G(\vec{X})$ are polynomials with coefficients in VF, $T(\vec{X}, \vec{Y}), S(\vec{X}, \vec{Y})$ are $\overline{\mathrm{K}}$-terms, $r \in \mathrm{RV}$, and $\square$ is one of the symbols $=, \neq, \leq$, and $>$.

Note that if $T(\vec{X}, \vec{Y})$ is a $\overline{\mathrm{K}}$-term, $\vec{a} \in \mathrm{VF}$, and $\vec{t} \in \mathrm{RV}$ then $T(\vec{a}, \vec{t})$ is defined if and only if each summand in $T(\vec{a}, \vec{t})$ is either of value 1 or is equal to 0. Also, since the value of $\overline{\mathrm{K}}$-terms are 0, we may assume that they do not occur in RV-sort inequalities.

Any $\mathcal{L}_{\mathrm{RV}}$-formula with parameters is provably equivalent to a disjunction of conjunctions of RV-literals. This follows from QE of ACVF and routine syntactical inductions.

Let ACVF(0, 0) denote ACVF with pure characteristic 0. From now on we shall work in a sufficiently saturated model $\mathfrak{C}$ of ACVF(0, 0). Let $S \subseteq \mathfrak{C}$ be a small substructure such that $\Gamma(S)$ *is nontrivial*. Let $\mathrm{ACVF}_S(0, 0)$ be the theory that extends ACVF(0, 0) with the atomic diagram of S. For notational simplicity we shall still refer to the language of $\mathrm{ACVF}_S(0, 0)$ as $\mathcal{L}_{\mathrm{RV}}$. Although we do not include the multiplicative inverse function in the VF-sort and the RV-sort, we always assume that, without loss of generality, $\mathrm{VF}(S)$ is a field and $\mathrm{RV}^{\times}(S)$ is a group.

Convention 2.5. By a definable subset of $\mathfrak{C}$ we mean a $\emptyset$-definable subset in the theory $\mathrm{ACVF}_S(0, 0)$. If additional parameters are used in defining a subset then we shall spell them out explicitly if necessary.

The substructure generated by a subset A is denoted by $\langle A \rangle$ or $\mathrm{dcl}(A)$. The model-theoretic algebraic closure of A is denoted by $\mathrm{acl}(A)$. A substructure S is VF-*generated* if there is a subset $A \subseteq \mathrm{VF}(S)$ such that $S = \langle A \rangle$; similarly for (VF, RV)-generated substructures, (VF, Γ)-generated substructures, etc.

Definition 2.6. A subset $\mathfrak{b}$ of VF is an *open ball* if there is a $\gamma \in \Gamma$ and a $b \in \mathfrak{b}$ such that $a \in \mathfrak{b}$ if and only if $\mathrm{val}(a - b) > \gamma$. It is a *closed ball* if $a \in \mathfrak{b}$ if and only if $\mathrm{val}(a - b) \geq \gamma$. It is an rv-*ball* if $\mathfrak{b} = \mathrm{rv}^{-1}(t)$ for some $t \in \mathrm{RV}$. The value γ is the *radius* of $\mathfrak{b}$, which is denoted as $\mathrm{rad}(\mathfrak{b})$. Each point in VF is a closed ball of radius ∞ and VF is a clopen ball of radius $-\infty$.

If val is constant on $\mathfrak{b}$ — that is, $\mathfrak{b}$ is contained in an rv-ball — then $\mathrm{val}(\mathfrak{b})$ is the *valuative center* of $\mathfrak{b}$; if val is not constant on $\mathfrak{b}$, that is, $0 \in \mathfrak{b}$, then the *valuative center* of $\mathfrak{b}$ is ∞. The valuative center of $\mathfrak{b}$ is denoted by $\mathrm{vcr}(\mathfrak{b})$.

A subset $\mathfrak{p} \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$ is an (*open, closed,* rv-) *polydisc* if it is of the form $(\prod_{i \leq n} \mathfrak{b}_i) \times \{\vec{t}\}$, where each $\mathfrak{b}_i$ is an (open, closed, rv-) ball and $\vec{t} \in \mathrm{RV}^m$. If $\mathfrak{p}$ is a polydisc then the *radius* of $\mathfrak{p}$, denoted as $\mathrm{rad}(\mathfrak{p})$, is $\min\{\mathrm{rad}(\mathfrak{b}_i) : i \leq n\}$. The open and closed polydiscs centered at a sequence of elements $\vec{a} = (a_1, \ldots, a_n) \in \mathrm{VF}^n$ with radii $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \Gamma^n$ are respectively denoted as $\mathfrak{o}(\vec{a}, \vec{\gamma})$ and $\mathfrak{c}(\vec{a}, \vec{\gamma})$.

An rv-polydisc $\mathrm{rv}^{-1}(t_1, \ldots, t_n) \times \{\vec{s}\}$ is *degenerate* if $t_i = \infty$ for some i.

Definition 2.7. Let $\mathcal{L}$ be a language expanding $\mathcal{L}_{\mathrm{RV}}$. Let M be a structure of $\mathcal{L}$ that satisfies the axioms for valued fields. We say that M is *C-minimal* if every parametrically definable subset of $\mathrm{VF}(M)$ is a boolean combination of balls. An $\mathcal{L}$-theory T is *C-minimal* if every model of T is *C-minimal*.

Theorem 2.8 (*[16, Theorem 4.2]*)**.** *The theory* ACVF *is C-minimal.*

Notation 2.9. We sometimes write $\vec{a} \in \mathrm{VF}$ to mean that every element in the tuple $\vec{a}$ is in VF; similarly for RV, Γ, etc. We often write $(\vec{a}, \vec{t})$ for a tuple of elements with the understanding that $\vec{a} \in \mathrm{VF}$ and $\vec{t} \in \mathrm{RV}$. For such a tuple $(\vec{a}, \vec{t}) = (a_1, \ldots, a_n, t_1, \ldots t_m)$, let

$$\mathrm{rv}(\vec{a}, \vec{t}) = (\mathrm{rv}(a_1), \ldots, \mathrm{rv}(a_n), \vec{t}), \quad \mathrm{rv}^{-1}(\vec{a}, \vec{t}) = \{\vec{a}\} \times \mathrm{rv}^{-1}(t_1) \times \cdots \times \mathrm{rv}^{-1}(t_m),$$

similarly for other functions.

Let $\vec{a} = (a_1, \ldots, a_n), \vec{a}' = (a'_1, \ldots, a'_n)$ be tuples in VF. We write $\mathrm{val}(\vec{a} - \vec{a}')$ for the element

$$\min\{\mathrm{val}(a_i - a'_i) : 1 \leq i \leq n\} \in \Gamma.$$

For any $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \Gamma$, the open polydisc $\{(b_1, \ldots, b_n) : \mathrm{val}(b_i - a_i) > \gamma_i\}$ is denoted by $\mathfrak{o}(\vec{a}, \vec{\gamma})$ and the closed polydisc $\{(b_1, \ldots, b_n) : \mathrm{val}(b_i - a_i) \geq \gamma_i\}$ is denoted by $\mathfrak{c}(\vec{a}, \vec{\gamma})$. We set $\mathfrak{o}(\vec{a}, \infty) = \mathfrak{c}(\vec{a}, \infty) = \{\vec{a}\}$.

Notation 2.10. Coordinate projection maps are ubiquitous in this paper. To facilitate the discussion, certain notational conventions about them are adopted.

Let $A \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$. For any $n \in \mathbb{N}$, let $I_n = \{1, \ldots, n\}$. First of all, the VF-coordinates and the RV-coordinates of A are indexed separately. It is cumbersome to actually distinguish them notationally, so we just assume that the set of the VF-indices is I_n and the set of the RV-indices is I_m. This should never cause confusion in context.

Let $I = I_n \uplus I_m$, $E \subseteq I$, and $\tilde{E} = I \smallsetminus E$. If E is a singleton $\{i\}$ then we always write E as i and $\tilde{E}$ as $\tilde{i}$. We write $\mathrm{pr}_E(A)$ for the projection of A to the coordinates in E. For any $\vec{a} \in \mathrm{pr}_{\tilde{E}}(A)$, the fiber $\{\vec{b} : (\vec{b}, \vec{a}) \in A\}$ is denoted by $\mathrm{fib}(A, \vec{a})$. Note that we shall often tacitly identify the two subsets $\mathrm{fib}(A, \vec{a})$ and $\mathrm{fib}(A, \vec{a}) \times \{\vec{a}\}$. Also, it is often more convenient to use simple descriptions as subscripts. For example, if $E = \{1, \ldots, k\}$ etc. then we may write $\mathrm{pr}_{\leq k}$ etc. If E contains exactly the VF-indices (respectively RV-indices) then pr_E is written as pvf (respectively prv). If E' is a subset of the coordinates of $\mathrm{pr}_E(A)$ then the composition $\mathrm{pr}_{E'} \circ \mathrm{pr}_E$ is written as $\mathrm{pr}_{E,E'}$. Naturally $\mathrm{pr}_{E'} \circ \mathrm{pvf}$ and $\mathrm{pr}_{E'} \circ \mathrm{prv}$ are written as $\mathrm{pvf}_{E'}$ and $\mathrm{prv}_{E'}$, respectively.

3. Some structural properties

In this section we shall list a number of structural properties concerning the relation among the three sorts VF, RV, and Γ. Some simple ones are just consequences of variations of compactness, for example:

Lemma 3.1. *Let A be a definable subset and s an element such that $s \in \mathrm{acl}(a)$ for every $a \in A$, then $s \in \mathrm{acl}(\emptyset)$.*

Proof. By compactness, there are a definable partition $A_1, \ldots, A_m$ of A, integers $k_1, \ldots, k_m$, formulas $\phi_1(X, Y), \ldots, \phi_m(X, Y)$, such that if $a \in A_i$ then the subset U_a defined by the formula $\phi_i(a, Y)$ contains s and its size is at most k_i. Then $\bigcap_{a \in A} U_a$ is a definable finite subset that contains s. $\square$

Corollary 3.2. *For any $\vec{t} \in \mathrm{RV}$, any $\vec{t}$-definable subset $A \subseteq \mathrm{rv}^{-1}(\vec{t})$, and any element x, if $x \in \mathrm{acl}(\vec{a})$ for every $\vec{a} \in A$ then $x \in \mathrm{acl}(\vec{t})$. Similarly, for any $\vec{\gamma} \in \Gamma$, any $\vec{\gamma}$-definable subset $B \subseteq \mathrm{vrv}^{-1}(\vec{\gamma})$, and any element x, if $x \in \mathrm{acl}(\vec{t})$ for every $\vec{t} \in B$ then $x \in \mathrm{acl}(\vec{\gamma})$.*

For any $A \subseteq \mathrm{VF}$ let A^{ac} be the field-theoretic algebraic closure of A. The field generated by $\vec{a} \in \mathrm{VF}$ is written as $\mathrm{VF}(S)(\vec{a})$.

Lemma 3.3. *For any $\vec{a}, b \in \mathrm{VF}$ and $\vec{t} \in \mathrm{RV}$, if $b \in \mathrm{acl}(\vec{a}, \vec{t})$ then $b \in \mathrm{VF}(S)(\vec{a})^{\mathrm{ac}}$.*

Proof. Suppose for contradiction $b \notin \mathrm{VF}(S)(\vec{a})^{\mathrm{ac}}$. Let $\phi(X, \vec{a}, \vec{t})$ be a formula that defines a finite subset containing b. Then, for any occurring polynomial $F(X, \vec{a})$ of $\phi(X, \vec{a}, \vec{t})$, we have $F(b, \vec{a}) \neq 0$. We see that, for any $d \in \mathrm{VF}$, if $\mathrm{val}(d - b)$ is sufficiently large then $\mathrm{rv}(F(d, \vec{a})) = \mathrm{rv}(F(b, \vec{a}))$ for all occurring polynomials $F(X, \vec{a})$ and hence $\phi(d, \vec{a}, \vec{t})$ holds, which is a contradiction. □

Corollary 3.4. *For any $\vec{a} \in \mathrm{VF}$ and $B \subseteq \mathrm{RV}$, the transcendental degrees of $\mathrm{VF}(S)(\vec{a})$, $\mathrm{VF}(\langle \vec{a}, B\rangle)$, and $\mathrm{VF}(\mathrm{acl}(\vec{a}, B))$ over $\mathrm{VF}(S)$ are all equal.*

Corollary 3.5 ([16, Lemma 4.12])**.** *Let $A \subseteq \mathrm{RV}^m$ and $f : A \longrightarrow \mathrm{VF}^n$ a definable function. Then $f(A)$ is finite.*

Proof. We may assume $n = 1$. Since for any $\vec{t} \in A$ we have $f(\vec{t}) \in \langle \vec{t} \rangle$, by Lemma 3.3, $f(\vec{t}) \in \mathrm{VF}(S)^{\mathrm{ac}}$. By compactness $f(A)$ must be finite. □

Lemma 3.6 ([16, Lemma 4.3])**.** *The exchange principle holds in both sorts:*

(1) *For any $a, b \in \mathrm{VF}$, if $a \in \mathrm{acl}(b) \smallsetminus \mathrm{acl}(\emptyset)$ then $b \in \mathrm{acl}(a)$.*
(2) *For any $t, s \in \mathrm{RV}$, if $t \in \mathrm{acl}(s) \smallsetminus \mathrm{acl}(\emptyset)$ then $s \in \mathrm{acl}(t)$.*

Corollary 3.7. *If $a \in \mathrm{VF}$ is such that $a \notin \mathrm{acl}(\emptyset)$, then for any $t \in \mathrm{RV}$ we have $a \notin \mathrm{acl}(t)$. Similarly, if $t \in \mathrm{RV}$ is such that $t \notin \mathrm{acl}(\emptyset)$, then for any $\gamma \in \Gamma$ we have $t \notin \mathrm{acl}(\gamma)$.*

Proof. For the first claim, suppose for contradiction that $a \in \mathrm{acl}(t)$. Then $a \in \mathrm{acl}(b)$ for every $b \in \mathrm{rv}^{-1}(t)$. So by the exchange principle we have $b \in \mathrm{acl}(a)$ for every $b \in \mathrm{rv}^{-1}(t)$, which is impossible. The other claim is proved in the same way. □

Lemma 3.8 ([16, Lemma 4.9])**.** *Let $c_1, \ldots, c_k \in \mathrm{VF}$ be distinct elements of the same value α such that their average is 0. Then for some $c_i \neq c_j$ we have $\mathrm{val}(c_i - c_j) = \alpha$ and hence rv is not constant on the set $\{c_1, \ldots, c_k\}$.*

Lemma 3.9 ([16, Lemma 4.10])**.** *Let A be a definable finite subset of VF^n. Then there is a definable injection $f : A \longrightarrow \mathrm{RV}^m$ for some m.*

Lemma 3.10 ([16, Lemma 4.15])**.** *Let $\mathfrak{B}$ be an algebraic set of closed balls. Then $\mathfrak{B}$ has centers.*

Lemma 3.11. *If a ball contains a definable proper subset then it contains a definable point.*

Proof. The proof of [16, Lemma 4.16] works almost verbatim here. □

Corollary 3.12. *Let $B \subseteq \mathrm{RV}$ and $f : \mathrm{rv}^{-1}(B) \longrightarrow \mathrm{RV}^m$ a definable function. Then, for all but finitely many $t \in B$, $f \upharpoonright \mathrm{rv}^{-1}(t)$ is constant.*

Proof. For any $t \in B$, if $f \upharpoonright \mathrm{rv}^{-1}(t)$ is not constant then, by Lemma 3.11, for each $\vec{s} \in \mathrm{ran}(f \upharpoonright \mathrm{rv}^{-1}(t))$, $\mathrm{rv}^{-1}(t)$ contains a $(\vec{t}, \vec{s})$-definable point $a_{\vec{t},\vec{s}}$. By Corollary 3.5, the image of the function given by $(\vec{t}, \vec{s}) \longmapsto a_{\vec{t},\vec{s}}$ is finite. □

Lemma 3.13 ([16, Lemma 4.17])**.** *Suppose that S is (VF, Γ)-generated. Let $\mathfrak{B}$ be an algebraic set of balls. Then $\mathfrak{B}$ has centers.*

Corollary 3.14 ([16, Corollary 4.18])**.** *Suppose that S is VF-generated. If the value group $\Gamma(\mathrm{acl}(S))$ is nontrivial then $\mathrm{acl}(S)$ is a model of $\mathrm{ACVF}_S(0, 0)$.*

Lemma 3.15. *Let A be a definable subset of RV. Let $V \subseteq \Gamma$ be the subset such that $\gamma \in V$ if and only if $\mathrm{vrv}^{-1}(\gamma) \cap A$ is nonempty and finite. Then V is finite and definable.*

Proof. By C-minimality each $\mathrm{vrv}^{-1}(\gamma) \cap A$ is either finite or cofinite. By compactness there is a number k such that if $\mathrm{vrv}^{-1}(\gamma) \cap A$ is finite then it has at most k elements. So V is definable. By C-minimality again V must be finite. □

Let A be a subset and $B \subseteq A \times \mathrm{VF}^n \times \mathrm{RV}^m$. We say that B is a *subset over A* if the projection of B to A is surjective.

Notation 3.16. Let A_1, A_2 be subsets and R_1, R_2 equivalence relations on them, respectively. A subset $B \subseteq A_1 \times A_2$ over A_1 may be considered as a function from A_1/R_1 into the powerset $\mathcal{P}(A_2/R_2)$ if, for each equivalence class $C \in A_1/R_1$ and every $c_1, c_2 \in C$, there is a $U \in \mathcal{P}(A_2/R_2)$ such that $\mathrm{fib}(B, c_1) = \mathrm{fib}(B, c_2) = \bigcup U$. In this case, we sometime do write B as a function $A_1/R_1 \longrightarrow \mathcal{P}(A_2/R_2)$. We are of course only interested in definable objects. For example, we will discuss functions of the forms

$$\mathrm{VF}/\mathcal{M} \longrightarrow \mathcal{P}(\mathrm{RV}^m), \quad \mathrm{VF}^n \times \Gamma^l \longrightarrow \mathcal{P}(\mathrm{RV}^m).$$

More elaborate syntactical analysis using the normal forms in Definition 2.4 can sometimes reveal finer details.

Lemma 3.17. *Let $f : \mathrm{VF}^{\times} \longrightarrow \mathcal{P}(\mathrm{RV}^m)$ be a definable function such that the subset $\mathrm{vrv}(\bigcup f(\mathrm{VF}^{\times}))$ is bounded from both above and below. Then for any sufficiently large $\delta \in \Gamma$ the restriction $f \upharpoonright \mathfrak{o}(0, \delta) \smallsetminus \{0\}$ is constant.*

Proof. Let $\phi(X,\vec{Y})$ be a disjunction of conjunctions of RV-literals that defines f. For any $\delta \in \Gamma$ let $\phi_\delta(X,\vec{Y})$ be the formula $\phi(X,\vec{Y}) \wedge \mathrm{val}(X) > \delta$. Any term of the form $\mathrm{rv}(F(X))$ in $\phi(X,\vec{Y})$ may be written as $\mathrm{rv}(X^m F^*(X))$, where $F^*(0) \neq 0$. So if $\mathrm{val}(a)$ is sufficiently large then

$$\mathrm{rv}(a^m F^*(a)) = \mathrm{rv}(a^m)\,\mathrm{rv}(F^*(0)).$$

Since $\mathrm{vrv}(\bigcup f(\mathrm{VF}^\times))$ is bounded from below, if δ is sufficiently large then we may assume that no $\overline{\mathrm{K}}$-term in $\phi_\delta(X,\vec{Y})$ contains X. Since $\mathrm{vrv}(\bigcup f(\mathrm{VF}^\times))$ is also bounded from above, it is not hard to see that $\phi_\delta(X,\vec{Y})$ is actually equivalent to a formula of the form $\psi(\vec{Y}) \wedge \mathrm{val}(X) > \delta$, where $\psi(\vec{Y})$ does not contain X. $\square$

It is not hard to see that the same argument shows that the above lemma also holds for functions $f : \mathrm{VF}^\times \longrightarrow \mathcal{P}(\mathrm{RV}^m)$ that satisfy the obvious condition.

Lemma 3.18. *Let G be a definable additive subgroup of* VF *(hence G is either an open ball around* 0 *or a closed ball around* 0*). Let $f : \mathrm{VF} \longrightarrow \mathcal{P}(\mathrm{RV}^m)$ be a definable function. Then*

(1) *There are G-cosets $D_1, \ldots, D_n$ such that $f \upharpoonright (\mathrm{VF} \smallsetminus \bigcup_i D_i)$ is a function from $(\mathrm{VF} \smallsetminus \bigcup_i D_i)/G$ into $\mathcal{P}(\mathrm{RV}^m)$.*
(2) *If either G is a closed ball or S is (VF, Γ)-generated then there is a definable function $f_\downarrow : \mathrm{VF}/G \longrightarrow \mathcal{P}(\mathrm{RV}^m)$ such that for any G-coset D there is a $d \in D$ such that $f(d) = f_\downarrow(D)$.*

Proof. For any $D \in \mathrm{VF}/G$ and any $\vec{t} \in \mathrm{RV}^m$ let $U_{\vec{t}}(D) = \left\{d \in D : \vec{t} \in f(d)\right\}$. Let

$$E_{\vec{t}} = \{D \in \mathrm{VF}/G : U_{\vec{t}}(D) \neq \emptyset \text{ and } U_{\vec{t}}(D) \neq D\}.$$

Note that $E_{\vec{t}}$ is $\vec{t}$-definable. Let $A = \{\vec{t} \in \mathrm{RV}^m : E_{\vec{t}} \neq \emptyset\}$, which is definable. If $D \notin E_{\vec{t}}$ for any $\vec{t}$ then $f \upharpoonright D$ is constant. So, without loss of generality, $A \neq \emptyset$. For any $\vec{t} \in A$, by C-minimality and compactness, there is a $\vec{t}$-definable function $h_{\vec{t}}$ on $E_{\vec{t}}$ such that, for each $D \in E_{\vec{t}}$,

(1) $h_{\vec{t}}(D)$ is either the union of the positive boolean components of $U_{\vec{t}}(D)$ or the union of the negative boolean components of $U_{\vec{t}}(D)$,
(2) there is a D-definable closed ball $\mathfrak{b}_D \subseteq D$ that properly contains $h_{\vec{t}}(D)$.

Since $h_{\vec{t}}(E_{\vec{t}})$ is $\vec{t}$-definable, by C-minimality again, $E_{\vec{t}}$ must be finite. By Lemma 3.10, there is a $\vec{t}$-definable subset $A_{\vec{t}}$ such that $|A_{\vec{t}} \cap \mathfrak{b}_D| = 1$. Let $g_D : A \longrightarrow \mathrm{VF}$ be the D-definable function given by $\vec{t} \longmapsto A_{\vec{t}} \cap \mathfrak{b}_D$ if $D \in E_{\vec{t}}$ and $\vec{t} \longmapsto 0$ otherwise. By Corollary 3.5, $g_D(A)$ is finite. Since $g_D(A) \subseteq D \cup \{0\}$, by C-minimality, the definable subset $\bigcup_{D \in \mathrm{VF}/G} g_D(A)$ must be finite and hence $\bigcup_{\vec{t} \in A} E_{\vec{t}}$ is finite. This establishes (1). By Lemma 3.10 or Lemma 3.13, $\bigcup_{\vec{t} \in A} E_{\vec{t}}$ has definable centers. This establishes (2). $\square$

Remark 3.19. Let G be a definable multiplicative subgroup of $\mathrm{VF}^\times$. Then G is an open ball around 1 or a closed ball around 1 or $\mathcal{O} \smallsetminus \mathcal{M}$. It is easy to see that if G is not $\mathcal{O} \smallsetminus \mathcal{M}$ then the proof of Lemma 3.18 also works with respect to G. If G is $\mathcal{O} \smallsetminus \mathcal{M}$ then we can modify the proof as follows: in the construction of $h_{\vec{t}}$, $\mathfrak{b}_D \subseteq D$ is a finite union of rv-balls and contains $h_{\vec{t}}(D)$.

4. Categories of definable subsets

4.1. Dimensions

For the categories of definable sets associated with $\mathrm{ACVF}_S(0,0)$ and their Grothendieck groups, two notions of dimension with respect to the two sorts are needed. Some basic properties of them are stated below.

Let $A \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$ be a definable subset.

Definition 4.1. The VF-*dimension* of A, denoted by $\dim_{\mathrm{VF}}(A)$, is the smallest number k such that there is a definable finite-to-one function $f : A \longrightarrow \mathrm{VF}^k \times \mathrm{RV}^l_\Gamma$.

Lemma 4.2. *For any natural number k, $\dim_{\mathrm{VF}}(A) \leq k$ if and only if there is a definable injection $f : A \longrightarrow \mathrm{VF}^k \times \mathrm{RV}^l_\Gamma$ for some l.*

Proof. Suppose that $\dim_{\mathrm{VF}}(A) \leq k$. Let $g : A \longrightarrow \mathrm{VF}^k \times \mathrm{RV}^l_\Gamma$ be a definable finite-to-one function. For every $(\vec{a},\vec{t}) \in g(A)$, since $g^{-1}(\vec{a},\vec{t})$ is finite, by Lemma 3.9, there is an $(\vec{a},\vec{t})$-definable injection $h_{\vec{a},\vec{t}} : g^{-1}(\vec{a},\vec{t}) \longrightarrow \mathrm{RV}^j_\Gamma$ for some j. By compactness, there is a definable function $h : A \longrightarrow \mathrm{RV}^j_\Gamma$ for some j such that $h \upharpoonright g^{-1}(\vec{a},\vec{t})$ is injective for every $(\vec{a},\vec{t}) \in g(A)$. Then the function f on A given by

$$(\vec{b},\vec{s}) \longmapsto (g(\vec{b},\vec{s}), h(\vec{b},\vec{s}))$$

is as desired. The other direction is trivial. $\square$

Lemma 4.3. *Let $f : A \longrightarrow \mathrm{RV}^l_\Gamma$ be a definable function. Then*

$$\dim_{\mathrm{VF}}(A) = \max\{\dim_{\mathrm{VF}}(f^{-1}(\vec{t})) : \vec{t} \in \mathrm{RV}^l_\Gamma\}.$$

Proof. Let $\max\{\dim_{\mathrm{VF}}(f^{-1}(\vec{t})) : \vec{t} \in \mathrm{RV}^l_\Gamma\} = k$. By Lemma 4.2, for every $\vec{t} \in \mathrm{ran}(f)$, there is a $\vec{t}$-definable injective function $h_{\vec{t}} : f^{-1}(\vec{t}) \longrightarrow \mathrm{VF}^k \times \mathrm{RV}^j_\Gamma$ for some j. By compactness, there is a definable function $h : A \longrightarrow \mathrm{VF}^k \times \mathrm{RV}^j_\Gamma$ for some j such that $h \upharpoonright f^{-1}(\vec{t})$ is injective for every $\vec{t} \in \mathrm{ran}(f)$. Then the function on A given by $(\vec{b}, \vec{s}) \longmapsto (h(\vec{b}, \vec{s}), f(\vec{b}, \vec{s}))$ is injective and hence $\dim_{\mathrm{VF}}(A) \le k$. The other direction is trivial. □

For any $(\vec{a}, \vec{t}) \in A$ let $\mathrm{tr\,deg}(\vec{a}, \vec{t})$ be the transcendental degree of $\mathrm{VF}(\langle \vec{a} \rangle)$ over $\mathrm{VF}(S)$. Let $\mathrm{tr\,deg}(A) = \max\{\mathrm{tr\,deg}(\vec{a}, \vec{t}) : (\vec{a}, \vec{t}) \in A\}$.

Lemma 4.4. $\dim_{\mathrm{VF}}(A) = \mathrm{tr\,deg}(A)$.

Proof. Let $\dim_{\mathrm{VF}}(A) = k$ and $\mathrm{tr\,deg}(A) = k'$. By Lemma 4.2, there is a definable injection $f : A \longrightarrow \mathrm{VF}^k \times \mathrm{RV}^l_\Gamma$ for some l. For any $(\vec{a}, \vec{t}) \in A$, if $f(\vec{a}, \vec{t}) = (\vec{b}, \vec{s})$ then, by Lemma 3.3, $\mathrm{VF}(\langle \vec{a} \rangle) \subseteq \mathrm{VF}(S)(\vec{b})^{\mathrm{ac}}$ and hence $\mathrm{tr\,deg}(\vec{a}, \vec{t}) \le k$. So $k' \le k$.

On the other hand, for any $\vec{a} = (a_1, \ldots, a_n) \in \mathrm{pvf}(A)$, there is a subset $E \subseteq \{1, \ldots, n\}$ of size k' such that for any $j \in \tilde{E}$ we have $a_j \in \mathrm{VF}(\langle \mathrm{pr}_E(\vec{a}) \rangle)^{\mathrm{ac}}$. Therefore, by compactness, there are a partition A_i of $\mathrm{pvf}(A)$, subsets $E_i \subseteq \{1, \ldots, n\}$ of size k', and formulas $\phi_i(\vec{X}, \vec{Y})$ such that, for every $\vec{a} \in A_i$, the subset $B_i \subseteq \mathrm{VF}^{n-k'}$ defined by $\phi_i(\vec{X}, \mathrm{pr}_{E_i}(\vec{a}))$ is finite and $\mathrm{pr}_{\tilde{E}_i}(\vec{a}) \in B_i$. By compactness and Lemma 3.9, there is a definable injection $A \longrightarrow \mathrm{VF}^{k'} \times \mathrm{RV}^l_\Gamma$ for some l and hence $k \le k'$. □

It follows that additional parameters cannot change the VF-dimension of a definable subset and hence there is no need to specify parameters when we discuss VF-dimension.

Corollary 4.5. *If $f : A \longrightarrow \mathcal{P}(\mathrm{VF}^{n'} \times \mathrm{RV}^{m'})$ is a definable function with finite images then* $\dim_{\mathrm{VF}}(A) \ge \dim_{\mathrm{VF}}(\bigcup f(A))$.

Lemma 4.6. $\dim_{\mathrm{VF}}(A) = n$ *if and only if there is a $\vec{t} \in \mathrm{RV}^m$ such that* $\mathrm{fib}(A, \vec{t})$ *contains an open polydisc.*

Proof. The "if" direction is immediate by Lemma 4.4. For the "only if" direction, by compactness, it is enough to show the case $A \subseteq \mathrm{VF}^n$. We do induction on n. For the base case $n = 1$, since A is infinite, the lemma simply follows from C-minimality. We proceed to the inductive step $n = m + 1$. For each $\vec{a} \in \mathrm{pr}_{\le m}(A) = B$, let $\Delta_{\vec{a}}$ be the subset of those $\gamma \in \Gamma$ such that $\mathrm{fib}(A, \vec{a})$ contains an open ball of radius γ (if $\mathrm{fib}(A, \vec{a})$ is finite then we set $\Delta_{\vec{a}} = \{\infty\}$). Since Γ is o-minimal, some element $\gamma_{\vec{a}}$ in $\Delta_{\vec{a}}$ is $\vec{a}$-definable. By compactness and the inductive hypothesis, we may assume that $\dim_{\mathrm{VF}}(B) = m$ and there is a quantifier-free formula $\phi(Z, \vec{X})$ such that, for every $\vec{a} \in B$, $\mathrm{fib}(A, \vec{a})$ contains an open ball whose radius $\gamma_{\vec{a}}$ is defined by the formula $\phi(Z, \vec{a})$.

Let $G_i(\vec{X})$ be the occurring polynomials of $\phi(Z, \vec{X})$. Let $f : B \longrightarrow \mathrm{RV}^k$ be the definable function given by

$$\vec{a} \longmapsto (\mathrm{rv}(G_1(\vec{a})), \ldots, \mathrm{rv}(G_k(\vec{a}))).$$

By Lemma 4.3, for some $\vec{t} \in \mathrm{RV}^k$, $\dim_{\mathrm{VF}}(f^{-1}(\vec{t})) = m$. By the inductive hypothesis, $f^{-1}(\vec{t})$ contains an open polydisc $\mathfrak{p}$. Note that, by the construction of f, for every $\vec{a} \in \mathfrak{p}$ the formula $\phi(Z, \vec{a})$ defines the same element $\delta \in \Gamma$. Let $\vec{b} \in \mathfrak{p}$. We may assume that $\mathfrak{p}$ is $\vec{b}$-definable. Note that, by Lemma 4.4, the VF-dimension of $\mathfrak{p}$ with respect to the substructure $\mathrm{dcl}(\vec{b})$ is still m. Consider the $\vec{b}$-definable subset

$$W = \left\{(\vec{a}, c) \in A : \vec{a} \in \mathfrak{p} \text{ and } \mathfrak{o}(c, \delta) \subseteq \mathrm{fib}(A, \vec{a})\right\}.$$

Since there is a $\vec{d} \in W$ such that the transcendental degree of $\mathrm{VF}(\mathrm{dcl}(\vec{d}, \vec{b}))$ over $\mathrm{VF}(\mathrm{dcl}(\vec{b}))$ is $m + 1$, by Lemma 4.4 again, $\dim_{\mathrm{VF}}(W) = m + 1$. By compactness, for some $c \in \mathrm{pr}_{m+1}(W)$, $\dim_{\mathrm{VF}}(\mathrm{fib}(W, c)) = m$. By the inductive hypothesis (with respect to the substructure $\mathrm{dcl}(\vec{b}, c)$), $\mathrm{fib}(W, c)$ contains an open polydisc $\mathfrak{q}$. So $\mathfrak{o}(c, \delta) \times \mathfrak{q} \subseteq A$, as required. □

Corollary 4.7. *Suppose that A contains an* rv*-polydisc of the form*

$$\{(0, \ldots, 0)\} \times \mathrm{rv}^{-1}(\vec{t}) \times \left\{\vec{s}\right\},$$

where $\vec{t} \in (\mathrm{RV}^\times)^k$. Then $\dim_{\mathrm{VF}}(A) \ge k$.

Proposition 4.8. *Suppose that $A \subseteq \mathrm{VF}^n$. Let $\overline{A}$ be the Zariski closure of A and k the Zariski dimension of $\overline{A}$. Then* $\dim_{\mathrm{VF}}(A) = k$.

Proof. Let D be an irreducible component of $\overline{A}$ and $\vec{a} \in D \cap A$. Let P be the prime ideal of $\mathrm{VF}(S)^{\mathrm{ac}}[X_1, \ldots, X_n]$ such that $D = Z(P)$. Let K_P be the corresponding quotient field. By general facts of commutative algebra (see, for example, [1, Chapter 11]), the dimension of D is equal to the transcendental degree of K_P over $\mathrm{VF}(S)$. Since the latter is no less than the transcendental degree of $\mathrm{VF}(S)^{\mathrm{ac}}(\vec{a})$ over $\mathrm{VF}(S)$, we see that, by Lemma 4.4, $k \ge \dim_{\mathrm{VF}}(A)$.

Let $\dim_{\mathrm{VF}}(A) = \mathrm{tr\,deg}(A) = k'$. If $k' = n$ then obviously $\overline{A} = \mathrm{VF}^n$ and hence $k = n$. Suppose $\dim_{\mathrm{VF}}(A) < n$. By compactness, there are Zariski closed subsets D_i given by formulas of the form

$$\bigwedge_{j \notin I_i} F_j(X_{i(1)}, \ldots, X_{i(k')}, X_j) = 0,$$

where $I_i = \left\{i(1), \ldots, i(k')\right\}$ and each F_j is a nonzero polynomial with coefficients in $\mathrm{VF}(S)$, such that $A \subseteq \bigcup_i D_i$. Then $\overline{A} \subseteq \bigcup_i D_i$ and hence each irreducible component of $\overline{A}$ is contained in some D_i, which implies $k \le k'$. □

Definition 4.9. Let $B \subseteq \mathrm{RV}^m$ be a definable subset. The RV-*dimension* of B, denoted by $\dim_{\mathrm{RV}}(B)$, is the smallest number k such that there is a definable finite-to-one function $f : B \longrightarrow \mathrm{RV}^k$ (RV^0 is taken to be the singleton $\{\infty\}$).

By the exchange principle (Lemma 3.6), if $\dim_{\mathrm{RV}}(B) = k$ then for every $\vec{t} \in B$ there is a subsequence $\vec{t}' \subseteq \vec{t}$ of length k such that $\vec{t} \in \mathrm{acl}(\vec{t}')$. Also, by compactness, there is a $\vec{t} \in B$ that contains an algebraically independent subsequence of length k (in the model-theoretic sense); that is, for some subsequence $(t_{i(1)}, \dots, t_{i(k)}) \subseteq \vec{t}$ of length k, no $t_{i(j)}$ is in the algebraic closure of the other $k-1$ elements. So additional parameters cannot change the RV-dimension of B as well. Also, if $f : B \longrightarrow \mathcal{P}(\mathrm{RV}^l)$ is a definable function then $\dim_{\mathrm{RV}}(B) \geq \dim_{\mathrm{RV}}(\bigcup f(B))$.

Lemma 4.10. *Let $\vec{s} = (s_1, \dots, s_m) \in \mathrm{RV}$, $\vec{\gamma} = \mathrm{vrv}(\vec{s})$, and $B \subseteq \mathrm{vrv}^{-1}(\vec{\gamma})$ a definable subset. Let*

$$B_{\vec{s}} = \{(t_1/s_1, \dots, t_m/s_m) : (t_1, \dots, t_m) \in B\}.$$

Then $\dim_{\mathrm{RV}}(B)$ agrees with the Zariski dimension of $B_{\vec{s}}$.

Proof. The proof of Proposition 4.8 works almost verbatim here. □

Lemma 4.11. *Let $B \subseteq \mathrm{RV}^m$ with $\dim_{\mathrm{RV}}(B) = k$. Then there is a definable sequence $\vec{\gamma} \in \Gamma^m$ such that $\dim_{\mathrm{RV}}(B \cap \mathrm{vrv}^{-1}(\vec{\gamma})) = k$.*

Proof. By compactness, without loss of generality, we may assume that, for every $\vec{t} \in B$, $\vec{t} \in \mathrm{acl}(t_1, \dots, t_k)$. Let

$$B_0 = \left\{(\mathrm{pr}_{\leq k}(\vec{t}), \mathrm{val}(\mathrm{pr}_{>k}(\vec{t}))) : \vec{t} \in B\right\} \subseteq \mathrm{RV}^k \times \Gamma^{m-k}.$$

Clearly there is a natural number q such that $\left|\mathrm{fib}(B_0, \vec{t})\right| \leq q$ for every $\vec{t} \in \mathrm{pr}_{\leq k}(B)$. For every $(\vec{t}, \vec{\gamma}) \in \mathrm{pr}_{>1}(B_0)$ let $D_{\vec{t},\vec{\gamma}} \subseteq \Gamma$ be the subset such that $\alpha \in D_{\vec{t},\vec{\gamma}}$ if and only if $\mathrm{vrv}^{-1}(\alpha) \cap \mathrm{fib}(B_0, (\vec{t}, \vec{\gamma}))$ is infinite. Since $\dim_{\mathrm{RV}}(B) = k$, by Corollary 3.7, we see that $D_{\vec{t},\vec{\gamma}}$ is not empty for some $(\vec{t}, \vec{\gamma}) \in \mathrm{pr}_{>1}(B_0)$. Also, by Lemma 3.15, $D_{\vec{t},\vec{\gamma}}$ is $(\vec{t}, \vec{\gamma})$-definable. So, by compactness, the subset

$$B_1 = \bigcup_{(\vec{t},\vec{\gamma}) \in \mathrm{pr}_{>1}(B_0)} D_{\vec{t},\vec{\gamma}} \times \left\{(\vec{t}, \vec{\gamma})\right\} \subseteq \mathrm{RV}^{k-1} \times \Gamma^{m-k+1}$$

is nonempty and definable. We may repeat this procedure with respect to B_1 and get a definable subset $B_2 \subseteq \mathrm{RV}^{k-2} \times \Gamma^{m-k+2}$, and so on. Eventually we obtain a nonempty definable subset $B_k \subseteq \Gamma^m$ with the following property: if $\vec{\gamma} \in B_k$ then there is a $(t_1, \dots, t_k, \dots, t_m) \in \mathrm{vrv}^{-1}(\vec{\gamma}) \cap B$ such that $t_1, \dots, t_k$ are algebraically independent and hence $\dim_{\mathrm{RV}}(\mathrm{vrv}^{-1}(\vec{\gamma}) \cap B) = k$. Now, since Γ is o-minimal, some $\vec{\gamma} \in B_k$ is definable. □

Definition 4.12. The RV-*fiber dimension* of A, denoted by $\dim^{\mathrm{fib}}_{\mathrm{RV}}(A)$, is

$$\max\left\{\dim_{\mathrm{RV}}(\mathrm{fib}(A, \vec{a})) : \vec{a} \in \mathrm{pvf}(A)\right\}.$$

Lemma 4.13. *Suppose that $f : A \longrightarrow A'$ is a definable bijection. Then $\dim^{\mathrm{fib}}_{\mathrm{RV}}(A) = \dim^{\mathrm{fib}}_{\mathrm{RV}}(A')$.*

Proof. Let $\dim^{\mathrm{fib}}_{\mathrm{RV}}(A) = k_1$ and $\dim^{\mathrm{fib}}_{\mathrm{RV}}(A') = k_2$. Since for every $\vec{b} \in \mathrm{pvf}(A')$ there is a $\vec{b}$-definable finite-to-one function $h_{\vec{b}} : \mathrm{fib}(A', \vec{b}) \longrightarrow \mathrm{RV}^{k_2}$, by compactness, there is a definable function $h : A' \longrightarrow \mathrm{RV}^{k_2}$ such that $h \restriction \mathrm{fib}(A', \vec{b})$ is finite-to-one for every $\vec{b} \in \mathrm{pvf}(A')$. For every $\vec{a} \in \mathrm{pvf}(A)$, by Corollary 3.5, the subset $(\mathrm{pvf} \circ f)(\mathrm{fib}(A, \vec{a}))$ is finite. So the function $g_{\vec{a}}$ on $\mathrm{fib}(A, \vec{a})$ given by

$$(\vec{a}, \vec{t}) \longmapsto (h \circ f)(\vec{a}, \vec{t})$$

is $\vec{a}$-definable and finite-to-one. So $k_1 \leq k_2$. Symmetrically we also have $k_1 \geq k_2$ and hence $k_1 = k_2$. □

4.2. Categories of definable subsets

The class of objects and the class of morphisms of any category $\mathcal{C}$ are denoted by $\mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Mor}\,\mathcal{C}$, respectively. By $A \in \mathcal{C}$ we usually mean that A is an object of $\mathcal{C}$.

Definition 4.14 (VF-*categories*)**.** The objects of the category $\mathrm{VF}[k, \cdot]$ are the definable subsets of VF-dimension $\leq k$. The morphisms in this category are the definable functions between the objects.

The category $\mathrm{VF}[k]$ is the full subcategory of $\mathrm{VF}[k, \cdot]$ of the definable subsets that have RV-fiber dimension 0 (that is, all the RV-fibers are finite). The category $\mathrm{VF}_*[\cdot]$ is the union of the categories $\mathrm{VF}[k, \cdot]$. The category VF_* is the union of the categories $\mathrm{VF}[k]$.

Note that, for any definable subset A, by Lemmas 3.9 and 4.3, $\mathrm{fib}(A, \vec{t})$ is finite for every $\vec{t} \in \mathrm{prv}(A)$ if and only if $A \in \mathrm{VF}[0, \cdot]$. Also, by Lemma 4.13, $A \in \mathrm{VF}[k]$ if and only if there is a definable finite-to-one map $A \longrightarrow \mathrm{VF}^k$.

Definition 4.15. For any tuple $\vec{t} = (t_1, \dots, t_n) \in \mathrm{RV}$, the *weight* of $\vec{t}$ is the number $|\{i \leq n : t_i \neq \infty\}|$, which is denoted by $\mathrm{wgt}(\vec{t})$.

Definition 4.16 (RV-*categories*)**.** An object of the category $\mathrm{RV}[k, \cdot]$ is a definable pair (U, f), where $U \subseteq \mathrm{RV}^m$ for some m and $f : U \longrightarrow \mathrm{RV}^k$ is a function (RV^0 is taken to be the singleton $\{\infty\}$). We often denote the projections $\mathrm{pr}_i \circ f$ as f_i and write f as $(f_1, \dots, f_k)$. The *companion* U_f of (U, f) is the subset $\left\{(f(\vec{u}), \vec{u}) : \vec{u} \in U\right\}$.

For any two objects (U, f), (U', f') in $\mathrm{RV}[k, \cdot]$ and any function $F : U \longrightarrow U'$, if $\mathrm{wgt}(f(\vec{u})) \leq \mathrm{wgt}((f' \circ F)(\vec{u}))$ for every $\vec{u} \in U$ then we say that F is *volumetric*. If F is definable, volumetric, and, for every $\vec{t} \in \mathrm{RV}^k$ the subset $(f' \circ F)(f^{-1}(\vec{t}))$ is finite, then it is a morphism in $\mathrm{Mor}\,\mathrm{RV}[k, \cdot]$.

The category $\mathrm{RV}[k]$ is the full subcategory of $\mathrm{RV}[k,\cdot]$ of the pairs (U,f) such that $f : U \longrightarrow \mathrm{RV}^k$ is finite-to-one.

Direct sums (coproducts) over these categories are formed naturally:

$$\mathrm{RV}[\leq i,\cdot] = \coprod_{0\leq k\leq i} \mathrm{RV}[k,\cdot], \quad \mathrm{RV}[*,\cdot] = \coprod_{0\leq k} \mathrm{RV}[k,\cdot],$$

and similarly for $\mathrm{RV}[\leq i]$ and $\mathrm{RV}[*]$.

We usually just write A for the object $(A, \mathrm{id}) \in \mathrm{RV}[k,\cdot]$. Also, for any object in $\mathrm{RV}[k,\cdot]$ of the form (U, pr_E), we may assume that (U, pr_E) is $(U, \mathrm{pr}_{\leq k})$ if this is more convenient. This should not cause any confusion in context.

One of the main reasons for the peculiar forms of the objects and the morphisms in the RV-categories is that each isomorphism class in these categories may be "lifted" to an isomorphism class in the corresponding VF-category. See Proposition 7.6 and Corollary 7.7 for details.

A *subobject* of an object A of a VF-category is just a definable subset. A *subobject* of an object (U,f) of an RV-category is a definable pair (A,g) with A a subset of U and $g = f \upharpoonright A$. Note that the inclusion map is a morphism in both cases.

Notice that the cartesian product of two objects $A, B \in \mathrm{VF}[k,\cdot]$ may or may not be in $\mathrm{VF}[k,\cdot]$. On the other hand, the cartesian product of two objects $(U,f), (U',f') \in \mathrm{RV}[k,\cdot]$ is the object $(U \times U', f \times f') \in \mathrm{RV}[2k,\cdot]$, which is definitely not in $\mathrm{RV}[k,\cdot]$ if $k > 0$. Hence, in $\mathrm{RV}[*,\cdot]$ or $\mathrm{RV}[*]$, multiplying with a singleton in general changes isomorphism class.

The categories $\mathrm{VF}_*[\cdot]$ and VF_* are formed through union instead of direct sum or other means that induces more complicated structure. The reason for this is that the main goal of the Hrushovski–Kazhdan integration theory is to assign motivic volumes, that is, elements in the Grothendieck groups of the RV-categories, to the definable subsets, or rather, the isomorphism classes of the definable subsets, in the VF-categories, and the simplest categories that contain all the definable subsets that may be "measured" in this motivic way are $\mathrm{VF}_*[\cdot]$ and VF_*. In contrast, the unions of the RV-categories are naturally endowed with the structure of direct sum, which gives rise to graded Grothendieck semirings. The ring homomorphisms are obtained by "passing to the limit". These will be made precise in a sequel.

Definition 4.17. For any $(U,f) \in \mathrm{RV}[k,\cdot]$ and any $F \in \mathrm{Mor}\,\mathrm{RV}[k,\cdot]$, let $\mathbb{E}_k(f)$ be the function on U given by $\vec{u} \longmapsto (f(\vec{u}), \infty)$, $\mathbb{E}_k(U,f) = (U, \mathbb{E}_k(f))$, and $\mathbb{E}_k(F) = F$. Obviously

$$\mathbb{E}_k : \mathrm{RV}[k,\cdot] \longrightarrow \mathrm{RV}[k+1,\cdot]$$

is a functor that is faithful, full, and injective on objects. For any $i < j$ let $\mathbb{E}_{i,j} = \mathbb{E}_{j-1} \circ \cdots \circ \mathbb{E}_i$ and $\mathbb{E}_{i,i} = \mathrm{id}$.

Homomorphisms between Grothendieck groups shall be induced by the following fundamental maps:

Definition 4.18. For any $(U,f) \in \mathrm{RV}[k,\cdot]$, let

$$\mathbb{L}_k(U,f) = \bigcup \left\{ \mathrm{rv}^{-1}(f(\vec{u})) \times \{\vec{u}\} : \vec{u} \in U \right\}.$$

The map $\mathbb{L}_k : \mathrm{Ob}\,\mathrm{RV}[k,\cdot] \longrightarrow \mathrm{Ob}\,\mathrm{VF}[k,\cdot]$ is called the kth *canonical* RV-*lift*. The map $\mathbb{L}_{\leq k} : \mathrm{Ob}\,\mathrm{RV}[\leq k,\cdot] \longrightarrow \mathrm{Ob}\,\mathrm{VF}[k,\cdot]$ is given by

$$((U_1,f_1), \ldots, (U_k,f_k)) \longmapsto \biguplus_{i\leq k} (\mathbb{L}_k \circ \mathbb{E}_{i,k})(U_i,f_i).$$

The map $\mathbb{L} : \mathrm{Ob}\,\mathrm{RV}[*,\cdot] \longrightarrow \mathrm{Ob}\,\mathrm{VF}_*[\cdot]$ is simply the union of the maps $\mathbb{L}_{\leq k}$.

For notational convenience, when there is no danger of confusion, we shall drop the subscripts and simply write $\mathbb{E}$ and $\mathbb{L}$ for these maps.

Observe that if $(U,f) \in \mathrm{RV}[k]$ then $\mathbb{L}(U,f) \in \mathrm{VF}[k]$ and hence the restriction $\mathbb{L} : \mathrm{Ob}\,\mathrm{RV}[k] \longrightarrow \mathrm{Ob}\,\mathrm{VF}[k]$ is well-defined. Similarly we have the maps

$$\mathbb{L} : \mathrm{Ob}\,\mathrm{RV}[\leq k] \longrightarrow \mathrm{Ob}\,\mathrm{VF}[k], \qquad \mathbb{L} : \mathrm{Ob}\,\mathrm{RV}[*] \longrightarrow \mathrm{Ob}\,\mathrm{VF}_*.$$

Also note that $\mathrm{rv}(\mathbb{L}(U,f)) = U_f$ for $(U,f) \in \mathrm{RV}[k,\cdot]$.

For any two objects $(U,f), (U',f') \in \mathrm{RV}[k,\cdot]$ and any definable function $F : U \longrightarrow U'$ there is a naturally induced function $F_{f,f'} : U_f \longrightarrow U'_{f'}$ given by

$$(f(\vec{u}), \vec{u}) \longmapsto ((f' \circ F)(\vec{u}), F(\vec{u})).$$

We have:

Lemma 4.19. *Suppose that F is volumetric and there is a definable function $F^{\uparrow} : \mathbb{L}(U,f) \longrightarrow \mathbb{L}(U',f')$ such that the diagram*

$$\begin{tikzcd}
\mathbb{L}(U,f) \arrow[r, "\mathrm{rv}"] \arrow[d, "F^{\uparrow}"'] & U_f \arrow[r, "\mathrm{pr}_{>k}"] \arrow[d, "F_{f,f'}"] & U \arrow[d, "F"] \\
\mathbb{L}(U',f') \arrow[r, "\mathrm{rv}"'] & U'_{f'} \arrow[r, "\mathrm{pr}_{>k}"'] & U'
\end{tikzcd}$$

commutes. Then F is a morphism in $\mathrm{RV}[k,\cdot]$.

Proof. It is enough to show that, for every $\vec{u} \in U$ and every $i \leq k$,

$$(f'_i \circ F)(\vec{u}) \in \operatorname{acl}(f(\vec{u})),$$

which is equivalent to $(\mathrm{pr}_i \circ F_{f,f'})(f(\vec{u}), \vec{u}) \in \operatorname{acl}(f(\vec{u}))$. To that end, fix a $\vec{u} \in U$. Let $\vec{a} \in \mathrm{rv}^{-1}(f(\vec{u}))$ and $F^{\uparrow}(\vec{a}, \vec{u}) = (b_1, \ldots, b_k, \vec{u}')$. By Lemma 3.3, $b_i \in \operatorname{acl}(\vec{a})$ and hence

$$(\mathrm{pr}_i \circ F_{f,f'})(f(\vec{u}), \vec{u}) = \mathrm{rv}(b_i) \in \operatorname{acl}(\vec{a})$$

for each $i \leq k$. By Corollary 3.2, $\mathrm{rv}(b_i) \in \operatorname{acl}(f(\vec{u}))$. □

Remark 4.20. In Lemma 4.19, if both F and $F^{\uparrow}$ are bijections then we may drop the assumption that F is volumetric, since it is guaranteed by the commutative diagram and Corollary 4.7.

4.3. Grothendieck groups

We now introduce the Grothendieck groups associated with the categories defined above. The construction is of course the same for any reasonable category of definable sets of a first-order theory. For concreteness, we shall limit our attention to the present context.

Let $\mathcal{C}$ be a VF-category or an RV-category. For any $A \in \mathrm{Ob}\,\mathcal{C}$, let $[A]$ denote the isomorphism class of A. The *Grothendieck semigroup* of $\mathcal{C}$, denoted by $\mathbf{K}_+\,\mathcal{C}$, is the semigroup generated by the isomorphism classes $[A]$ of $\mathcal{C}$, subject to the relation

$$[A] + [B] = [A \cup B] + [A \cap B].$$

It is easy to check that $\mathbf{K}_+\,\mathcal{C}$ is actually a commutative monoid, the identity element being $[\emptyset]$ or $([\emptyset], \ldots)$. Since $\mathcal{C}$ always has disjoint unions, the elements of $\mathbf{K}_+\,\mathcal{C}$ are precisely the isomorphism classes of $\mathcal{C}$. If $\mathcal{C}$ is one of the categories $\mathrm{VF}_*[\cdot]$, VF_*, $\mathrm{RV}[*, \cdot]$, and $\mathrm{RV}[*]$ then it is closed under cartesian product. In this case, $\mathbf{K}_+\,\mathcal{C}$ has a semiring structure with multiplication given by

$$[A][B] = [A \times B].$$

Since the symmetry isomorphisms $A \times B \longrightarrow B \times A$ and the association isomorphisms $(A \times B) \times C \longrightarrow A \times (B \times C)$ are always present in these categories, $\mathbf{K}_+\,\mathcal{C}$ is always a commutative semiring.

Remark 4.21. If $\mathcal{C}$ is either $\mathrm{VF}_*[\cdot]$ or VF_* then the isomorphism class of definable singletons is the multiplicative identity of $\mathbf{K}_+\,\mathcal{C}$. If $\mathcal{C}$ is $\mathrm{RV}[*, \cdot]$ then we adjust multiplication when $\mathrm{RV}[0, \cdot]$ is involved as follows. For any $(U, f) \in \mathrm{RV}[0, \cdot]$ and $(V, g) \in \mathrm{RV}[k, \cdot]$, let

$$[(U, f)][(X, g)] = [(X, g)][(U, f)] = [(U \times V, g^*)],$$

where g^* is the function on $U \times V$ given by $(\vec{t}, \vec{s}) \longmapsto g(\vec{s})$. It is easily seen that, with this adjustment, $\mathbf{K}_+\,\mathrm{RV}[*, \cdot]$ becomes a filtrated semiring and its multiplicative identity element is the isomorphism class of (∞, id) in $\mathrm{RV}[0, \cdot]$. Multiplication in $\mathbf{K}_+\,\mathrm{RV}[*]$ is adjusted in the same way.

Definition 4.22. A *semigroup congruence relation* on $\mathbf{K}_+\,\mathcal{C}$ is a sub-semigroup R of the semigroup $\mathbf{K}_+\,\mathcal{C} \times \mathbf{K}_+\,\mathcal{C}$ such that R is an equivalence relation on $\mathbf{K}_+\,\mathcal{C}$. Similarly, a *semiring congruence relation* on $\mathbf{K}_+\,\mathcal{C}$ is a sub-semiring R of the semiring $\mathbf{K}_+\,\mathcal{C} \times \mathbf{K}_+\,\mathcal{C}$ such that R is an equivalence relation on $\mathbf{K}_+\,\mathcal{C}$.

Let R be a semigroup congruence relation on $\mathbf{K}_+\,\mathcal{C}$ and $(x, y), (v, w) \in R$. Then $(x + v, y + v), (y + v, y + w) \in R$ and hence $(x + v, y + w) \in R$. Therefore the equivalence classes of R has a semigroup structure induced by that of $\mathbf{K}_+\,\mathcal{C}$. This semigroup is denoted by $\mathbf{K}_+\,\mathcal{C}/R$ and is also referred to as a Grothendieck semigroup. Similarly, if R is a semiring congruence relation on $\mathbf{K}_+\,\mathcal{C}$ then $\mathbf{K}_+\,\mathcal{C}/R$ is actually a Grothendieck semiring.

Remark 4.23. Let R be an equivalence relation on the semiring $\mathbf{K}_+\,\mathcal{C}$. If for every $(x, y) \in R$ and every $z \in \mathbf{K}_+\,\mathcal{C}$ we have $(x + z, y + z) \in R$ and $(xz, yz) \in R$ then R is a semiring congruence relation.

Let $(\mathbb{Z}^{\mathbf{K}_+\,\mathcal{C}}, \oplus)$ be the free abelian group generated by the elements of $\mathbf{K}_+\,\mathcal{C}$ and C the subgroup of $(\mathbb{Z}^{\mathbf{K}_+\,\mathcal{C}}, \oplus)$ generated by all elements of $(\mathbb{Z}^{\mathbf{K}_+\,\mathcal{C}}, \oplus)$ of the types

$$(1 \cdot x) \oplus ((-1) \cdot x), \quad (1 \cdot x) \oplus (1 \cdot y) \oplus ((-1) \cdot (x + y)),$$

where $x, y \in \mathbf{K}_+\,\mathcal{C}$. The *Grothendieck group* of $\mathcal{C}$, denoted by $\mathbf{K}\,\mathcal{C}$, is the formal groupification $(\mathbb{Z}^{(\mathbf{K}_+\,\mathcal{C})}, \oplus)/C$ of $\mathbf{K}_+\,\mathcal{C}$, which is essentially unique by the universal mapping property. In general the natural homomorphism from $\mathbf{K}_+\,\mathcal{C}$ into $\mathbf{K}\,\mathcal{C}$ is not injective. Note that if $\mathbf{K}_+\,\mathcal{C}$ is a commutative semiring then $\mathbf{K}\,\mathcal{C}$ is naturally a commutative ring.

It is easily checked that $\mathbb{E}_k$ induces an injective semigroup homomorphism

$$\mathbf{K}_+\,\mathrm{RV}[k, \cdot] \longrightarrow \mathbf{K}_+\,\mathrm{RV}[k + 1, \cdot],$$

which is also denoted by $\mathbb{E}_k$.

5. RV-pullbacks and special bijections

We shall adopt [16, Convention 4.20]: Since definably bijective subsets are to be identified, for a subset A, we shall tacitly substitute its canonical image $\mathbf{c}(A)$ for it in the discussion if it is necessary or is just more convenient.

For any subset U, recall from [16, Definition 4.21] that the RV-*hull* of U is the union of the rv-polydiscs that have a nonempty intersection with U. If U is equal to its RV-hull then U is an RV-*pullback*. An RV-pullback is *degenerate* if it contains a degenerate rv-polydisc and is *strictly degenerate* if it only contains degenerate rv-polydiscs.

Here comes the general version of [16, Definition 4.22]:

Definition 5.1. Let $A \subseteq \mathrm{VF} \times \mathrm{VF}^n \times \mathrm{RV}^m$. Let $C \subseteq \mathrm{RVH}(A)$ be an RV-pullback and $\lambda : \mathrm{pr}_{>1}(C \cap A) \longrightarrow \mathrm{VF}$ a function such that every $(\lambda(\vec{a}_1, \vec{t}), \vec{a}, \vec{t})$ is in C. Let

$$C^\sharp = \bigcup_{(\vec{a}_1, t_1, \vec{t}_1) \in \mathrm{pr}_{>1} C} \left(\left(\bigcup \left\{ \mathrm{rv}^{-1}(t) : \mathrm{vrv}(t) > \mathrm{vrv}(t_1) \right\} \right) \times \left\{ (\vec{a}_1, t_1, \vec{t}_1) \right\} \right),$$

$$\mathrm{RVH}(A)^\sharp = C^\sharp \uplus (\mathrm{RVH}(A) \smallsetminus C).$$

The *centripetal transformation* $\eta : A \longrightarrow \mathrm{RVH}(A)^\sharp$ *with respect to* λ is defined by

$$\begin{cases} \eta(a_1, \vec{a}_1, \vec{t}) = (a_1 - \lambda(\vec{a}_1, \vec{t}), \vec{a}_1, \vec{t}), & \text{on } C \cap A, \\ \eta = \mathrm{id}, & \text{on } A \smallsetminus C. \end{cases}$$

Note that η is injective. The inverse of η is naturally called the *centrifugal transformation with respect to* λ. The function λ is called a *focus map of* X. The RV-pullback C is called the *locus* of λ. A *special bijection* T is an alternating composition of centripetal transformations and the canonical bijection. The *length* of a special bijection T, denoted by $\mathrm{lh}\, T$, is the number of centripetal transformations in T. The image $T(A)$ is sometimes denoted by $A^\sharp$.

Note that we should have included the index of the targeted VF-coordinate as a part of the data of a focus map. Since it should not cause confusion in context, we shall suppress mentioning it for notational ease.

We shall only be concerned with definable special bijections.

Clearly if A is an RV-pullback and T is a special bijection on A then $T(A)$ is an RV-pullback. Recall that a subset A is called a *deformed* RV-*pullback* if there is a special bijection T such that $T(A)$ is an RV-pullback.

Lemma 5.2. *Every definable subset* $A \subseteq \mathrm{VF} \times \mathrm{RV}^m$ *is a deformed* RV-*pullback.*

Proof. See [16, Lemma 4.26]. □

Remark 5.3. Let $A \subseteq \mathrm{VF} \times \mathrm{RV}^m$ be a deformed RV-pullback and $T : A \longrightarrow U$ a special bijection that witnesses this. By a routine induction, we see that if $\mathrm{rv}^{-1}(s) \times \left\{(s, \vec{t})\right\} \subseteq U$ with $s \neq \infty$ then $T^{-1}(\mathrm{rv}^{-1}(s) \times \left\{(s, \vec{t})\right\})$ is an open polydisc that is contained in an rv-polydisc.

Let $f : A \longrightarrow B$ be a function. We say that f is *contractible* if for every rv-polydisc $\mathfrak{p} \subseteq \mathrm{RVH}(A)$ the subset $f(\mathfrak{p} \cap A)$ is contained in one rv-polydisc. Clearly, if $f : A \longrightarrow B$ is a (definable) contractible function then there is a unique (definable) function $f_\downarrow : \mathrm{rv}(A) \longrightarrow \mathrm{rv}(B)$ such that the diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{rv}"'] & B \arrow[d, "\mathrm{rv}"] \\ \mathrm{rv}(A) \arrow[r, "f_\downarrow"'] & \mathrm{rv}(B) \end{tikzcd}$$

In this case we say that $f_\downarrow$ is the *contraction* of f.

The following technical result is a major tool for the Hrushovski–Kazhdan construction as presented in [15].

Theorem 5.4. *Let* $F(\vec{X}) = F(X_1, \ldots, X_n)$ *be a polynomial with coefficients in* $\mathrm{VF}(S)$, $\vec{u} \in \mathrm{RV}^n$ *a definable tuple,* $\tau : \mathrm{rv}^{-1}(\vec{u}) \longrightarrow A$ *a special bijection, and* $f = F \circ \tau^{-1}$. *Then there is a special bijection* T *on* A *such that* $f \circ T^{-1}$ *is contractible.*

Proof. First observe that if the assertion holds for one polynomial $F(\vec{X})$ then it holds simultaneously for any finite number of polynomials. We do induction on n. For the base case $n = 1$, we simply write X for $\vec{X}$. Let T be a special bijection on A. For any rv-polydisc $\mathfrak{p} \subseteq T(A)$, let $k_T(\mathfrak{p})$ be the size of the set $\{\vec{x} \in \mathfrak{p} : (f \circ T^{-1})(\vec{x}) = 0\}$.

Claim. *There is a special bijection* T^* *on* $T(A)$ *such that* $f \circ (T^* \circ T)^{-1}$ *is contractible.*

Proof. By compactness, we may concentrate on one rv-polydisc $\mathfrak{p} = \mathrm{rv}^{-1}(s) \times \left\{(s, \vec{r})\right\} \subseteq T(A)$. We do induction on $k_T(\mathfrak{p})$. For the base case $k_T(\mathfrak{p}) = 1$, consider the focus map $\lambda : \{(s, \vec{r})\} \longrightarrow \mathrm{VF}$ such that $f(T^{-1}(\lambda(s, \vec{r}), s, \vec{r})) = 0$ and the special bijection T^* on $\mathfrak{p}$ given by

$$(b, s, \vec{r}) \longmapsto (b - \lambda(s, \vec{r}), \mathrm{rv}(b - \lambda(s, \vec{r})), s, \vec{r}).$$

By Remark 5.3, for every rv-polydisc $\mathfrak{r} \subseteq T^*(\mathfrak{p})$, $(T^* \circ T \circ \tau)^{-1}(\mathfrak{r})$ is either the root of $F(X)$ in question or an open ball that contains no roots of any $F(X)$. So T^* is as required.

For the inductive step $k_T(\mathfrak{p}) = m > 1$, let $(d_1, s, \vec{r}), \ldots, (d_m, s, \vec{r}) \in \mathfrak{p}$ be the points in question and d the average of $d_1, \ldots, d_m$. Consider the special bijection T^* on $\mathfrak{p}$ given by

$$(b, s, \vec{r}) \longmapsto (b - d, \mathrm{rv}(b - d), s, \vec{r}).$$

By Lemma 3.8, rv is not constant on $\{d_1 - d, \ldots, d_m - d\}$ and hence $k_{T^* \circ T}(\mathfrak{r}) < m$ for every rv-polydisc $\mathfrak{r} \subseteq T^*(\mathfrak{p})$. So we are done by compactness and the inductive hypothesis. □

This completes the base case of the induction.

We now proceed to the inductive step. As above, we may concentrate on one rv-polydisc $\mathfrak{p} = \mathrm{rv}^{-1}(\vec{s}) \times \left\{(\vec{s}, \vec{r})\right\} \subseteq A$. Let $\phi(\vec{X}, Y)$ be a quantifier-free formula that defines the function $(\mathrm{rv} \circ f) \upharpoonright \mathfrak{p}$, where Y is the free RV-sort variable. Let $G_i(\vec{X})$ enumerate the occurring polynomials of $\phi(\vec{X}, Y)$. For each $a \in \mathrm{rv}^{-1}(s_1)$ let $G_{i,a} = G_i(a, X_2, \ldots, X_n)$. By the inductive hypothesis, there is a special bijection R_a on $\mathrm{rv}^{-1}(s_2, \ldots, s_n)$ such that every function $G_{i,a} \circ R_a^{-1}$ is contractible. Let $U_{j,a}$ enumerate the loci used in R_a and $\lambda_{j,a}$ the corresponding focus maps. By compactness,

(1) for each i there is a quantifier-free formula $\psi_i(X_1, \vec{Z}', Z)$ such that $\psi_i(a, \vec{Z}', Z)$ defines the contraction of $G_{i,a} \circ R_a^{-1}$,
(2) there is a quantifier-free formula $\theta(X_1, \vec{Z}'')$ such that $\theta(a, \vec{Z}'')$ determines the sequence $\mathrm{rv}(U_{j,a})$ and the VF-coordinates targeted by $\lambda_{j,a}$.

Let $H_k(X_1)$ enumerate the occurring polynomials of the formulas $\psi_i(X_1, \vec{Z}', Z), \theta(X_1, \vec{Z}'')$. Applying the inductive hypothesis again, we obtain a special bijection T_1 on $\mathrm{rv}^{-1}(s_1)$ such that every function $H_k \circ T_1^{-1}$ is contractible. This means that, for every rv-polydisc $\mathfrak{q} \subseteq T_1(\mathrm{rv}^{-1}(s_1))$ and every $a_1, a_2 \in T_1^{-1}(\mathfrak{q})$, the formulas $\psi_i(a_1, \vec{W}, Z)$, $\psi_i(a_2, \vec{W}, Z)$ define the same function and the special bijections R_{a_1}, R_{a_2} may be naturally glued together to form one special bijection on $\{a_1, a_2\} \times \mathrm{rv}^{-1}(s_2, \ldots, s_n)$. Consequently, T_1 and R_a naturally induce a special bijection T on $\mathfrak{p}$ such that each function $G_i \circ T^{-1}$ is contractible. This implies that $f \circ T^{-1}$ is contractible. □

We immediately give a slightly more general version of Theorem 5.4, which is easier to use:

Theorem 5.5. *Let $F(\vec{X}) = F(X_1, \ldots, X_n)$ be a polynomial with coefficients in $\mathrm{VF}(S)$, $B \subseteq \mathrm{VF}^n$ a definable subset, $\tau : B \longrightarrow A$ a special bijection, and $f = F \circ \tau^{-1}$. Then there is a special bijection T on A such that $T(A)$ is an RV-pullback and $f \circ T^{-1}$ is contractible.*

Proof. By compactness, we may concentrate on a subset of the form $A_{\mathfrak{p}} = \mathfrak{p} \cap A$, where $\mathfrak{p}$ is an rv-polydisc. Let $\phi(\vec{X}, Z)$ be a quantifier-free formula that defines the function $(\mathrm{rv} \circ f) \upharpoonright A_{\mathfrak{p}}$. Let $F_i(\vec{X})$ enumerate the occurring polynomials of $\phi(\vec{X}, Z)$. By Theorem 5.4 there is a special bijection T on $\mathfrak{p}$ such that each function $F_i \circ T^{-1}$ is contractible. This means that, for each rv-polydisc $\mathfrak{q} \subseteq T(\mathfrak{p})$,

(1) either $T^{-1}(\mathfrak{q}) \subseteq A_{\mathfrak{p}}$ or $T^{-1}(\mathfrak{q}) \cap A_{\mathfrak{p}} = \emptyset$,
(2) if $T^{-1}(\mathfrak{q}) \subseteq A_{\mathfrak{p}}$ then $(\mathrm{rv} \circ f \circ T^{-1})(\mathfrak{q})$ is a singleton.

So $T \upharpoonright A_{\mathfrak{p}}$ is as required. □

Now Lemma 5.2 may be easily generalized to all dimensions:

Corollary 5.6. *Every definable subset $A \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$ is a definable deformed RV-pullback.*

Proof. By compactness, we may assume that A is contained in an rv-polydisc. Then the assertion simply follows from Theorem 5.5. □

Applying Lemmas 4.2 and 4.13, we get the following corollary:

Corollary 5.7. *The map $\mathbb{L} : \mathrm{Ob}\,\mathrm{RV}[k, \cdot] \longrightarrow \mathrm{Ob}\,\mathrm{VF}[k, \cdot]$ is surjective on the isomorphism classes of $\mathrm{VF}[k, \cdot]$. The map $\mathbb{L} : \mathrm{Ob}\,\mathrm{RV}[k] \longrightarrow \mathrm{Ob}\,\mathrm{VF}[k]$ is surjective on the isomorphism classes of $\mathrm{VF}[k]$.*

6. Interlude: quantifier elimination for henselian fields

The analysis on special transformations in Section 5 leads to a general quantifier elimination result for henselian fields. Pas' quantifier elimination result [12, Theorem 4.1] may be recovered from it.

Definition 6.1. A substructure M is *functionally closed* if, for any definable subset A and any definable function f on A, $f(A \cap M) \subseteq M$.

Lemma 6.2. *Let M be a substructure such that $(\mathrm{VF}(M), \mathcal{O}(M))$ is a nontrivially valued henselian field and $\mathrm{rv}(\mathrm{VF}(M)) = \mathrm{RV}(M)$. Then M is functionally closed.*

Proof. By Corollary 3.14, $\mathrm{acl}(M) \models \mathrm{ACVF}_S(0, 0)$. Note that $\mathrm{VF}(\mathrm{acl}(M)) = \mathrm{VF}(M)^{\mathrm{ac}}$. Since the valued field $(\mathrm{VF}(M), \mathcal{O}(M))$ is henselian, it is the fixed field under the valued field automorphisms of $(\mathrm{VF}(M)^{\mathrm{ac}}, \mathcal{O}(\mathrm{acl}(M)))$ over $(\mathrm{VF}(M), \mathcal{O}(M))$. On the other hand, these valued field automorphisms are in one-to-one correspondence with the $\mathcal{L}_{\mathrm{RV}}$-automorphisms of $\mathrm{acl}(M)$ over M. So $\mathrm{VF}(\mathrm{dcl}(M)) = \mathrm{VF}(M)$. Since M is VF-generated, by Lemma 3.13, every $t \in \mathrm{RV}(\mathrm{dcl}(M))$ has an M-definable point in VF. So $M = \mathrm{dcl}(M)$ and the lemma follows. □

Let $\mathrm{HEN}_S(0,0)$ be the theory of henselian fields of pure characteristic 0 in a language $\mathcal{L}_{\mathrm{H}}$ that expands $\mathcal{L}_{\mathrm{RV}}$, where the expansion happens only in the RV-sort. Such a theory may be formulated as in Definition 2.2, with obvious modifications. Note that $\mathrm{HEN}_S(0,0)$ includes the statement that the function rv is surjective.

Lemma 6.3. *Let $\phi(\vec{X})$ be a* VF*-quantifier-free $\mathcal{L}_{\mathrm{H}}$-formula, where $\vec{X} = (X_1, \ldots, X_n)$ are the free* VF*-sort variables. Then* $\mathrm{HEN}_S(0,0)$ *proves that $\exists \vec{X}\, \phi(\vec{X})$ is equivalent to a* VF*-quantifier-free $\mathcal{L}_{\mathrm{H}}$-formula.*

Proof. Without loss of generality, we may assume that $\phi(\vec{X})$ contains no VF-sort literals. Let $F_i(\vec{X})$ be the occurring polynomials of $\phi(\vec{X}, \vec{Y})$. Let $\phi^*(\vec{Z})$ be the formula obtained from $\phi(\vec{X})$ by replacing each term $\mathrm{rv}(F_i(\vec{X}))$ with a new RV-sort variable Z_i. Let $M \models \mathrm{HEN}_S(0,0)$ such that its reduct to $\mathcal{L}_{\mathrm{RV}}$ is a substructure of $\mathfrak{C}$.

By Theorem 5.5, there is an RV-pullback A and an $\mathcal{L}_{\mathrm{RV}}$-definable bijection $T : A \longrightarrow \mathrm{VF}^n$ such that, for every rv-polydisc $\mathfrak{p} \subseteq A$, every subset of the form $\mathrm{rv}(F_i(T(\mathfrak{p})))$ is a singleton. This induces functions $f_i : \mathrm{rv}(A) \longrightarrow \mathrm{RV}$, defined by quantifier-free $\mathcal{L}_{\mathrm{RV}}$-formulas $\psi_i(\vec{Y}, Z_i)$ (hence no VF-sort quantifiers). By Lemma 6.2, $T^{-1}(M \cap \mathrm{VF}^n) \subseteq M$ and hence $T^{-1}(M \cap \mathrm{VF}^n) = A \cap M$. Similarly $f_i(\mathrm{rv}(A \cap M)) \subseteq M$ for every i. Therefore

$$M \models \exists \vec{X}\, \phi(\vec{X}) \leftrightarrow \exists \vec{Y}, \vec{Z} \left(\bigwedge_i \psi_i(\vec{Y}, Z_i) \wedge \phi^*(\vec{Z}) \right).$$

The lemma follows. $\square$

By elementary logic this lemma yields:

Proposition 6.4. *The theory* $\mathrm{HEN}_S(0,0)$ *admits elimination of* VF*-quantifiers.*

If angular component map exists then $\mathrm{RV}^\times$ may be understood as $\overline{\mathrm{K}}^\times \oplus \Gamma$. Hence we have the following:

Corollary 6.5 (*[12, Theorem 4.1]*). *The theory of henselian fields of pure characteristic* 0 *in any Denef–Pas language admits elimination of field sort quantifiers.*

The "descent" technique in this section can also be applied to theories of henselian fields with sections, which are formulated in a natural way as in [16]. This will be explained elsewhere.

7. Lifting functions from RV to VF

We shall show in this section that the map $\mathbb{L}$ actually induces homomorphisms between various Grothendieck semigroups when S is a (VF, Γ)-generated substructure.

Any polynomial in $\mathcal{O}[\vec{X}]$ corresponds to a polynomial in $\overline{\mathrm{K}}[\vec{X}]$ via the canonical quotient map. The following definition generalizes this phenomenon.

Definition 7.1. Let $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \Gamma$. A polynomial $F(\vec{X}) = \sum_{\vec{ij}} a_{\vec{ij}} \vec{X}^{\vec{i}}$ with coefficients $a_{\vec{ij}} \in \mathrm{VF}$ is a *$\vec{\gamma}$-polynomial* if there is an $\alpha \in \Gamma$ such that

$$\alpha = \mathrm{val}(a_{\vec{ij}}) + i_1\gamma_1 + \cdots + i_n\gamma_n$$

for each $\vec{ij} = (i_1, \ldots, i_n, j)$. In this case we say that α is a *residue value* of $F(\vec{X})$ (with respect to $\vec{\gamma}$). For a $\vec{\gamma}$-polynomial $F(\vec{X})$ with residue value α and a $\vec{t} \in \mathrm{RV}$ with $\mathrm{vrv}(\vec{t}) = \vec{\gamma}$, if $\mathrm{val}(F(\vec{a})) > \alpha$ for some (hence all) $\vec{a} \in \mathrm{rv}^{-1}(\vec{t})$ then $\vec{t}$ is a *residue root* of $F(\vec{X})$.

If $\vec{t} \in \mathrm{RV}$ is a common residue root of the $\vec{\gamma}$-polynomials $F_1(\vec{X}), \ldots, F_n(\vec{X})$ but is not a residue root of the $\vec{\gamma}$-polynomial

$$\det \partial(F_1, \ldots, F_n)/\partial \vec{X},$$

then we say that $F_1(\vec{X}), \ldots, F_n(\vec{X})$ are *minimal* for $\vec{t}$ and $\vec{t}$ is a *simple* common residue root of $F_1(\vec{X}), \ldots, F_n(\vec{X})$.

Therefore, according to this definition, every polynomial in $\overline{\mathrm{K}}[\vec{X}]$ is the projection of a $\vec{0}$-polynomial $F(\vec{X})$ with residue value 0, where $\vec{0} = (0, \ldots, 0)$.

Hensel's lemma is accordingly generalized as follows.

Lemma 7.2 (*Generalized Hensel's Lemma*). *Let $F_1(\vec{X}), \ldots, F_n(\vec{X})$ be $\vec{\gamma}$-polynomials with residue values $\alpha_1, \ldots, \alpha_n$, where $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \Gamma$. For every simple common residue root $\vec{t} = (t_1, \ldots, t_n) \in \mathrm{RV}$ of $F_1(\vec{X}), \ldots, F_n(\vec{X})$ there is a unique $\vec{a} \in \mathrm{rv}^{-1}(\vec{t})$ such that $F_i(\vec{a}) = 0$ for every i.*

Proof. Without loss of generality we may work in a topologically complete submodel of ACVF of rank 1.

Fix a simple common residue root $\vec{t} = (t_1, \ldots, t_n) \in \mathrm{RV}$ of $F_1(\vec{X}), \ldots, F_n(\vec{X})$. Choose a $c_i \in \mathrm{rv}^{-1}(t_i)$. Changing the coefficients accordingly we may rewrite each $F_i(\vec{X})$ as $F_i(X_1/c_1, \ldots, X_n/c_n)$. Write Y_i for X_i/c_i. Note that, for each i, the coefficients of the $\vec{0}$-polynomial $F_i(\vec{Y})$ are all of the same value α_i. For each i choose an $e_i \in \mathrm{VF}$ with $\mathrm{val}(e_i) = -\alpha_i$. We have that each $\vec{0}$-polynomial $F_i^*(\vec{Y}) = e_i F_i(\vec{Y})$ has residue value 0 (that is, the coefficients of $F_i^*(\vec{Y})$ is of value 0). Clearly $(1, \ldots, 1)$ is a common residue root of $F_1^*(\vec{Y}), \ldots, F_n^*(\vec{Y})$; that is, for every $\vec{a} \in \mathrm{rv}^{-1}(1, \ldots, 1)$ and every i we have $\mathrm{val}(F_i^*(\vec{a})) > 0$. It is actually a simple root because for every $\vec{a} \in \mathrm{rv}^{-1}(1, \ldots, 1)$ we have

$$\det \partial(F_1^*, \ldots, F_n^*)/\partial \vec{Y}(\vec{a}) = \left(\prod_i e_i c_i \right) \cdot \det \partial(F_1, \ldots, F_n)/\partial \vec{X}(\vec{a}\vec{c}),$$

where $\vec{ac} = (a_1c_1, \dots, a_nc_n)$, and hence

$$\operatorname{val}(\det \partial(F_1^*, \dots, F_n^*)/\partial \vec{Y}(\vec{a})) = \sum_i (-\alpha_i + \gamma_i) + \sum_i \alpha_i - \sum_i \gamma_i = 0.$$

Now the lemma follows from the multivariate version of Hensel's lemma (for example, see [2, Corollary 2, p. 224]). □

Definition 7.3. Let B, C be two RV-pullbacks, A a subset of $B \times C$, and U a subset of $\operatorname{rv}(B \times C)$. We say that A is a (B, C)-*lift of U from* RV *to* VF, or just a *lift of U* for short, if $A \cap (\mathfrak{p} \times \mathfrak{q})$ is a bijection from $\mathfrak{p}$ onto $\mathfrak{q}$ for any rv-polydiscs $\mathfrak{p} \subseteq B$ and $\mathfrak{q} \subseteq C$ with $\operatorname{rv}(\mathfrak{p} \times \mathfrak{q}) \in U$. A *partial lift of U* is a lift of any subset of U.

For any RV$[k, \cdot]$-isomorphism $F : (U, f) \longrightarrow (V, g)$, a lift $F^\uparrow$ of F is actually an $(\mathbb{L}(U, f), \mathbb{L}(V, g))$-lift of the induced function $F_{f,g} : U_f \longrightarrow V_g$; that is, $F^\uparrow$ is a function on $\mathbb{L}(U, f)$ such that each restriction

$$F^\uparrow : \operatorname{rv}^{-1}(f(\vec{u}), \vec{u}) \longrightarrow \operatorname{rv}^{-1}((g \circ F)(\vec{u}), F(\vec{u}))$$

is a bijection.

It would be ideal to lift all definable subsets of $\mathrm{RV}^n \times \mathrm{RV}^n$ with finite-to-finite correspondence for any substructure S. However, the following crucial lemma fails when S is not (VF, Γ)-generated.

Lemma 7.4. *Suppose that S is* (VF, Γ)*-generated. Let $\vec{t} = (t_1, \dots, t_n) \in$ RV with $t_n \in \operatorname{acl}(t_1, \dots, t_{n-1})$. Let $\operatorname{vrv}(\vec{t}) = (\gamma_1, \dots, \gamma_n) = \vec{\gamma} \in \Gamma$. Then there is a $\vec{\gamma}$-polynomial $F(X_1, \dots, X_n)$ with coefficients in* VF(S) *such that $\vec{t}$ is a residue root of $F(\vec{X})$ but is not a residue root of $\partial F(\vec{X})/\partial X_n$.*

Proof. Write $(t_1, \dots, t_{n-1})$ as $\vec{t}_n$. Let $\phi(\vec{X})$ be a quantifier-free formula such that $\phi(\vec{t}_n, X_n)$ defines a finite subset that contains t_n. Without loss of generality we may assume that $\phi(\vec{X})$ is an RV-sort equality such that $\phi(\vec{t}_n, X_n)$ defines a finite subset. Since S is (VF, Γ)-generated, we may assume that $\phi(\vec{X})$ does not contain parameters from $\mathrm{RV}(S) \smallsetminus \operatorname{rv}(\mathrm{VF}(S))$. Hence it is of the form

$$\vec{X}^{\vec{k}} \cdot \sum_{\vec{i}} (\operatorname{rv}(a_{\vec{i}}) \cdot \vec{X}^{\vec{i}}) = \operatorname{rv}(a) \cdot \vec{X}^{\vec{l}} \cdot \sum_{\vec{j}} (\operatorname{rv}(a_{\vec{j}}) \cdot \vec{X}^{\vec{j}}),$$

where $a_{\vec{i}}, a, a_{\vec{j}} \in \mathrm{VF}(S)$. Fix an $s \in \mathrm{RV}$ such that $\operatorname{vrv}(s \cdot \vec{t}^{\vec{k}}) = \operatorname{vrv}(s \cdot \operatorname{rv}(a) \cdot \vec{t}^{\vec{l}}) = 0$. Let $\operatorname{vrv}(s) = \delta$. Note that δ is $\vec{t}_n$-definable. Let

$$T_1(\vec{X}, s) = \sum_{\vec{i}} (s \cdot \operatorname{rv}(a_{\vec{i}}) \cdot \vec{X}^{\vec{i}+\vec{k}}), \quad T_2(\vec{X}, s) = \sum_{\vec{j}} (-s \cdot \operatorname{rv}(a a_{\vec{j}}) \cdot \vec{X}^{\vec{j}+\vec{l}}).$$

Consider the RV-sort polynomial $H(\vec{X}, s) = T_1(\vec{X}, s) + T_2(\vec{X}, s)$. For any $r \in \mathrm{RV}$, $H(\vec{t}_n, r, s) = 0$ if and only if

$$\text{either} \quad \sum_{\vec{i}} (\operatorname{rv}(a_{\vec{i}}) \cdot (\vec{t}_n, r)^{\vec{i}}) = \sum_{\vec{j}} (\operatorname{rv}(a_{\vec{j}}) \cdot (\vec{t}_n, r)^{\vec{j}}) = 0 \quad \text{or} \quad \operatorname{rv}(T_1(\vec{t}_n, r, s)/s) = \operatorname{rv}(-T_2(\vec{t}_n, r, s)/s).$$

So the equation $H(\vec{t}_n, X_n, s) = 0$ defines a finite subset that contains t_n and is actually $\vec{t}_n$-definable.

Let m be the maximal exponent of X_n in $H(\vec{X}, s)$. For each $i \leq m$ let $H_i(\vec{X}, s)$ be the sum of all the monomials $M(\vec{X}, s)$ in $H(\vec{X}, s)$ such that the exponent of X_n in $M(\vec{X}, s)$ is i. Replacing s with a variable Y and each $\operatorname{rv}(a)$ with a in $H_i(\vec{X}, s)$, we obtain a VF-sort polynomial $H_i^*(\vec{X}, Y)$ for each $i \leq m$. Let

$$E = \{i \leq m : \operatorname{val}(H_i^*(\vec{b}, c)) = 0 \text{ for all } (\vec{b}, c) \in \operatorname{rv}^{-1}(\vec{t}, s)\}.$$

Since $H(\vec{t}, s) = 0$, clearly $|E| \neq 1$. Since the equation $H(\vec{t}_n, X_n, s) = 0$ defines a finite subset, we actually have $|E| > 1$. Now let

$$H^*(\vec{X}, Y) = \sum_{i \in E} H_i^*(\vec{X}, Y) = \sum_{i \in E} Y X_n^i G_i(\vec{X}_n) = Y G(\vec{X}).$$

Since $(\vec{t}, s)$ is a residue root of $H^*(\vec{X}, Y)$, clearly $G(\vec{X})$ is a $\vec{\gamma}$-polynomial with residue value $-\delta$ and $\vec{t}$ is a residue root of $G(\vec{X})$. Also, $\vec{t}_n$ is not a residue root of any $G_i(\vec{X}_n)$. It follows that, for some $k < \max E$, $\vec{t}$ is a residue root of the $\vec{\gamma}$-polynomial $\partial G(\vec{X})/\partial^k X_n$ but is not a residue root of the $\vec{\gamma}$-polynomial $\partial G(\vec{X})/\partial^{k+1} X_n$. □

Remark 7.5. For definable subsets of the residue field, the situation may be further simplified. Suppose that $A \subseteq \overline{\mathrm{K}}^n$ is definable. Let $\phi(\vec{X})$ be a quantifier-free formula in disjunctive normal form that defines A. It is easily seen by inspection that each conjunct in each disjunct of $\phi(\vec{X})$ is either an RV-sort equality or an RV-sort disequality, with coefficients in $\overline{\mathrm{K}}(S)$. So the geometry of definable subsets in the residue field coincides with its algebraic geometry. In other words, each definable subset in the residue field is a constructible subset (in the sense of algebraic geometry) of a Zariski topological space $\operatorname{Spec} \overline{\mathrm{K}}(S)[\vec{X}]$.

Theorem 7.6. *Suppose that the substructure S is* (VF, Γ)*-generated. Let $C \subseteq (\mathrm{RV}^\times)^n \times (\mathrm{RV}^\times)^n$ be a definable subset such that both $\mathrm{pr}_{\leq n} \upharpoonright C$ and $\mathrm{pr}_{>n} \upharpoonright C$ are finite-to-one. Then there is a definable subset $C^\uparrow \subseteq \mathrm{VF}^n \times \mathrm{VF}^n$ that lifts C.*

Proof. By compactness, the lemma is reduced to showing that for every $(\vec{t}, \vec{s}) \in C$ there is a definable lift of some subset of C that contains $(\vec{t}, \vec{s})$. Fix a $(\vec{t}, \vec{s}) \in C$ and set $(\vec{\gamma}, \vec{\delta}) = \mathrm{vrv}(\vec{t}, \vec{s})$. Let $\phi(\vec{X}, \vec{Y})$ be a formula that defines C. By Lemma 7.4, for each Y_i there is a $(\vec{\gamma}, \delta_i)$-polynomial $F_i(\vec{X}, Y_i)$ with coefficients in $\mathrm{VF}(S)$ such that $(\vec{t}, s_i)$ is a residue root of $F_i(\vec{X}, Y_i)$ but is not a residue root of $\partial F_i(\vec{X}, Y_i)/\partial Y_i$. Similarly we obtain such a $(\gamma_i, \vec{\delta})$-polynomial $G_i(X_i, \vec{Y})$ for each X_i. For each i, let $a_i(\vec{X}\vec{Y})^{\vec{k}_i}$ and $b_i(\vec{X}\vec{Y})^{\vec{l}_i}$ be two monomials with $a_i, b_i \in \mathrm{VF}(S)$ such that

$$F_i^*(\vec{X}, \vec{Y}) + G_i^*(\vec{X}, \vec{Y}) = a_i(\vec{X}\vec{Y})^{\vec{k}_i} F_i(\vec{X}, Y_i) + b_i(\vec{X}\vec{Y})^{\vec{l}_i} G_i(X_i, \vec{Y})$$

is a $(\vec{\gamma}, \vec{\delta})$-polynomial. Let α_i be the residue value of $F_i^*(\vec{X}, \vec{Y}) + G_i^*(\vec{X}, \vec{Y})$. Note that for any $(\vec{a}, \vec{b}) \in \mathrm{rv}^{-1}(\vec{t}, \vec{s})$ we have

$$\mathrm{val}(\partial F_i^*/\partial Y_i(\vec{a}, \vec{b})) = \mathrm{val}(a_i(\vec{a}\vec{b})^{\vec{k}_i}) + \mathrm{val}(\partial F_i/\partial Y_i(\vec{a}, \vec{b})) = \alpha_i - \delta_i$$

and for $j \neq i$ we have

$$\mathrm{val}(\partial F_i^*/\partial Y_j(\vec{a}, \vec{b})) = \mathrm{val}(a_i) + \mathrm{val}(\partial(\vec{X}\vec{Y})^{\vec{k}_i}/\partial Y_j(\vec{a}, \vec{b})) + \mathrm{val}(F_i(\vec{a}, b_i)) > \alpha_i - \delta_j.$$

Therefore,

$$\mathrm{val}(\det \partial(F_1^*, \ldots, F_n^*)/\partial\vec{Y}(\vec{a}, \vec{b})) = \mathrm{val}\left(\prod_i \partial F_i^*/\partial Y_i(\vec{a}, \vec{b})\right) = \sum_i \alpha_i - \sum_i \delta_i.$$

This shows that $\vec{s}$ is a simple common residue root of $F_1^*(\vec{a}, \vec{Y}), \ldots, F_n^*(\vec{a}, \vec{Y})$ for any $\vec{a} \in \mathrm{rv}^{-1}(\vec{t})$. Similarly $\vec{t}$ is a simple common residue root of $G_1^*(\vec{X}, \vec{b}), \ldots, G_n^*(\vec{X}, \vec{b})$ for any $\vec{b} \in \mathrm{rv}^{-1}(\vec{s})$.

Now for each i we choose a pair of integers p_i, q_i. Consider the $(\vec{\gamma}, \vec{\delta})$-polynomials

$$H_i(\vec{X}, \vec{Y}) = p_i F_i^*(\vec{X}, \vec{Y}) + q_i G_i^*(\vec{X}, \vec{Y}).$$

Let $\sigma \in S_n$ be a permutation and $\tau(\vec{X}, \vec{Y})$ a term in the expansion of the product $\prod_i \partial H_i(\vec{X}, \vec{Y})/\partial Y_{\sigma(i)}$. The coefficient c_τ of $\tau(\vec{X}, \vec{Y})$ is of the form $\prod_i m_i$, where m_i is either p_i or q_i. Suppose that

$$\mathrm{val}(\tau(\vec{a}, \vec{b})) = \sum_i \alpha_i - \sum_i \delta_i$$

for some (hence all) $(\vec{a}, \vec{b}) \in \mathrm{rv}^{-1}(\vec{t}, \vec{s})$. Then $\mathrm{rv}(\tau(\vec{X}, \vec{Y}))$ is constant on $\mathrm{rv}^{-1}(\vec{t}, \vec{s})$, which is denoted by $\mathrm{rv}(\tau)$. Observe that there is only one such term with coefficient $\prod_i p_i$, namely $\prod_i \partial(p_i F_i^*)/\partial Y_i$. Let τ_i enumerate all such terms other than $\prod_i \partial(p_i F_i^*)/\partial Y_i$. It is not hard to see that p_i, q_i may be chosen so that

$$1 + \sum_i \mathrm{rv}(\tau_i)/\,\mathrm{rv}\left(\prod_i \partial(p_i F_i^*)/\partial Y_i\right) \neq 0.$$

This implies that, for all $(\vec{a}, \vec{b}) \in \mathrm{rv}^{-1}(\vec{t}, \vec{s})$,

$$\mathrm{val}(\det \partial(H_1, \ldots, H_n)/\partial\vec{Y}(\vec{a}, \vec{b})) = \sum_i \alpha_i - \sum_i \delta_i$$

and hence $\vec{s}$ is a simple common residue root of the $\vec{\delta}$-polynomials $H_1(\vec{a}, \vec{Y}), \ldots, H_n(\vec{a}, \vec{Y})$ for any $\vec{a} \in \mathrm{rv}^{-1}(\vec{t})$. In fact the choice of p_i, q_i can be improved so that we also have, for all $(\vec{a}, \vec{b}) \in \mathrm{rv}^{-1}(\vec{t}, \vec{s})$,

$$\mathrm{val}(\det \partial(H_1, \ldots, H_n)/\partial\vec{X}(\vec{a}, \vec{b})) = \sum_i \alpha_i - \sum_i \gamma_i$$

and hence $\vec{t}$ is a simple common residue root of the $\vec{\gamma}$-polynomials $H_1(\vec{X}, \vec{b}), \ldots, H_n(\vec{X}, \vec{b})$ for any $\vec{b} \in \mathrm{rv}^{-1}(\vec{s})$. By Lemma 7.2, for each $\vec{a} \in \mathrm{rv}^{-1}(\vec{t})$ there is a unique $\vec{b} \in \mathrm{rv}^{-1}(\vec{s})$ such that $\bigwedge_i H_i(\vec{a}, \vec{b}) = 0$, and vice versa. □

Corollary 7.7. *Suppose that the substructure S is* (VF, Γ)*-generated. The map $\mathbb{L}$ induces surjective homomorphisms between various Grothendieck semigroups, for example:*

$$\mathbf{K}_+ \,\mathrm{RV}[k, \cdot] \longrightarrow \mathbf{K}_+ \,\mathrm{VF}[k, \cdot], \quad \mathbf{K}_+ \,\mathrm{RV}[k] \longrightarrow \mathbf{K}_+ \,\mathrm{VF}[k].$$

Proof. For any $\mathrm{RV}[k, \cdot]$-isomorphism $F : (U, f) \longrightarrow (V, g)$ and any $\vec{u} \in U$, by definition, $\mathrm{wgt}(f(\vec{u})) = \mathrm{wgt}((g \circ F)(\vec{u}))$. Let

$$C = \left\{(f(\vec{u}), (g \circ F)(\vec{u})) : \vec{u} \in U\right\} \subseteq \mathrm{RV}^k \times \mathrm{RV}^k.$$

By Theorem 7.6 there is a lift $C^\uparrow$ of C, which induces a $\mathrm{VF}[k, \cdot]$-isomorphism between $\mathbb{L}(U, f)$ and $\mathbb{L}(V, g)$. So $\mathbb{L}$ induces a map on the isomorphism classes, which is clearly a semigroup homomorphism. By Corollary 5.7 it is surjective. The other cases are handled similarly. □

8. More on structural properties

Lemma 8.1. *Let $A \subseteq \mathrm{VF}^n$ be a definable subset. Suppose that there is a $\gamma \in \Gamma$ such that $\mathfrak{o}(\vec{a}', \gamma) \cap \mathfrak{o}(\vec{a}'', \gamma) = \emptyset$ for every $\vec{a}'$, $\vec{a}'' \in A$. Then A is finite.*

Proof. We do induction on n. The base case $n = 1$ just follows from C-minimality. For the inductive step, consider the subset $\mathrm{pr}_1(A) = A_1$. If A_1 is finite then by the inductive hypothesis $\mathrm{fib}(A, a)$ is finite for every $a \in A_1$ and hence A is finite. If A_1 is infinite then by C-minimality there is an open ball $\mathfrak{b} \subseteq A_1$ with $\mathrm{rad}(\mathfrak{b}) > \gamma$. For any $a' \in \mathfrak{b}$, $a'' \in \mathfrak{b}$, $\vec{b}' \in \mathrm{fib}(A, a')$, and $\vec{b}'' \in \mathrm{fib}(A, a'')$, if $\mathfrak{o}(\vec{b}', \gamma) \cap \mathfrak{o}(\vec{b}'', \gamma) \neq \emptyset$ then $\mathfrak{o}((a', \vec{b}'), \gamma) \cap \mathfrak{o}((a'', \vec{b}''), \gamma) \neq \emptyset$, contradicting the assumption. Therefore, by the inductive hypothesis again, $\bigcup_{a\in\mathfrak{b}} \mathrm{fib}(A, a)$ is finite. So there is a $\vec{b} \in \bigcup_{a\in\mathfrak{b}} \mathrm{fib}(A, a)$ such that $\mathrm{fib}(A, \vec{b}) \cap \mathfrak{b}$ is infinite, contradiction again. □

Lemma 8.2. *Let $f : \mathrm{VF}^n \longrightarrow \mathrm{VF}^m$ be a definable function. Let $A \subseteq \mathrm{VF}^n$ be the definable subset of those $\vec{a} \in \mathrm{VF}^n$ such that there are $\epsilon, \delta \in \Gamma$ with*

$$\mathfrak{o}(\vec{a}, \delta) \cap f^{-1}(\mathfrak{o}(f(\vec{a}), \epsilon)) = \left\{\vec{a}\right\}.$$

Then $\dim_{\mathrm{VF}}(A) < n$.

Proof. For each $\vec{a} \in A$ let $(\epsilon_{\vec{a}}, \delta_{\vec{a}}) \in \Gamma^2$ be an $\vec{a}$-definable pair that satisfies the condition above, which exists by o-minimality. Let $h : A \longrightarrow \Gamma^2$ be the definable function given by $\vec{a} \longmapsto (\epsilon_{\vec{a}}, \delta_{\vec{a}})$. Suppose for contradiction that $\dim_{\mathrm{VF}}(A) = n$. Then, by compactness and Lemma 4.6, there is a pair $(\epsilon_{\vec{a}}, \delta_{\vec{a}}) \in \Gamma^2$ such that $h^{-1}(\epsilon_{\vec{a}}, \delta_{\vec{a}})$ contains an open polydisc $\mathfrak{p}$. Without loss of generality we may assume $\vec{a} \in \mathfrak{p}$. Fix an $\vec{a}$-definable $\gamma \geq \delta_{\vec{a}}$. If $\vec{a}', \vec{a}'' \in \mathfrak{o}(\vec{a}, \gamma)$ are distinct then $\mathfrak{o}(f(\vec{a}'), \epsilon_{\vec{a}}) \cap \mathfrak{o}(f(\vec{a}''), \epsilon_{\vec{a}}) = \emptyset$. By Lemma 8.1, $f(\mathfrak{o}(\vec{a}, \gamma))$ is finite, which is a contradiction. □

Let A be a definable subset with $\dim_{\mathrm{VF}}(A) = n$. A property holds *almost everywhere on A* or *for almost every element in A* if there is a definable subset $B \subseteq A$ with $\dim_{\mathrm{VF}}(B) < n$ such that the property holds with respect to $A \smallsetminus B$. For example, if $f : \mathrm{VF}^n \longrightarrow \mathrm{VF}^m$ is a definable function, then the property that defines the subset A in Lemma 8.2 *does not* hold almost everywhere on VF^n. This terminology is also used with respect to RV-dimension.

Lemma 8.3. *Let $f : \mathrm{VF} \times \mathrm{VF}^k \longrightarrow \mathrm{VF}^m$ be a definable function. Then there are a definable subset $A \subseteq \mathrm{VF} \times \mathrm{VF}^k$ over VF^k and a finite set E of positive rational numbers such that*

(1) $\mathrm{VF} \smallsetminus \mathrm{fib}(A, \vec{b})$ *is finite for all* $\vec{b} \in \mathrm{VF}^k$,
(2) *for every $\vec{a} = (a, \vec{b}) \in A$ there are $\vec{a}$-definable $\epsilon, \delta \in \Gamma$ and a number $k \in E$ such that either $f \restriction \mathfrak{o}(a, \delta) \times \{\vec{b}\}$ is constant or, for any $a' \in \mathfrak{o}(a, \delta)$,*

$$\mathrm{val}(f(a', \vec{b}) - f(a, \vec{b})) = \epsilon + k\,\mathrm{val}(a' - a).$$

Proof. For every $\vec{b} \in \mathrm{VF}^k$ let $B_{\vec{b}} \subseteq \mathrm{VF} \times \{\vec{b}\}$ be as given by Lemma 8.2 with respect to the function $f \restriction \mathrm{VF} \times \{\vec{b}\}$. By compactness $A = \mathrm{VF}^{k+1} \smallsetminus \bigcup_{\vec{b}\in\mathrm{VF}^k} B_{\vec{b}}$ is definable. Let $\phi(X_1, X_2, \vec{Y}, Z)$ be a quantifier-free $\mathcal{L}_{\mathrm{v}}$-formula, possibly with additional parameters from VF, that defines the function on $\mathrm{VF}^2 \times \mathrm{VF}^k$ given by

$$(a', a, \vec{b}) \longmapsto \mathrm{val}(f(a', \vec{b}) - f(a, \vec{b})).$$

Fix an $\vec{a} = (a, \vec{b}) \in A$ such that $f \restriction \mathrm{VF} \times \{\vec{b}\}$ is not constant on any open ball around a. For any term of the form $\mathrm{val}(G(X_1, X_2, \vec{Y}))$ in $\phi(X_1, X_2, \vec{Y}, Z)$ there is an $\vec{a}$-definable $\alpha \in \Gamma \cup \{\infty\}$ and an integer $l \geq 0$ such that, for any $a + d \in \mathrm{VF}$, if $\mathrm{val}(d)$ is sufficiently large then

$$\mathrm{val}(G(a + d, a, \vec{b})) = \alpha + l\,\mathrm{val}(d).$$

Therefore, there is an $\epsilon \in \Gamma \cup \{\infty\}$ and a rational number $k \geq 0$ such that for any sufficiently large $\delta \in \Gamma$, the formula

$$\mathrm{val}(X) > \delta \wedge \phi(a + X, a, \vec{b}, Z)$$

defines a function on $\mathfrak{o}(a, \delta) \times \{\vec{b}\}$ that is given by the equation $Z = \epsilon + k\,\mathrm{val}(X)$. Note that, by the choice of $\vec{a}$, we actually must have $k > 0$ and $\epsilon \neq \infty$. Since Γ is o-minimal, ϵ and some δ are $\vec{a}$-definable. Now it is easy to see that the number k is provided by the exponents of X_1 in $\phi(X_1, X_2, \vec{Y}, Z)$ and hence there are only finitely many choices. □

Lemma 8.4. *Let $\mathfrak{a}, \mathfrak{b}$ be open balls around 0 and $f : \mathfrak{a} \longrightarrow \mathfrak{b}$ a definable bijection that takes open balls around 0 to open balls around 0. Then there are definable $\gamma, \epsilon \in \Gamma$ such that $\mathrm{val}(f(a)) = \epsilon + \mathrm{val}(a)$ for every $a \in \mathfrak{o}(0, \gamma)$.*

Proof. By the proof of Lemma 8.3 we may assume that there is a definable $\epsilon \in \Gamma$ and a positive rational number k such that $\mathrm{val}(f(a)) = \epsilon + k\,\mathrm{val}(a)$. We need to show that $k = 1$.

Suppose for contradiction $k \neq 1$. Let $\phi(X, Y)$ be a quantifier-free $\mathcal{L}_{\mathrm{v}}$-formula, possibly with additional parameters from VF, that defines f. Let $F_i(X, Y)$ be the occurring polynomials of $\phi(X, Y)$. If $a \in \mathfrak{a}$ then $F_i(a, f(a)) = 0$ for some i, since otherwise $f^{-1}(f(a))$ would be infinite. By C-minimality, we may shrink $\mathfrak{a}$ if necessary so that, for every $a \in \mathfrak{a}$, $F_i(a, f(a)) = 0$ if and only if $i \leq m$. For every $F_j(X, Y)$ with $j > m$, since $k \neq 1$, we may shrink $\mathfrak{a}$ again so that, for some monomial cX^lY^n, for every $a \in \mathfrak{a}$, and for every $r, s \in \mathrm{VF}$ with $\mathrm{val}(r) = \mathrm{val}(a)$ and $\mathrm{val}(s) = \mathrm{val}(f(a))$, we have

$$\mathrm{val}(F_j(r, s)) = \mathrm{val}(cr^l s^n) = \mathrm{val}(ca^l f(a)^n) = \mathrm{val}(F_j(a, f(a))).$$

Now, using the division algorithm, there are rational functions $G(X, Y) \in \mathrm{VF}(X)[Y]$ and $H(X, Y) \in \mathrm{VF}(Y)[X]$ such that, possibly after shrinking $\mathfrak{a}$ again,

(1) every solution of $G(a, Y) = 0$ is a solution of $\bigwedge_{i \le m} F_i(a, Y) = 0$ for every $a \in \mathfrak{a}$,
(2) every solution of $H(X, b) = 0$ is a solution of $\bigwedge_{i \le m} F_i(X, b) = 0$ for every $b \in \mathfrak{b}$,
(3) taking derivatives and using the division algorithm again if necessary, for every $a \in \mathfrak{a}$, $f(a)$ is not a repeated root of $G(a, Y)$ and a is not a repeated root of $H(X, f(a))$,

Moreover, we may assume that, if we write $G(X, Y)$ as $\sum_i G_i(X) Y^i$ then there are indices $i < i'$ such that for every $a \in \mathfrak{a}$

$$\mathrm{val}(f(a))^{i'-i} = \mathrm{val}(G_{i'}(a) / \mathrm{val}(G_i(a))) > 0;$$

Similarly for $H(X, Y)$. Observe that if $i' - i > 1$ then for every $a \in \mathfrak{a}$ there is a root $r \neq f(a)$ of $G(a, Y)$ such that $\mathrm{val}(F_j(a, f(a))) = \mathrm{val}(F_j(a, r))$ for all $j > m$ and hence $\phi(a, r)$ holds, which is a contradiction. So $i' = i + 1$. Since the radius of $\mathfrak{a}$ is sufficiently large, we conclude that k must be a positive integer. Symmetrically $1/k$ is also a positive integer and hence $k = 1$, contradicting the assumption $k \neq 1$. □

Lemma 8.5. *Let $A, B \subseteq \mathrm{VF}$ be infinite subsets and $f : A \longrightarrow B$ a definable bijection. Then for almost all $a \in A$ there are a-definable $\delta \in \Gamma$ and $t \in \mathrm{RV}^\times$ such that, for any $b, b' \in \mathfrak{o}(a, \delta)$,*

$$\mathrm{rv}(f(b) - f(b')) = t\, \mathrm{rv}(b - b').$$

Proof. Let $A' \subseteq A$ be a definable subset such that $A \smallsetminus A'$ is finite and for every $a \in A'$ there are $\epsilon_a, \delta_a \in \Gamma$ given as in Lemma 8.3. Translating A, B to $A - a$, $B - f(a)$ and applying Lemma 8.4, we see that δ_a may be chosen so that

$$\mathrm{val}(f(b) - f(a)) = \epsilon_a + \mathrm{val}(b - a)$$

for any $b \in \mathfrak{o}(a, \delta_a)$. Let $D_a = (\mathfrak{o}(a, \delta_a) - a) \smallsetminus \{0\}$ and $g_a : D_a \longrightarrow \mathrm{RV}$ the function given by

$$d \longmapsto \mathrm{rv}(f(d + a) - f(a)) / \mathrm{rv}(d).$$

Since $\mathrm{vrv}(g_a(D_a))$ is bounded from both above and below, by Lemma 3.17, there is a $\beta_a \in \Gamma$ such that $g_a(\mathfrak{o}(0, \beta_a) \smallsetminus \{0\}) = t_a$. Let $h : A' \longrightarrow \Gamma \times \mathrm{RV}$ be the function given by $a \longmapsto (\delta_a, t_a)$. By compactness and Corollary 3.5, there are only finitely many $a \in A'$ that is isolated in $h^{-1}(\delta_a, t_a)$. On the other hand, if $\mathfrak{o}(a, \gamma) \subseteq h^{-1}(\delta, t)$ with $\gamma \geq \delta$ then clearly for any $b, b' \in \mathfrak{o}(a, \gamma)$,

$$\mathrm{rv}(f(b) - f(b')) = t\, \mathrm{rv}(b - b'),$$

as required. □

Lemma 8.3 can be generalized to multivariate functions, but only with inequality:

Lemma 8.6. *Let $f : \mathrm{VF}^n \times \mathrm{VF}^k \longrightarrow \mathrm{VF}^m$ be a definable function. Then there are a definable subset $A \subseteq \mathrm{VF}^n \times \mathrm{VF}^k$ over VF^k and a positive rational number k such that*

(1) *$\dim_{\mathrm{VF}}(\mathrm{VF}^n \smallsetminus \mathrm{fib}(A, \vec{b})) < n$ for all $\vec{b} \in \mathrm{VF}^k$,*
(2) *for every $\vec{x} = (\vec{a}, \vec{b}) \in A$ there are $\vec{x}$-definable $\epsilon, \delta \in \Gamma$ such that for any $\vec{a}' \in \mathfrak{o}(\vec{a}, \delta)$,*

$$\mathrm{val}(f(\vec{a}', \vec{b}) - f(\vec{a}, \vec{b})) \geq \epsilon + k\, \mathrm{val}(\vec{a}' - \vec{a}).$$

Proof. We do induction on n. The base case $n = 1$ is readily implied by Lemma 8.3.

We proceed to the inductive step. By the inductive hypothesis, there are a definable subset $A_1 \subseteq \mathrm{VF}^{n-1} \times \mathrm{VF}^{k+1}$ over VF^{k+1} and a positive rational number k_1 with respect to which the conclusion of the lemma holds. Similarly, there are a definable subset $A_2 \subseteq \mathrm{VF}^{n-1} \times \mathrm{VF} \times \mathrm{VF}^k$ over VF^{k+n-1} and a positive rational number k_2 with respect to which the conclusion of the lemma holds.

Let $k = \min\{k_1, k_2\}$. Fix a $\vec{c} \in \mathrm{VF}^k$. We shall concentrate on the subsets $\mathrm{fib}(A_1, \vec{c})$, $\mathrm{fib}(A_2, \vec{c})$, which, for simplicity, are respectively written as C_1, C_2. Also we shall suppress mentioning $\vec{c}$ as parameters. Set $C = C_1 \cap C_2$. Note that, by compactness, $\dim_{\mathrm{VF}}(\mathrm{VF}^n \smallsetminus C) < n$. Consider any $(\vec{a}, b) \in C_1$. Let $(\epsilon_b, \delta_b) \in \Gamma^2$ be an $(\vec{a}, b)$-definable pair such that, for any $\vec{a}' \in \mathfrak{o}(\vec{a}, \delta_b)$,

$$\mathrm{val}(f(\vec{a}', b) - f(\vec{a}, b)) \geq \epsilon_b + k\, \mathrm{val}(\vec{a}' - \vec{a}).$$

Let $h_{\vec{a}} : \mathrm{fib}(C_1, \vec{a}) \longrightarrow \Gamma^2$ be the $\vec{a}$-definable function given by $(\vec{a}, b) \longmapsto (\epsilon_b, \delta_b)$. For each $(\epsilon, \delta) \in \Gamma^2$ let $B_{\epsilon,\delta}$ be the topological interior of $h_{\vec{a}}^{-1}(\epsilon, \delta)$. Let

$$B_{\vec{a}} = \bigcup_{(\epsilon,\delta) \in \Gamma^2} B_{\epsilon,\delta} \quad \text{and} \quad B = \bigcup_{\vec{a} \in \mathrm{pr}_{<n}(C_1)} (\{\vec{a}\} \times (\mathrm{fib}(C_1, \vec{a}) \smallsetminus B_{\vec{a}})).$$

By C-minimality, $\dim_{\mathrm{VF}}(h_{\vec{a}}^{-1}(\epsilon, \delta) \smallsetminus B_{\epsilon,\delta}) = 0$ for every $(\epsilon, \delta) \in \Gamma^2$ and hence, by Lemmas 4.3 and 4.2, $\dim_{\mathrm{VF}}(\mathrm{fib}(C_1, \vec{a}) \smallsetminus B_{\vec{a}}) = 0$ and $\dim_{\mathrm{VF}}(B) < n$.

Let $(\vec{a}_1, b_1) \in C \smallsetminus B$ and $h_{\vec{a}_1}(b_1) = (\epsilon_1, \delta_1)$. Since the corresponding interior B_{ϵ_1,δ_1} is nonempty, there are $(\vec{a}_1, b_1)$-definable $\delta_2, \epsilon_2 \in \Gamma$ such that $\mathfrak{o}(b_1, \delta_2) \subseteq B_{\epsilon_1,\delta_1}$ and, for any $b_2 \in \mathfrak{o}(b_1, \delta_2)$,

$$\operatorname{val}(f(\vec{a}_1, b_2) - f(\vec{a}_1, b_1)) \geq \epsilon_2 + k \operatorname{val}(b_2 - b_1).$$

On the other hand, for any $b_2 \in \mathfrak{o}(b_1, \delta_2)$ and any $\vec{a}_2 \in \mathfrak{o}(\vec{a}_1, \delta_1)$,

$$\operatorname{val}(f(\vec{a}_2, b_2) - f(\vec{a}_1, b_2)) \geq \epsilon_1 + k \operatorname{val}(\vec{a}_2 - \vec{a}_1).$$

We then have

$$\begin{aligned} \operatorname{val}(f(\vec{a}_2, b_2) - f(\vec{a}_1, b_1)) &\geq \min\left\{\operatorname{val}(f(\vec{a}_1, b_2) - f(\vec{a}_1, b_1)), \operatorname{val}(f(\vec{a}_2, b_2) - f(\vec{a}_1, b_2))\right\} \\ &\geq \min\{\epsilon_1, \epsilon_2\} + \min\left\{k \operatorname{val}(b_2 - b_1), k \operatorname{val}(\vec{a}_2 - \vec{a}_1)\right\} \\ &= \min\{\epsilon_1, \epsilon_2\} + k \operatorname{val}((\vec{a}_2, b_2) - (\vec{a}_1, b_1)). \end{aligned}$$

Now the lemma follows from compactness. □

Clearly this lemma holds with respect to any definable function $f : A \longrightarrow \mathrm{VF}^m$ with $A \subseteq \mathrm{VF}^n$ and $\dim_{\mathrm{VF}}(A) = n$, since f may be extended to VF^n by sending $\mathrm{VF}^n \smallsetminus A$ to any definable tuple in VF^m. In application we usually take $k = 0$.

Lemma 8.7. *Let $f : \mathrm{VF}^n \longrightarrow \mathrm{VF}^m$ be a definable function. Then there is a definable closed subset $A \subseteq \mathrm{VF}^n$ with $\dim_{\mathrm{VF}}(A) < n$ such that $f \restriction (\mathrm{VF}^n \smallsetminus A)$ is continuous with respect to the valuation topology.*

Proof. Let $A \subseteq \mathrm{VF}^n$ be the definable subset of "discontinuous points" of f; that is, $\vec{a} \in A$ if and only if there is a $\gamma \in \Gamma$ such that $f^{-1}(\mathfrak{o}(f(\vec{a}), \gamma))$ fails to contain any open polydisc around $\vec{a}$. Let $\vec{A}$ be the topological closure of A, which is definable, and set $f_1 = f \restriction (\mathrm{VF}^n \smallsetminus \vec{A})$. For any $\vec{a} \in \mathrm{VF}^n \smallsetminus \vec{A}$ and any $\gamma \in \Gamma$, since $f^{-1}(\mathfrak{o}(f(\vec{a}), \gamma))$ contains an open polydisc around $\vec{a}$, $f_1^{-1}(\mathfrak{o}(f(\vec{a}), \gamma))$ must also contain an open polydisc around $\vec{a}$. So it is enough to show that $\dim_{\mathrm{VF}}(\vec{A}) < n$, which, by Lemma 4.6, is equivalent to showing that $\dim_{\mathrm{VF}}(A) < n$.

Suppose for contradiction that $\dim_{\mathrm{VF}}(A) = n$. Let $A' \subseteq A$ be the definable subset given by Lemma 8.6 with respect to f. Since $\dim_{\mathrm{VF}}(A') = n$, by Lemma 4.6 again, A' contains an open polydisc $\mathfrak{p}$. Fix an $\vec{a} \in \mathfrak{p}$ and let $\gamma \in \Gamma$ be such that $f^{-1}(\mathfrak{o}(f(\vec{a}), \gamma))$ fails to contain any open ball around $\vec{a}$. By Lemma 8.6, there are $\epsilon, \delta \in \Gamma$ such that

(1) $\mathfrak{o}(\vec{a}, \delta) \subseteq \mathfrak{p}$,
(2) $\epsilon + \delta > \gamma$,
(3) for any $\vec{b} \in \mathfrak{o}(\vec{a}, \delta)$ with $\vec{b} \neq \vec{a}$, $\operatorname{val}(f(\vec{b}) - f(\vec{a})) \geq \epsilon + \delta$.

So $\mathfrak{o}(\vec{a}, \delta) \subseteq f^{-1}(\mathfrak{o}(f(\vec{a}), \gamma))$, contradiction. □

Definition 8.8. A function $f : \mathrm{VF}^n \longrightarrow \mathscr{P}(\mathrm{RV}^m)$ is *locally constant at $\vec{a}$* if there is an open subset $U_{\vec{a}} \subseteq \mathrm{VF}^n$ containing $\vec{a}$ such that $f \restriction U_{\vec{a}}$ is constant. If f is locally constant at every point in an open subset A then f is locally constant on A.

Lemma 8.9. *Let $f : \mathrm{VF}^n \longrightarrow \mathscr{P}(\mathrm{RV}^m)$ be a definable function. Then f is locally constant almost everywhere.*

Proof. We do induction on n. For the base case $n = 1$, let $A \subseteq \mathrm{VF}$ be the definable subset of those $a \in \mathrm{VF}$ such that f is not constant on any $\mathfrak{o}(a, \gamma)$. Let $\vec{A}$ be the topological closure of A. It is enough to show that $\dim_{\mathrm{VF}}(\vec{A}) = 0$, which, by C-minimality, is equivalent to showing that A is finite. Suppose for contradiction that A is infinite. By C-minimality again there is a definable $\gamma \in \Gamma$ such that A contains infinitely many cosets of $\mathfrak{o}(0, \gamma)$. By Lemma 3.18, f fails to be constant on only finitely many cosets of $\mathfrak{o}(0, \gamma)$, contradiction.

We proceed to the inductive step. For any $\vec{a} = (a_1, \vec{a}_1) \in \mathrm{VF}^n$, let $(\alpha_{\vec{a}}, \beta_{\vec{a}}) \in \Gamma^2$ be an $\vec{a}$-definable pair such that f is constant on both $\mathfrak{o}(a_1, \alpha_{\vec{a}}) \times \left\{\vec{a}_1\right\}$ and $\{a_1\} \times \mathfrak{o}(\vec{a}_1, \beta_{\vec{a}})$. If no such pair exists then set $\alpha_{\vec{a}} = \beta_{\vec{a}} = \infty$. Let $g : \mathrm{VF}^n \longrightarrow \Gamma^2$ be the function given by $\vec{a} \longmapsto (\alpha_{\vec{a}}, \beta_{\vec{a}})$. By the inductive hypothesis and compactness, $\dim_{\mathrm{VF}}(g^{-1}(\infty, \infty)) < n$. For each $(\alpha, \beta) \in \Gamma^2$ let $B_{\alpha,\beta}$ be the topological interior of $g^{-1}(\alpha, \beta)$. By Lemma 4.6,

$$\dim_{\mathrm{VF}}(g^{-1}(\alpha, \beta) \smallsetminus B_{\alpha,\beta}) < n.$$

Let $B = \bigcup_{(\alpha,\beta)\in\Gamma^2} B_{\alpha,\beta}$. By compactness, $\dim_{\mathrm{VF}}(\mathrm{VF}^n \smallsetminus B) < n$. For any $\vec{a} = (a_1, \vec{a}_1) \in B$, since $B_{\alpha_{\vec{a}},\beta_{\vec{a}}}$ contains an open polydisc around $\vec{a}$, clearly for any sufficiently large γ and any $(a_1', \vec{a}_1') \in \mathfrak{o}(\vec{a}, \gamma)$ we have $f(a_1, \vec{a}_1) = f(a_1, \vec{a}_1') = f(a_1', \vec{a}_1')$. So f is locally constant on B. □

9. Differentiation

We shall extend the results in Sections 5 and 7 to finer categories of definable subsets with volume forms. To define these categories we first need a notion of the Jacobian in the VF-sort. There are two approaches, which essentially give the same data. The first one is an analogue of the classical analytic approach, where we first define differentiation and the notion of "approaching a point" is expressed via valuation. This method makes certain computations very easy (see Lemmas 9.11 and 9.12). The second approach is an algebraic one, where we are reduced to the case of computing the Jacobian of a regular map between varieties over VF. The Jacobian in the RV-sort will also be defined in this way. This makes the compatibility of the Jacobian in both sorts transparent.

In the discussion below it is convenient to think that there is a "point at infinity" in the VF-sort, denoted by p_∞. The set $\mathrm{VF} \cup \{p_\infty\}$ is denoted by $\mathbb{P}(\mathrm{VF})$. Balls around p_∞ are defined in a reversed way. For example, for any $\gamma \in \Gamma$, the open ball $\mathfrak{o}(p_\infty, \gamma)$ around p_∞ of radius γ is the subset $\mathrm{VF} \smallsetminus \mathfrak{c}(0, -\gamma)$. Note the negative sign in front of γ. We emphasize that p_∞ will not be treated as a real point. It is merely a notational device that allows us to discuss complements of balls around 0 more efficiently.

Definition 9.1. Let $A \subseteq \mathrm{VF}^n$, $f : A \longrightarrow \mathscr{P}(\mathrm{VF}^m)$ a definable function, $\vec{a} \in \mathrm{VF}^n$, and $L \subseteq \mathbb{P}(\mathrm{VF})^m$. We say that L is a *limit set of f at $\vec{a}$*, written as $\lim_{A\to\vec{a}} f \subseteq L$, if for every $\epsilon \in \Gamma$ there is a $\delta \in \Gamma$ such that if $\vec{c} \in \mathfrak{o}(\vec{a}, \delta) \cap (A \smallsetminus \vec{a})$ then $f(\vec{c}) \subseteq \bigcup_{\vec{b}\in L'} \mathfrak{o}(\vec{b}, \epsilon)$ for some $L' \subseteq L$.

A limit set L of f at $\vec{a}$ is *minimal* if no proper subset of L is a limit set of f at $\vec{a}$. Observe that if $\lim_{A\to\vec{a}} f \subseteq L$ and $\vec{b} \in L$ is not isolated in L then actually $\lim_{A\to\vec{a}} f \subseteq L \smallsetminus \{\vec{b}\}$. So in a minimal limit set every element is isolated. Moreover, if a minimal limit set L exists then its topological closure $\vec{L}$ is unique:

Lemma 9.2. *Let $L_1, L_2 \subseteq \mathrm{VF}^m$ be two minimal limit sets of f at $\vec{a}$ and $\vec{L}_1, \vec{L}_2$ their topological closures. Then $\vec{L}_1 = \vec{L}_2$.*

Proof. Suppose for contradiction that, say, $\vec{L}_1 \smallsetminus \vec{L}_2 \neq \emptyset$ and hence there is a $\vec{b} \in L_1 \smallsetminus \vec{L}_2$. So there is an $\epsilon \in \Gamma$ such that $\mathfrak{o}(\vec{b}, \epsilon) \cap L_2 = \emptyset$. Let $\delta \in \Gamma$ be such that, for all $\vec{c} \in \mathfrak{o}(\vec{a}, \delta) \cap (A \smallsetminus \vec{a})$, $f(\vec{c}) \subseteq \bigcup_{\vec{d}\in L_2'} \mathfrak{o}(\vec{d}, \epsilon)$ for some $L_2' \subseteq L_2$. Since $\mathfrak{o}(\vec{b}, \epsilon) \cap \mathfrak{o}(\vec{d}, \epsilon) = \emptyset$ for any $\vec{d} \in L_2$, we see that $L_1 \smallsetminus \{\vec{b}\}$ is a limit set of f at a, contradicting the minimality condition on L_1. So $\vec{L}_1 \subseteq \vec{L}_2$ and symmetrically $\vec{L}_2 \subseteq \vec{L}_1$. $\square$

This lemma justifies the equality $\lim_{A\to\vec{a}} f = L$ when L is a closed (hence the unique) minimal limit set of f at $\vec{a}$.

Lemma 9.3. *Let $f_1, f_2 : A \longrightarrow \mathscr{P}(\mathrm{VF}^m)$ be definable functions with $\lim_{A\to\vec{a}} f_i = L_i$, then $\lim_{A\to\vec{a}}(f_1 \cup f_2) = L_1 \cup L_2$.*

Proof. Let $f = f_1 \cup f_2$ and $L = L_1 \cup L_2$. Clearly L is a closed limit set of f at $\vec{a}$. We need to show that it is minimal. To that end, fix a $\vec{b} \in L_1$. If $\vec{b} \in L_1 \cap L_2$ then, since $\vec{b}$ is isolated in both L_1 and L_2, there is an $\epsilon \in \Gamma$ such that $\mathfrak{o}(\vec{b}, \epsilon) \cap (L \smallsetminus \{\vec{b}\}) = \emptyset$. If $\vec{b} \in L_1 \smallsetminus L_2$ then, since L_2 is closed, there is again an $\epsilon \in \Gamma$ such that $\mathfrak{o}(\vec{b}, \epsilon) \cap (L \smallsetminus \{\vec{b}\}) = \emptyset$. Now, since L_1 is a limit set of f_1 at $\vec{a}$ but $L_1 \smallsetminus \{\vec{b}\}$ is not, we see that $L \smallsetminus \{\vec{b}\}$ cannot be a limit set of f at $\vec{a}$. This shows that L is minimal. $\square$

Lemma 9.4. *Let $f : A \longrightarrow \mathscr{P}(\mathrm{VF}^m)$ be a definable function with finite images. Let k be the maximal size of $f(c)$. Let $a \in \mathrm{VF}$ and suppose that there is an open ball $\mathfrak{b}$ containing a such that $\mathfrak{b} \smallsetminus \{a\} \subseteq A$. Suppose that $\lim_{A\to a} f = L$. If L is finite then $|L| \leq k$.*

Proof. Let $L = \{\vec{b}_1, \dots, \vec{b}_l\}$ and suppose for contradiction that $l > k$. Let $\alpha \in \Gamma$ be such that $\mathfrak{o}(\vec{b}_i, \alpha) \cap \mathfrak{o}(\vec{b}_j, \alpha) = \emptyset$ whenever $i \neq j$. Without loss of generality we may assume that $A \smallsetminus \{a\} = \mathfrak{b} \smallsetminus \{a\}$ and $\bigcup f(A) \subseteq \bigcup_i \mathfrak{o}(\vec{b}_i, \alpha)$. For each $D \subseteq L$ with $|D| = k$ let

$$A_D = \left\{ c \in A : f(c) \subseteq \bigcup_{\vec{b}_i \in D} \mathfrak{o}(\vec{b}_i, \alpha) \right\}.$$

Each A_D is $\langle L, \alpha \rangle$-definable. By C-minimality, some $A_D \cup \{a\}$ contains an open ball around a and hence $\lim_{A\to a} f = \lim_{A_D\to a}(f \upharpoonright A_D) \subseteq D$, contradicting the assumption that $\lim_{A\to a} f = L$. $\square$

Here is the key lemma that makes the definition of differentiation in VF below work. It is essentially a variation on a fundamental property of henselian fields, see [9, Proposition, p. 70].

Lemma 9.5. *Let $\mathfrak{b} \subseteq \mathrm{VF}$ be a ball containing 0 and $A \subseteq (\mathfrak{b} \smallsetminus \{0\}) \times \mathrm{VF}^m$ a definable function $\mathfrak{b} \smallsetminus \{0\} \longrightarrow \mathscr{P}(\mathrm{VF}^m)$ with finite images. Then there is a definable finite subset $L \subseteq \mathbb{P}(\mathrm{VF})^m$ such that $\lim_{\mathfrak{b}\smallsetminus\{0\}\to 0} A = L$.*

Proof. Without loss of generality we may assume that $\mathfrak{b}$ is an open ball. Let $\mathfrak{a} = \mathfrak{b} \smallsetminus \{0\}$. We first consider the basic case: $m = 1$ and there is a polynomial $G(X, Y) \in \mathrm{VF}(S)[X, Y]$ such that $(a, b) \in A$ if and only if $a \in \mathfrak{a}$ and $G(a, b) = 0$.

Fix an $\epsilon \in \Gamma$. Write $G(X, Y)$ as $Y^m H(X) G^*(X, Y)$, where $H(X) \in \mathrm{VF}(S)[X]$ and $G^*(X, Y) \in \mathrm{VF}(S)[X, Y]$ is of the form

$$H_n(X)Y^n + \cdots + H_0(X),$$

where the polynomials $H_j(X)Y^j \in \mathrm{VF}(S)[X, Y]$ are relatively prime. Shrinking $\mathfrak{a}$ if necessary, we may assume that $\mathfrak{a}$ does not contain any root of $H(X)$ or nonzero $H_j(X)$. If $n = 0$ then clearly $L = \{0\}$ is as required. If $m > 0$ then $(a, 0) \in A$ for every $a \in \mathfrak{a}$. So let us assume $n > 0$ and $m = 0$. Let $E \subseteq \{0, 1, \dots, n\}$ be the subset such that $i \in E$ if and only if X divides $H_i(X)$. Let

$$G_1(X, Y) = \sum_{i\in E} H_i(X)Y^i, \quad G_2(X, Y) = \sum_{i\notin E} (H_i^*(X) + H_i(0))Y^i.$$

Note that X also divides each $H_i^*(X)$. For any sufficiently large $\delta \in \Gamma$, $\mathrm{val}(H_i(X))$ has a sufficiently large lower bound on $\mathfrak{o}(0, \delta) \smallsetminus \{0\}$ for every $i \in E$; similarly for every $H_i^*(X)$. On the other hand, let $d_1, \dots, d_k$ be the distinct roots of $G_2(0, Y) \in \mathrm{VF}(S)[Y]$ ($k = 0$ if $G_2(0, Y)$ is a nonzero constant) then, for any sufficiently large $\alpha \in \Gamma$, $\mathrm{val}(G_2(0, b)) > \alpha$

only if $b \in \mathfrak{o}(d_i, \epsilon)$ for some i. Therefore, if $\delta \in \Gamma$ is sufficiently large then for every $a \in \mathfrak{o}(0, \delta) \smallsetminus \{0\}$ and every $b \notin \mathfrak{o}(p_\infty, \epsilon) \cup \bigcup_i \mathfrak{o}(d_i, \epsilon)$ we must have

$$\mathrm{val}(G^*(a, b) - G_2(0, b)) > \mathrm{val}(G_2(0, b))$$

and hence $G^*(a, b) \neq 0$. This concludes the basic case.

More generally, by compactness, $A \subseteq \mathrm{VF}^2$ is a union of finitely many subsets of the form $A_i \cap D_i$, where each A_i is given by a VF-sort equality as above. Since the lemma holds for each $A_i \cap D_i$, it holds for A by Lemma 9.3.

For the case $m > 1$, let $A_i = \{(b, \mathrm{pr}_i(\vec{a})) : (b, \vec{a}) \in A\}$ for each $i \leq m$ and $\lim_{\mathfrak{b} \smallsetminus \{0\} \to 0} A_i = L_i$. It is easy to see that

$$\lim_{\mathfrak{b} \smallsetminus \{0\} \to 0} A \subseteq L_1 \times \cdots \times L_m$$

and hence, as in the proof of Lemma 9.4, there is a definable $L \subseteq L_1 \times \cdots \times L_m$ such that $\lim_{\mathfrak{b} \smallsetminus \{0\} \to 0} A = L$. □

Definition 9.6. Let $f : \mathrm{VF}^n \longrightarrow \mathrm{VF}^m$ be a definable function. For any $\vec{a} \in \mathrm{VF}^n$, we say that f is *differentiable at* $\vec{a}$ if there is a linear map $\lambda : \mathrm{VF}^n \longrightarrow \mathrm{VF}^m$ (of VF-vector spaces) such that, for any $\epsilon \in \Gamma$, if $\vec{b} \in \mathrm{VF}^n$ and $\mathrm{val}(\vec{b})$ is sufficiently large then

$$\mathrm{val}(f(\vec{a} + \vec{b}) - f(\vec{a}) - \lambda(\vec{b})) - \mathrm{val}(\vec{b}) > \epsilon.$$

It is straightforward to check that if such a linear function λ (a matrix with entries in VF) exists then it is unique and hence may be called *the derivative of f at $\vec{a}$*, which shall be denoted by $\mathrm{d}_{\vec{a}} f$.

For each $1 \leq j \leq m$ let $f_j = \mathrm{pr}_j \circ f$. For any $\vec{a} = (a_i, \tilde{a}_i) \in \mathrm{VF}^n$, if the derivative of the function $f_j \upharpoonright (\mathrm{VF} \times \{\tilde{a}_i\})$ at a_i exists then we call it the ijth *partial derivative of f at $\vec{a}$* and denote it by $\partial^{ij}_{\vec{a}} f$.

The classical differentiation rules, such as the product rule and the chain rule, hold with respect to this definition. Here we only check the chain rule:

Lemma 9.7 (*The Chain Rule*)**.** *Let $f : \mathrm{VF}^n \longrightarrow \mathrm{VF}^m$ be differentiable at $\vec{a} \in \mathrm{VF}^n$ and $g : \mathrm{VF}^m \longrightarrow \mathrm{VF}^l$ differentiable at $f(\vec{a})$. Then $g \circ f$ is differentiable at $\vec{a}$ and*

$$\mathrm{d}_{\vec{a}}(g \circ f) = (\mathrm{d}_{f(\vec{a})} g) \times (\mathrm{d}_{\vec{a}} f),$$

where the right-hand side is a product of matrices.

Proof. Fix an $\epsilon \in \Gamma$. Since $\mathrm{d}_{\vec{a}} f$ is a linear function, there is an $\alpha \in \Gamma$ such that, for every $\vec{b} \in \mathrm{VF}^n$, $\mathrm{val}(\mathrm{d}_{\vec{a}} f(\vec{b})) - \mathrm{val}(\vec{b}) \geq \alpha$. Similarly there is a $\beta \in \Gamma$ such that, for every $\vec{b} \in \mathrm{VF}^m$, $\mathrm{val}(\mathrm{d}_{f(\vec{a})} g(\vec{b})) - \mathrm{val}(\vec{b}) \geq \beta$. Let $s : \mathrm{VF}^n \longrightarrow \mathrm{VF}^m$ be the function given by

$$\vec{b} \longmapsto f(\vec{a} + \vec{b}) - f(\vec{a}) - \mathrm{d}_{\vec{a}} f(\vec{b}).$$

By assumption, for any $\vec{b} \in \mathrm{VF}^n$ with $\mathrm{val}(\vec{b})$ sufficiently large,

$$\mathrm{val}(\mathrm{d}_{f(\vec{a})} g(s(\vec{b}))) \geq \mathrm{val}(s(\vec{b})) + \beta > \mathrm{val}(\vec{b}) + (\epsilon - \beta) + \beta = \mathrm{val}(\vec{b}) + \epsilon.$$

Therefore, if $\mathrm{val}(\vec{b})$ is sufficiently large then either

$$\mathrm{val}(g(f(\vec{a} + \vec{b})) - g(f(\vec{a})) - \mathrm{d}_{f(\vec{a})} g(\mathrm{d}_{\vec{a}} f(\vec{b}))) > \mathrm{val}(\vec{b}) + \epsilon$$

or

$$\begin{aligned} \mathrm{val}(g(f(\vec{a} + \vec{b})) - g(f(\vec{a})) - \mathrm{d}_{f(\vec{a})} g(\mathrm{d}_{\vec{a}} f(\vec{b}))) &= \mathrm{val}(g(f(\vec{a} + \vec{b})) - g(f(\vec{a})) - \mathrm{d}_{f(\vec{a})} g(\mathrm{d}_{\vec{a}} f(\vec{b})) - \mathrm{d}_{f(\vec{a})} g(s(\vec{b}))) \\ &= \mathrm{val}(g(f(\vec{a}) + \mathrm{d}_{\vec{a}} f(\vec{b}) + s(\vec{b})) - g(f(\vec{a})) - \mathrm{d}_{f(\vec{a})} g(\mathrm{d}_{\vec{a}} f(\vec{b}) + s(\vec{b}))) \\ &> \mathrm{val}(\mathrm{d}_{\vec{a}} f(\vec{b}) + s(\vec{b})) + \min\{\beta, \epsilon - \alpha\} \\ &\geq \mathrm{val}(\vec{b}) + \epsilon. \end{aligned}$$

In either case the lemma follows. □

Lemma 9.8. *Let $f : \mathrm{VF}^n \longrightarrow \mathrm{VF}^m$ be a definable function. Then each partial derivative $\partial^{ij} f$ is defined almost everywhere.*

Proof. Let $\vec{a} = (a_i, \tilde{a}_i) \in \mathrm{VF}^n$. Let $g^{ij}_{\vec{a}} : \mathrm{VF}^{\times} \longrightarrow \mathrm{VF}$ be the $\vec{a}$-definable function given by

$$b \longmapsto (f_j(a_i + b, \tilde{a}_i) - f_j(\vec{a}))/b,$$

where $f_j = \mathrm{pr}_j \circ f$. By Lemma 8.6, for almost all $\vec{a} \in \mathrm{VF}^n$ there is an $\vec{a}$-definable open ball $\mathfrak{b}_{\vec{a}}$ punctured at 0 such that $\mathrm{val}(g^{ij}_{\vec{a}}(\mathfrak{b}_{\vec{a}}))$ is bounded from below. By Lemmas 9.5 and 9.4, $\lim_{\mathfrak{b}_{\vec{a}} \to 0} g^{ij}_{\vec{a}} = \zeta(\vec{a})$ for some $\zeta(\vec{a}) \in \mathrm{VF}$. The linear function is constructed in the usual way, taking $\zeta(\vec{a})$ as the slope. □

Corollary 9.9. *Let $f : \mathrm{VF}^n \longrightarrow \mathrm{VF}^m$ be a definable function. Then f is continuously partially differentiable almost everywhere.*

Proof. This is immediate by Lemmas 9.8 and 8.7. □

We would like to differentiate functions between arbitrary definable subsets. The simplest way to do this to be "forgetful" about the RV-coordinates. Let $f : \mathrm{VF}^n \times \mathrm{RV}^m \longrightarrow \mathrm{VF}^{n'} \times \mathrm{RV}^{m'}$ be a definable function. For each $\vec{t} \in \mathrm{RV}^m$ let $U_{\vec{t}} = \mathrm{prv}(f(\mathrm{VF}^n \times \{\vec{t}\}))$. For every $\vec{s} \in U_{\vec{t}}$ let $f_{\vec{t},\vec{s}}$ be the function on $\{\vec{a} : \mathrm{prv}(f(\vec{a},\vec{t})) = \vec{s}\}$ given by $\vec{a} \longmapsto \mathrm{pvf}(f(\vec{a},\vec{t}))$. Note that, by compactness, there is an $\vec{s} \in U_{\vec{t}}$ such that $\dim_{\mathrm{VF}}(\mathrm{dom}(f_{\vec{t},\vec{s}})) = n$ and hence, by Lemma 4.6, $\mathrm{dom}(f_{\vec{t},\vec{s}})$ contains an open polydisc. For such an $\vec{s}$ and each $\vec{a} \in \mathrm{dom}(f_{\vec{t},\vec{s}})$ we define the ijth *partial derivative of* f *at* $(\vec{a},\vec{t})$ to be the ijth partial derivative of $f_{\vec{t},\vec{s}}$ at $\vec{a}$. It follows from Corollary 9.9 and compactness that every partial derivative of f is defined almost everywhere.

Definition 9.10. If $n = n'$ and all the partial derivatives exist at a point $(\vec{a},\vec{t})$ then the *Jacobian of* f *at* $(\vec{a},\vec{t})$ is defined in the usual way, that is, the determinant of the Jacobian matrix, and is denoted by $\mathrm{Jcb}_{\mathrm{VF}} f(\vec{a},\vec{t})$.

Lemma 9.11. *For any special bijection* $T : A \longrightarrow A^{\sharp}$*, the Jacobians of* T *and* T^{-1} *are equal to* 1 *almost everywhere. If* A *is a nondegenerate* RV*-pullback then they are equal to* 1 *everywhere.*

Proof. We may assume that the length of T is 1. Then this is clear if we apply the proof of Lemma 9.8 to (additive) translation and canonical bijection (or its inverse). □

Lemma 9.12. *Let* $f : A \longrightarrow B$ *and* $g : B \longrightarrow C$ *be definable functions. Then for any* $\vec{x} \in A$,

$$\mathrm{Jcb}_{\mathrm{VF}}(g \circ f)(\vec{x}) = \mathrm{Jcb}_{\mathrm{VF}}\, g(f(\vec{x})) \cdot \mathrm{Jcb}_{\mathrm{VF}} f(\vec{x}),$$

if both sides are defined.

Proof. This is immediate by the chain rule. □

Next we describe the second approach to defining the Jacobian in VF. Let $f : \mathrm{VF}^n \longrightarrow \mathrm{VF}^n$ be a definable function, which in general is not a rational map. Let $D \subseteq \mathrm{VF}^{2n}$ be the Zariski closure of the graph of f. By Proposition 4.8 the dimension of D is n and hence $\mathrm{pr}_{\leq n} \upharpoonright D$ is finite-to-one. Let $D_1 = \mathrm{pr}_{\leq n}(D) = \mathrm{VF}^n$ and $D_2 = \mathrm{pr}_{>n}(D)$. For almost all $(\vec{a}_1, \vec{a}_2) = \vec{a} \in D$, $\mathrm{pr}_{\leq n}$ and $\mathrm{pr}_{>n}$ induce surjective linear maps of the tangent spaces (see [14, Lemma 2, p. 141]):

$$\mathrm{d}_{\vec{a}}\, \mathrm{pr}_{\leq n} : T_{\vec{a}}(D) \longrightarrow T_{\vec{a}_1}(D_1), \quad \mathrm{d}_{\vec{a}}\, \mathrm{pr}_{>n} : T_{\vec{a}}(D) \longrightarrow T_{\vec{a}_2}(D_2).$$

Since the dimension of D_1 is also n, we see that $\mathrm{d}_{\vec{a}}\, \mathrm{pr}_{\leq n}$ is an isomorphism of the tangent spaces for almost all $(\vec{a}_1, \vec{a}_2) = \vec{a} \in D$ and hence the composition

$$(\mathrm{d}_{\vec{a}}\, \mathrm{pr}_{>n}) \circ (\mathrm{d}_{\vec{a}}\, \mathrm{pr}_{\leq n})^{-1} : T_{\vec{a}_1}(D_1) \longrightarrow T_{\vec{a}_2}(D_2)$$

is defined and is given by an $n \times n$ matrix $\lambda_{\vec{a}}$ with entries in VF (not necessarily invertible). Suppose $f(\vec{a}_1) = \vec{a}_2$. Then $\lambda_{\vec{a}}$ satisfies the defining property in Definition 9.6 and hence $\det \lambda_{\vec{a}} = \mathrm{Jcb}_{\mathrm{VF}} f(\vec{a}_1)$. It is clear that this equality holds for almost all $\vec{a}_1 \in \mathrm{VF}^n$. Note that the construction can be carried out even if f is a partial function, as long as $\dim_{\mathrm{VF}}(\mathrm{dom}(f)) = n$.

Now the Jacobian in RV may be defined almost identically as above. But for clarity we shall repeat the whole procedure. Let $(U, f), (V, g) \in \mathrm{RV}[n, \cdot]$. Set $A = f(U) \cap (\mathrm{RV}^{\times})^n$ and $B = g(V) \cap (\mathrm{RV}^{\times})^n$.

Definition 9.13. An *essential isomorphism* between (U, f) and (V, g) is an isomorphism between $(f^{-1}(A), f \upharpoonright f^{-1}(A))$ and $(g^{-1}(B), g \upharpoonright g^{-1}(B))$.

Let $F : (U, f) \longrightarrow (V, g)$ be an essential isomorphism. Note that if $A \neq \emptyset$ then a lift of F is defined almost everywhere on $\mathbb{L}(U, f)$. Actually, since the parts $f(U) \smallsetminus A$ and $g(V) \smallsetminus B$ will not concern us, we may assume $f^{-1}(A) = U$ and $g^{-1}(B) = V$. We also assume that A, B are of RV-dimension n. Set

$$C = \left\{(f(\vec{u}), g(F(\vec{u}))) : \vec{u} \in U\right\} \subseteq A \times B.$$

Note that, since F is an isomorphism, both $\mathrm{pr}_{\leq n} \upharpoonright C$ and $\mathrm{pr}_{>n} \upharpoonright C$ are finite-to-one. We first consider the simple situation $A, B \subseteq (\overline{\mathrm{K}}^{\times})^n$. By Remark 7.5, A, B are unions of locally closed subsets (in the sense of Zariski topology). We may assume that A, B, C are varieties. Clearly the dimensions of A, B, C are all n. Since the projections π_A, π_B of C to A and B are dominant rational maps, for almost all $(f(\vec{u}), g(F(\vec{u}))) = \vec{c} \in C$ (that is, outside of a closed subset of dimension $< n$), π_A, π_B induce isomorphisms of the tangent spaces:

$$\mathrm{d}_{\vec{c}}\, \pi_A : T_{\vec{c}}(C) \longrightarrow T_{\pi_A(\vec{c})}(A), \quad \mathrm{d}_{\vec{c}}\, \pi_B : T_{\vec{c}}(C) \longrightarrow T_{\pi_B(\vec{c})}(B).$$

Therefore the composition

$$(\mathrm{d}_{\vec{c}}\, \pi_B) \circ (\mathrm{d}_{\vec{c}}\, \pi_A)^{-1} : T_{\pi_A(\vec{c})}(A) \longrightarrow T_{\pi_B(\vec{c})}(B)$$

is defined and is given by an invertible $n \times n$ matrix $\lambda_{\vec{u}}$ with entries in $\overline{\mathrm{K}}$. The determinant of $\lambda_{\vec{u}}$, denoted by $\mathrm{Jcb}_{\overline{\mathrm{K}}} F(f(\vec{u}), \vec{u})$, is the *Jacobian of* F *at* $\vec{u}$, which is a $\vec{u}$-definable element in $\overline{\mathrm{K}}^{\times}$. Note that $\mathrm{Jcb}_{\overline{\mathrm{K}}} F$ is defined almost everywhere in A, that is, the subset of those $f(\vec{u}) \in A$ such that $\mathrm{Jcb}_{\overline{\mathrm{K}}} F(f(\vec{u}), \vec{u})$ is not defined is of dimension $< n$.

In general, if $(f(\vec{u}), g(F(\vec{u}))) \in C$ is contained in a multiplicative coset O of $(\overline{\mathrm{K}}^{\times})^{2n}$ then we may translate A, B coordinate-wise by $f(\vec{u}), g(F(\vec{u}))$ respectively so that O is mapped into $(\overline{\mathrm{K}}^{\times})^{2n}$. Let (U, f'), (V, g') be the induced objects and F' the induced isomorphism on $f'^{-1}((\overline{\mathrm{K}}^{\times})^n)$.

Definition 9.14. The *Jacobian* $\operatorname{Jcb}_{\mathrm{RV}} F(f(\vec{u}), \vec{u})$ *of* F *at* $\vec{u}$ is a $\vec{u}$-definable element in $\mathrm{RV}^{\times}$ given by

$$(\Pi f(\vec{u}))^{-1}(\Pi g(F(\vec{u}))) \operatorname{Jcb}_{\overline{\mathrm{K}}} F'(1, \ldots, 1)$$

if it exists, where $\Pi(t_1, \ldots, t_n) = t_1 \times \cdots \times t_n$.

By Lemma 4.11 and compactness, the subset of those $f(\vec{u}) \in A$ such that $\operatorname{Jcb}_{\mathrm{RV}} F(f(\vec{u}), \vec{u})$ is defined is not empty and the subset of those $f(\vec{u}) \in A$ such that $\operatorname{Jcb}_{\mathrm{RV}} F(f(\vec{u}), \vec{u})$ is not defined is of dimension $< n$. Symmetrically this is also true for B.

We may further coarsen the data and define the *Γ-Jacobian*

$$\operatorname{Jcb}_{\Gamma} F(f(\vec{u}), \vec{u}) = \Sigma(\operatorname{vrv} \circ g \circ F)(\vec{u}) - \Sigma(\operatorname{vrv} \circ f)(\vec{u}),$$

where $\Sigma(\gamma_1, \ldots, \gamma_n) = \gamma_1 + \cdots + \gamma_n$. Obviously this always exists and

$$\operatorname{vrv}(\operatorname{Jcb}_{\mathrm{RV}} F(f(\vec{u}), \vec{u})) = \operatorname{Jcb}_{\Gamma} F(f(\vec{u}), \vec{u}).$$

Note that the chain rule clearly holds for both $\operatorname{Jcb}_{\mathrm{RV}}$ and $\operatorname{Jcb}_{\Gamma}$ whenever the things involved are defined.

For the rest of this section we do not need to assume that A, B are of RV-dimension n.

Lemma 9.15. *Let* $F^{\uparrow} : \mathbb{L}(U, f) \longrightarrow \mathbb{L}(V, g)$ *be a lift of* F. *Then for every* $f(\vec{u}) \in A$ *outside of a definable subset of* A *of dimension* $< n$ *and almost all* $(\vec{a}, \vec{u}) \in \mathrm{rv}^{-1}(f(\vec{u}), \vec{u})$,

$$\operatorname{rv}(\operatorname{Jcb}_{\mathrm{VF}} F^{\uparrow}(\vec{a}, \vec{u})) = \operatorname{Jcb}_{\mathrm{RV}} F(f(\vec{u}), \vec{u}).$$

Also, for almost all $(\vec{a}, \vec{u}) \in \mathbb{L}(U, f)$,

$$\operatorname{val}(\operatorname{Jcb}_{\mathrm{VF}} F^{\uparrow}(\vec{a}, \vec{u})) = \operatorname{Jcb}_{\Gamma} F(f(\vec{u}), \vec{u}).$$

Proof. Without loss of generality we may assume $\dim_{\mathrm{RV}}(A) = n$. Also, by Lemma 9.12 and compactness, we may assume $A, B \subseteq (\overline{\mathrm{K}}^{\times})^n$. For almost all $(\vec{a}, \vec{u}) \in \mathbb{L}(U, f)$, $\operatorname{Jcb}_{\mathrm{VF}} F^{\uparrow}(\vec{a}, \vec{u})$ may be obtained by running the construction described above with respect to $\mathrm{rv}^{-1}(A)$, $\mathrm{rv}^{-1}(B)$, $\mathrm{rv}^{-1}(C)$ and the projection maps. For almost all $f(\vec{u}) \in A$ this construction modulo the maximal ideal agrees with the construction that yields $\operatorname{Jcb}_{\mathrm{RV}} F(f(\vec{u}), \vec{u})$. The second assertion follows from Lemma 9.12. □

Let $a, b \in \mathcal{O}$ be definable units. Set $\mathrm{rv}(a) = t$ and $\mathrm{rv}(b) = s$. Clearly for any definable unit $c \in \mathcal{O}$ there is a definable bijection $f : \mathrm{rv}^{-1}(t) \longrightarrow \mathrm{rv}^{-1}(s)$ such that $\mathrm{d}_x f = c$ for all $x \in \mathrm{rv}^{-1}(t)$. This simple observation is used in the following analogue of Theorem 7.6, where we need to assume that f, g are finite-to-one, that is, $(U, f), (V, g) \in \mathrm{RV}[n]$ (for otherwise we may not have definable points in VF to work with).

Theorem 9.16. *Suppose that* S *is* (VF, Γ)*-generated and* f, g *are finite-to-one. Let* $\omega : U \longrightarrow \mathrm{RV}$ *be a definable function such that*

(1) $\omega(\vec{u}) = \operatorname{Jcb}_{\mathrm{RV}} F(f(\vec{u}), \vec{u})$ *for every* $\vec{u} \in U$ *outside of a definable subset of* U *of dimension* $< n$,
(2) $\operatorname{vrv}(\omega(\vec{u})) = \operatorname{Jcb}_{\Gamma} F(f(\vec{u}), \vec{u})$ *for* every $\vec{u} \in U$.

Then there is a lift $F^{\uparrow} : \mathbb{L}(U, f) \longrightarrow \mathbb{L}(V, g)$ *of* F *such that for almost all* $(\vec{a}, \vec{u}) \in \mathbb{L}(U, f)$,

$$\operatorname{rv}(\operatorname{Jcb}_{\mathrm{VF}} F^{\uparrow}(\vec{a}, \vec{u})) = \omega(\vec{u}).$$

Proof. As in the proof of Lemma 9.15 we may assume $A, B \subseteq (\overline{\mathrm{K}}^{\times})^n$ and hence $\operatorname{vrv} \circ \omega$ is the zero function. By Theorem 7.6 and Lemma 9.15 we are reduced to showing this for a definable subset $A_1 \subseteq A$ of RV-dimension $<n$. We do induction on $\dim_{\mathrm{RV}}(A_1)$. For the base case, since A_1 is finite, by Lemma 3.13 the rv-balls involved have centers, then it is easy to see that we may apply the simple observation above in one of the coordinates and use additive translation in the other coordinates.

We proceed to the inductive step. Let $f^{-1}(A_1) = U_1$, $F(U_1) = V_1$, and $B_1 = (g \circ F)(U_1)$. Since $\dim_{\mathrm{RV}}(A_1) = k < n$, without loss of generality, we may assume over a definable finite partition of A_1 that both $\mathrm{pr}_{\leq k} \upharpoonright A_1$ and $\mathrm{pr}_{\leq k} \upharpoonright B_1$ are finite-to-one. Let

$$f_1 : U_1 \longrightarrow \mathrm{pr}_{\leq k}(A_1), \quad g_1 : V_1 \longrightarrow \mathrm{pr}_{\leq k}(B_1), \quad F_1 : (U_1, f_1) \longrightarrow (V_1, g_1)$$

be the naturally induced definable functions and

$$C_1 = \{(f_1(\vec{u}), g_1(F_1(\vec{u}))) : \vec{u} \in U_1\} \subseteq \mathrm{pr}_{\leq k}(A_1) \times \mathrm{pr}_{\leq k}(B_1).$$

Clearly both $\mathrm{pr}_{\leq k} \upharpoonright C_1$ and $\mathrm{pr}_{>k} \upharpoonright C_1$ are finite-to-one and hence, by Theorem 7.6 and Lemma 9.15 again, there is a definable subset $A_2 \subseteq \mathrm{pr}_{\leq k}(A_1)$ and a lift $F_1^{\uparrow}$ of F_1 such that $\dim_{\mathrm{RV}}(\mathrm{pr}_{\leq k}(A_1) \smallsetminus A_2) < k$ and for all $f_1(\vec{u}) \in A_2$ and almost all $(\vec{a}, \vec{u}) \in \mathrm{rv}^{-1}(f_1(\vec{u}), \vec{u})$,

$$\operatorname{rv}(\operatorname{Jcb}_{\mathrm{VF}} F_1^{\uparrow}(\vec{a}, \vec{u})) = \operatorname{Jcb}_{\mathrm{RV}} F_1(f_1(\vec{u}), \vec{u}).$$

Let $U_2 = (\mathrm{pr}_{\leq k} \circ f)^{-1}(A_2)$. By the inductive hypothesis there is a lift of $F \upharpoonright (U_1 \smallsetminus U_2)$ as desired.

We construct a lift $F_2^{\uparrow}$ of $F \upharpoonright U_2$ as follows. Let $\vec{t} \in A_2$ and $U_{\vec{t}} = f^{-1}(\mathrm{fib}(A_1, \vec{t}))$. For any $\vec{a} \in \mathrm{rv}^{-1}(\vec{t})$ we have $\vec{a}$-definable centers

$$h_{\vec{a}} : \mathrm{fib}(A_1, \vec{t}) \cup \omega(U_{\vec{t}}) \longrightarrow \mathcal{O} \smallsetminus \mathcal{M}.$$

For any $(\vec{a}, \vec{u}) \in \mathrm{rv}^{-1}(\vec{t}, \vec{u})$, using the centers provided by $h_{\vec{a}}$ as above, we may construct an $(\vec{a}, \vec{u})$-definable bijection

$$F_{\vec{a},\vec{u}} : \mathrm{rv}^{-1}((\mathrm{pr}_{>k} \circ f)(\vec{u})) \longrightarrow \mathrm{rv}^{-1}((\mathrm{pr}_{>k} \circ g \circ F)(\vec{u}))$$

such that, for any $\vec{b} \in \mathrm{dom}(F_{\vec{a},\vec{u}})$,

$$\mathrm{Jcb}_{\mathrm{VF}} F_{\vec{a},\vec{u}}(\vec{b}) = (\mathrm{Jcb}_{\mathrm{VF}} F_1^{\uparrow}(\vec{a}, \vec{u}))^{-1} h_{\vec{a}}(\vec{u})$$

if the right-hand side is defined; otherwise let $F_{\vec{a},\vec{u}}$ be any $(\vec{a}, \vec{u})$-definable bijection. Now let $F_2^{\uparrow}$ be the lift of $F \upharpoonright U_2$ given by

$$(\vec{a}, \vec{b}, \vec{u}) \longmapsto (\vec{a}, F_{\vec{a},\vec{u}}(\vec{a}, \vec{b}), \vec{u}) \longmapsto (F_1^{\uparrow}(\vec{a}, \vec{u}), F_{\vec{a},\vec{u}}(\vec{a}, \vec{b})).$$

Multiplying the Jacobians of the two components (Lemma 9.12), we see that $F_2^{\uparrow}$ is as desired. □

10. Categories with volume forms

In this section we shall assume that the substructure S is (VF, Γ)-generated.

We shall define finer categories of definable subsets with the notion of the Jacobian factored in. This will make the homomorphisms between various Grothendieck groups compatible with the Jacobian transformation, as in the classical integration theory.

Definition 10.1 (VF-*categories With Volume Forms*). First set $\mu\mathrm{VF}[0, \cdot] = \mathrm{VF}[0, \cdot]$. Suppose $k > 0$. An object in the category $\mu\mathrm{VF}[k, \cdot]$ is a definable pair (A, ω), where $\mathrm{pvf}(A) \subseteq \mathrm{VF}^k$ and $\omega : A \longrightarrow \mathrm{RV}^{\times}$ is a function. The latter is understood as a *definable* RV-*volume form* on A, or simply a volume form on A. A *morphism* between two objects (A, ω), (A', ω') is a definable *essential bijection* $F : A \longrightarrow A'$, that is, a bijection that is defined outside of definable subsets of A, A' of VF-dimension $<k$, such that for every $\vec{x} \in \mathrm{dom}(F)$,

$$\omega(\vec{x}) = \omega'(F(\vec{x})) \cdot \mathrm{rv}(\mathrm{Jcb}_{\mathrm{VF}} F(\vec{x})).$$

We also say that such an F is an RV-*measure-preserving map*, or simply measure-preserving map.

An object in the category $\mu_{\Gamma}\mathrm{VF}[k, \cdot]$ is a pair (A, ω), where $A \in \mathrm{VF}[k, \cdot]$ and $\omega : A \longrightarrow \Gamma$ a definable function. The latter is understood as a *definable Γ-volume form* on A. A morphism between two objects (A, ω), (A', ω') is a definable essential bijection $F : A \longrightarrow A'$ such that for every $\vec{x} \in \mathrm{dom}(F)$,

$$\omega(\vec{x}) = \omega'(F(\vec{x})) + \mathrm{val}(\mathrm{Jcb}_{\mathrm{VF}} F(\vec{x})).$$

We also say that such an F is a *Γ-measure-preserving map*.

The category $\mathrm{VF}_1[k, \cdot]$ is the full subcategory of $\mu\mathrm{VF}[k, \cdot]$ such that $(A, \omega) \in \mathrm{VF}_1[k, \cdot]$ if and only of $\omega = 1$. The category $\mathrm{VF}_0[k, \cdot]$ is the full subcategory of $\mu\mathrm{VF}[k, \cdot]$ such that $(A, \omega) \in \mathrm{VF}_0[k, \cdot]$ if and only of $\omega = 0$.

The category $\mu\mathrm{VF}[k]$ is the full subcategory of $\mu\mathrm{VF}[k, \cdot]$ such that $(A, \omega) \in \mu\mathrm{VF}[k]$ if and only of $A \in \mathrm{VF}[k]$; similarly for the categories $\mu_{\Gamma}\mathrm{VF}[k]$, $\mathrm{VF}_1[k]$, $\mathrm{VF}_0[k]$.

The category $\mu\mathrm{VF}_*[\cdot]$ is defined to be the *direct sums* (coproducts) of the corresponding categories; similarly for the other ones.

Note that, for conceptual simplicity, we have allowed redundant objects in these categories. For example, if $(A, \omega) \in \mu\mathrm{VF}[k, \cdot]$ with $\dim_{\mathrm{VF}}(A) < k$ then (A, ω) is isomorphic to the empty object. Also, given how each $\mu\mathrm{VF}[k, \cdot]$ is defined, $\mu\mathrm{VF}_*[\cdot]$ is actually just the union of the corresponding categories.

Remark 10.2. Any two morphisms in $\mu\mathrm{VF}[k, \cdot]$ that agree almost everywhere may be naturally identified. It is conceptually more "correct" to define a morphism in $\mu\mathrm{VF}[k, \cdot]$ as such an equivalence class, although in practice it is more convenient to work with a representative. The "equivalence class" point of view is required when it comes to defining the Grothendieck semigroup. Consequently, since the Jacobian of the identity map is equal to 1 almost everywhere, by Lemma 9.12, every morphism is actually an isomorphism. This is very similar to birational maps in algebraical geometry. Below by a "morphism" we shall mean either an equivalence class or a representative of the class, depending on the context.

Definition 10.3 (RV-*categories With Volume Forms*). First set $\mu\mathrm{RV}[0] = \mathrm{RV}[0]$. Suppose $k > 0$. An object of the category $\mu\mathrm{RV}[k]$ is a definable triple (U, f, ω), where $(U, f) \in \mathrm{RV}[k]$ and $\omega : U \longrightarrow \mathrm{RV}^{\times}$ is a function, which is understood as a *volume form* on (U, f). A *morphism* between two objects (U, f, ω), (U', f', ω') is an essential isomorphism $F : (U, f) \longrightarrow (U', f')$ such that

(1) $\omega(\vec{u}) = \omega'(F(\vec{u})) \cdot \mathrm{Jcb}_{\mathrm{RV}} F(f(\vec{u}), \vec{u})$ for every $\vec{u} \in \mathrm{dom}(F)$ outside of a definable subset of $\mathrm{dom}(F)$ of dimension $< k$,
(2) $\mathrm{vrv}(\omega(\vec{u})) = (\mathrm{vrv} \circ \omega' \circ F)(\vec{u}) + \mathrm{Jcb}_{\Gamma} F(f(\vec{u}), \vec{u})$ for every $\vec{u} \in \mathrm{dom}(F)$.

It is easily seen from the definitions of $\mathrm{Jcb}_{\mathrm{RV}}$ and Jcb_{Γ} that every morphism here is actually an isomorphism.

The categories $\mu_{\Gamma}\mathrm{RV}[k]$, $\mathrm{RV}_1[k]$, $\mathrm{RV}_0[k]$ are similar to the corresponding VF-categories.

The categories $\mu\mathrm{RV}[\leq k]$, $\mu\mathrm{RV}[*]$ are defined to be the *direct sums* (coproducts) of the corresponding categories; similarly for the other ones.

Note that, as in the VF-categories with volume forms, we have allowed redundant objects in the RV-categories with volume forms. For example, for an object $(\mathbf{U}, \omega)$, if $\mathbb{L}\mathbf{U}$ is strictly degenerate then $(\mathbf{U}, \omega)$ is isomorphic to the empty object.

For any $(\mathbf{U}, \omega) \in \mu\mathrm{RV}[k]$, let $\mathbb{L}\omega$ be the function on $\mathbb{L}\mathbf{U}$ naturally induced by ω. The *lift* of $(\mathbf{U}, \omega)$ is the object $\mathbb{L}(\mathbf{U}, \omega) = (\mathbb{L}\mathbf{U}, \mathbb{L}\omega) \in \mu\mathrm{VF}[k]$.

For each $(A, \omega) \in \mu\mathrm{VF}[k]$ let $A_\omega = \{(\vec{a}, \omega(\vec{a})) : \vec{a} \in A)\}$. The function ω induces naturally a function on A_ω, which will also be denoted by ω for simplicity. Clearly (A, ω) and (A_ω, ω) are isomorphic.

Theorem 10.4. *Every object (A, ω) in $\mu\mathrm{VF}[k]$ is isomorphic to another object $\mathbb{L}(\mathbf{U}, \pi)$ in $\mu\mathrm{VF}[k]$, where $(\mathbf{U}, \pi) \in \mu\mathrm{RV}[k]$; similarly for other pairs of corresponding categories.*

Proof. By Corollary 5.6 there is a special bijection $T : A_\omega \longrightarrow A_1$ with A_1 an RV-pullback such that $(\mathrm{rv}(A_1), \mathrm{pr}_{\leq k}) \in \mathrm{RV}[k]$. Let $\omega_1 = \omega \circ T^{-1}$. So ω_1 is constant on every rv-polydisc. By Corollary 9.9 and Lemma 9.11, (A, ω) and (A_1, ω_1) are isomorphic. Let $\pi : \mathrm{rv}(A_1) \longrightarrow \mathrm{RV}$ be the function naturally induced by ω_1. Then $(\mathrm{rv}(A_1), \mathrm{pr}_{\leq k}, \pi)$ is as required.

The arguments for the other cases are essentially the same. □

Theorem 10.5. *Let $F : (\mathbf{U}, \omega) \longrightarrow (\mathbf{U}', \omega')$ be a $\mu\mathrm{RV}[k]$-isomorphism. Then there exists a measuring-preserving lift $F^\uparrow : \mathbb{L}(\mathbf{U}, \omega) \longrightarrow \mathbb{L}(\mathbf{U}', \omega')$ of F.*

Proof. Let $\omega^* : \mathrm{dom}(F) \longrightarrow \mathrm{RV}$ be the function given by $\vec{u} \longmapsto \omega(\vec{u})/\omega'(F(\vec{u}))$. By Theorem 9.16, there is a lift $F^\uparrow : \mathbb{L}\mathbf{U} \longrightarrow \mathbb{L}\mathbf{U}'$ such that $\mathrm{rv}(\mathrm{Jcb}_{\mathrm{VF}} F^\uparrow(\vec{a}, \vec{u})) = \omega^*(\vec{u})$ for almost all $(\vec{a}, \vec{u}) \in \mathbb{L}\mathbf{U}$, that is, $F^\uparrow$ is a $\mu\mathrm{VF}[k]$-isomorphism between $\mathbb{L}(\mathbf{U}, \omega)$ and $\mathbb{L}(\mathbf{U}', \omega')$. □

Corollary 10.6. *The map $\mathbb{L}$ induces surjective homomorphisms between the various Grothendieck semigroups associated with the categories with volume forms, for example:*

$$\mathbf{K}_+ \, \mu\mathrm{RV}[k] \longrightarrow \mathbf{K}_+ \, \mu\mathrm{VF}[k], \quad \mathbf{K}_+ \, \mu_\Gamma\mathrm{RV}[k] \longrightarrow \mathbf{K}_+ \, \mu_\Gamma\mathrm{VF}[k].$$

As mentioned in Step 3 in the introduction, various classical properties, in particular, special cases of Fubini's theorem and a change of variables formula, can already be verified for the inversions of the homomorphisms in Corollary 10.6 and hence we may complete the Hrushovski–Kazhdan construction of motivic integration right here. However, we choose to postpone this until we have achieved a more satisfying theory by putting forward a canonical description of the kernels of these homomorphisms in a sequel.

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