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Antipodal and Fixed Point Theorems for Sets in $\mathfrak{R}^n$ Bounded by a Finite Number of Spheres

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1. AN ANTIPODAL THEOREM FOR D^n

In $\mathfrak{R}^n$, let B_j^n $(1 \le j \le 2p + q)$ be pairwise disjoint n-balls (i.e., n-dimensional closed balls) all contained in the interior of another n-ball B_0^n, and let S_j^{n-1} denote the boundary of B_j^n $(0 \le j \le 2p + q)$. We are interested in the set

$$(1) \qquad D^n = \left\{ x \in B_0^n : x \text{ is not in the interior of } \bigcup_{j=1}^{2p+q} B_j^n \right\}.$$

We say that D^n is *the set bounded by the* $(n-1)$-*spheres* S_j^{n-1} $(0 \le j \le 2p + q)$. Here p and q are nonnegative integers. In case $p = q = 0$, D^n is just B_0^n. Each S_j^{n-1} is called a *boundary* $(n-1)$-*sphere* of D^n. Two points x, y of D^n are said to be *antipodal* if there is an index j, $0 \le j \le 2p + q$, such that $\{x, y\} \subset S_j^{n-1}$ and $(x + y)/2$ is the center of S_j^{n-1}. Our main result is the following antipodal theorem.

THEOREM 1. *Let D^n be a set in $\mathfrak{R}^n$ bounded by* $1 + 2p + q$ $(n-1)$-*spheres* S_j^{n-1} $(0 \le j \le 2p + q)$, *and let m be a positive integer independent of n. Let A_i, A_{-i} $(1 \le i \le m)$ be closed subsets of D^n satisfying conditions* (2), (3), *and* (4):

$$(2) \qquad \bigcup_{i=1}^{m} (A_i \cup A_{-i}) = D^n,$$

$$(3) \qquad A_i \cap A_{-i} = \varnothing \quad \textit{for } 1 \le i \le m,$$

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(4) *For any two antipodal points* x *and* y *on a boundary* $(n-1)$*-sphere* S_j^{n-1}, *there is an index* $i \in \{\pm 1, \pm 2, \ldots, \pm m\}$ *such that*

$$x \in A_i \quad \text{and} \quad y \in A_{-i} \quad \text{if } 0 \le j \le 2p;$$

$$\{x, y\} \subset A_i \quad \text{if } 2p + 1 \le j \le 2p + q.$$

Then we have the following conclusions:

$$m \ge n + 1. \tag{5}$$

(6) *There exist* $n+1$ *indices* $1 \le k_1 < k_2 < \cdots < k_{n+1} \le m$ *such that at least one of the two intersections*

$$\bigcap_{i=1}^{n+1} A_{(-1)^{i-1}k_i} \quad \text{and} \quad \bigcap_{i=1}^{n+1} A_{(-1)^{i}k_i}$$

is nonempty.

(7) *For each* j, $0 \le j \le 2p$, *there exist* n *indices* $1 \le h_1 < h_2 < \cdots < h_n \le m$ *such that*

$$S_j^{n-1} \cap \bigcap_{i=1}^{n} A_{(-1)^{i-1}h_i} \ne \varnothing.$$

When $p = q = 0$, Theorem 1 reduces to a result in [1], where we have seen that this result strengthens slightly the antipodal theorems of Lusternik–Schnirelmann–Borsuk and Borsuk–Ulam (see, e.g., [3, pp. 134–141]). The present Theorem 1 implies also immediately a fixed point theorem which we shall discuss before proving Theorem 1.

2. A FIXED POINT THEOREM FOR D^n

For a continuous mapping $g\colon D^n \to \mathfrak{R}^n$ and $x = (x_1, x_2, \ldots, x_n) \in D^n$, we write $g(x) = (g_1(x), g_2(x), \ldots, g_n(x))$ and $N(x) = \max_{1 \le i \le n} |g_i(x) - x_i|$.

THEOREM 2. *Let* D^n *be a set in* $\mathfrak{R}^n$ *bounded by* $1 + 2p + q$ $(n-1)$*-spheres* S_j^{n-1} $(0 \le j \le 2p + q)$. *Let* $g\colon D^n \to \mathfrak{R}^n$ *be a continuous mapping satisfying the following*:

(8) *For any index* j, $0 \le j \le 2p + q$, *and for any two antipodal points* x, y *on* S_j^{n-1}, *we can choose an index* i, $1 \le i \le n$, *and* $\epsilon = \pm 1$ *such that*

$$g_i(x) - x_i = \epsilon N(x) \quad \text{and} \quad g_i(y) - y_i = -\epsilon N(y) \quad \text{if } 0 \le j \le 2p;$$

$$g_i(x) - x_i = \epsilon N(x) \quad \text{and} \quad g_i(y) - y_i = \epsilon N(y)$$
$$\text{if } 2p + 1 \le j \le 2p + q.$$

Then g has a fixed point

Proof. Suppose g has no fixed point. Then $N(x) > 0$ for all $x \in D^n$. For $1 \le i \le n$, define

$$A_i = \{x \in D^n \colon g_i(x) - x_i = N(x)\},$$
$$A_{-i} = \{x \in D^n \colon g_i(x) - x_i = -N(x)\}.$$

These n pairs of closed sets obviously satisfy (2) and (3). Our hypothesis (8) says that condition (4) is also satisfied. But we have only n pairs of sets A_i, A_{-i}, contradicting conclusion (5) of Theorem 1. ∎

Remark. In Theorem 2, hypothesis (8) would become simpler if we replace it by

(9) For any ordered pair of antipodal points x, y on S_j^{n-1}, there is $t > 0$ such that

$$g(x) - x = t(y - g(y)) \quad \text{if } 0 \le j \le 2p;$$
$$g(x) - x = t(g(y) - y) \quad \text{if } 2p + 1 \le j \le 2p + q.$$

But this replacement would weaken Theorem 2 considerably, since (9) is much more restrictive than (8).

3. THREE COMBINATORIAL LEMMAS

In the proof of Theorem 1, we shall need Lemma 3, which depends on two other results proved many years ago in [1, 2]. For the convenience of the reader, we give their statements below as Lemma 1 and Lemma 2. Our first lemma is a combinatorial result for n-pseudomanifolds. Since this term may not be widely used, we recall its definition [4].

A finite simplicial complex M^n is called an *n-pseudomanifold* if the following conditions are satisfied:

(a) Every simplex of M^n is a face of at least one n-simplex of M^n.

(b) Every $(n-1)$-simplex of M^n is a face of at most two n-simplexes of M^n.

(c) If s and s' are n-simplexes of M^n, there is a finite sequence $s = s_1, s_2, \ldots, s_m = s'$ of n-simplexes of M^n such that s_i and s_{i+1} have an $(n-1)$-face in common for $1 \le i < m$.

An $(n-1)$-simplex of M^n is called a *boundary* $(n-1)$-*simplex* if it is a face of exactly one n-simplex of M^n.

LEMMA 1. *Let M^n be an n-pseudomanifold. To each vertex of M^n, let a nonzero integer, positive or negative, be assigned such that for any* 1-*simplex of M^n, the integers assigned to its two vertices have sum different from* 0. *Then we have the congruence*

$$\sum_{0<k_1<k_2<\cdots<k_{n+1}} \Big\{\alpha\big(k_1, -k_2, k_3, -k_4, \ldots, (-1)^n k_{n+1}\big) + \alpha\big(-k_1, k_2, -k_3, k_4, \ldots, (-1)^{n+1} k_{n+1}\big)\Big\} \equiv \sum_{0<k_1<k_2<\cdots<k_n} \beta\big(k_1, -k_2, k_3, -k_4, \ldots, (-1)^{n-1} k_n\big) \mod 2. \tag{10}$$

Here $\alpha(h_1, h_2, h_3, \ldots, h_{n+1})$ denotes the number of those n-simplexes of M^n whose vertices receive the integers $h_1, h_2, h_3, \ldots, h_{n+1}$ (in an arbitrary order of arrangement). Similarly, $\beta(h_1, h_2, h_3, \ldots, h_n)$ is the number of those boundary $(n-1)$-simplexes of M^n whose vertices receive the integers $h_1, h_2, h_3, \ldots, h_n$.

Lemma 1 is a special case of a theorem in [2], where the n-pseudomanifold is oriented and an equality replaces congruence (10).

In the statement of Lemma 2, an *octahedral subdivision* of an $(n-1)$-sphere S^{n-1} in $\mathfrak{R}^n$ is the subdivision of S^{n-1} into 2^n $(n-1)$-simplexes by n arbitrarily chosen orthogonal hyperplanes in $\mathfrak{R}^n$ passing through the center of S^{n-1}. A *barycentric derived octahedral triangulation* of S^{n-1} is the triangulation of S^{n-1} obtained by a finite number of successive barycentric subdivisions of an octahedral subdivision of S^{n-1}. The next lemma is already proved in [1] as a generalization of a result of Tucker (see [3, pp. 134–141]).

LEMMA 2. *Let M^{n-1} be the $(n-1)$-pseudomanifold obtained by a barycentric derived octahedral triangulation of an $(n-1)$-sphere S^{n-1}. To each vertex of M^{n-1} let one of the $2m$ integers $\pm 1, \pm 2, \ldots, \pm m$ be assigned such that the following conditions are fulfilled*:

(a) *The integers assigned to the two vertices of any* 1-*simplex of M^{n-1} have sum different from* 0.

(b) *The integers assigned to any two antipodal vertices of M^{n-1} have sum* 0.

Then the congruence

(11)
$$\sum_{1 \le k_1 < k_2 < \cdots < k_n \le m} \beta\left(k_1, -k_2, k_3, -k_4, \ldots, (-1)^{n-1} k_n\right) \equiv 1 \mod 2$$

holds. Here $\beta(k_1, -k_2, k_3, -k_4, \ldots, (-1)^{n-1} k_n)$ *is the number of those* $(n-1)$*-simplexes of* M^{n-1} *whose vertices receive the indicated integers. In particular,* $m \ge n$.

For a set D^n in $\Re^n$ bounded by $1 + 2p + q$ $(n-1)$-spheres S_j^{n-1} $(0 \le j \le 2p + q)$, a *barycentric derived octahedral triangulation* of D^n is a triangulation of D^n such that its restriction to each S_j^{n-1} $(0 \le j \le 2p+q)$ is a barycentric derived octahedral triangulation of S_j^{n-1}.

From Lemmas 1 and 2, we can derive the following.

LEMMA 3. *Let* D^n *be a set in* $\Re^n$ *bounded by* $1 + 2p + q$ $(n-1)$*-spheres* S_j^{n-1} $(0 \le j \le 2p+q)$. *Let* M^n *be the* n*-pseudomanifold obtained by a barycentric derived octahedral triangulation of* D^n. *Let* m *be a positive integer independent of* n. *For each vertex* v *of* M^n, *let* $\phi(v)$ *be an integer among* $\pm 1, \pm 2, \ldots, \pm m$ *satisfying*

(a) $\phi(v_1) + \phi(v_2) \ne 0$ *for the vertices* v_1, v_2 *of any* 1*-simplex of* M^n,

(b) $\phi(u) + \phi(v) = 0$ *for any two antipodal vertices* u, v *on* S_j^{n-1} *if* $0 \le j \le 2p$,

(c) $\phi(u) = \phi(v)$ *for any two antipodal vertices* u, v *on* S_j^{n-1} *if* $2p + 1 \le j \le 2p + q$.

Then we have the congruence

(12)
$$\sum_{1 \le k_1 < k_2 < \cdots < k_{n+1} \le m} \left\{\alpha\left(k_1, -k_2, k_3, -k_4, \ldots, (-1)^n k_{n+1}\right) + \alpha\left(-k_1, k_2, -k_3, k_4, \ldots, (-1)^{n+1} k_{n+1}\right)\right\} \equiv 1 \mod 2.$$

In particular, $m \ge n + 1$. *Here* $\alpha(h_1, h_2, \ldots, h_{n+1})$ *has the same meaning as in Lemma* 1.

Proof. By Lemma 1, congruence (12) is equivalent to

(13)
$$\sum_{1 \le k_1 < k_2 < \cdots < k_n \le m} \beta\left(k_1, -k_2, k_3, -k_4, \ldots, (-1)^{n-1} k_n\right) \equiv 1 \mod 2.$$

Here $\beta(k_1, -k_2, \ldots, (-1)^{n-1}k_n)$ is the number of those boundary $(n-1)$-simplexes of M^n whose vertices $v_1, v_2, \ldots, v_n$ can be so arranged that $\phi(v_i) = (-1)^{i-1}k_i$ $(1 \le i \le n)$. As every boundary $(n-1)$-simplex of M^n lies on one of the $1 + 2p + q$ boundary $(n-1)$-spheres S_j^{n-1} $(0 \le j \le 2p + q)$, we have

$$\beta\left(k_1, -k_2, \ldots, (-1)^{n-1}k_n\right) = \sum_{j=0}^{2p+q} \beta_j\left(k_1, -k_2, \ldots, (-1)^{n-1}k_n\right),$$

where β_j counts the number of relevant $(n-1)$-simplexes on S_j^{n-1}. If we define γ_j $(0 \le j \le 2p+q)$ by

(14)
$$\gamma_j = \sum_{1 \le k_1 < k_2 < \cdots < k_n \le m} \beta_j\left(k_1, -k_2, k_3, -k_4, \ldots, (-1)^{n-1}k_n\right),$$

then (13) may be written

$$\sum_{j=0}^{2p+q} \gamma_j \equiv 1 \mod 2. \tag{15}$$

In view of properties (a), (b) of ϕ, Lemma 2 shows that γ_j is odd if $0 \le j \le 2p$. On the other hand, property (c) of ϕ clearly implies that γ_j is even if $2p + 1 \le j \le 2p + q$. This proves (15) and therefore the desired congruence (12). ∎

Remark. Like Lemma 1, Lemmas 2 and 3 can be sharpened if we consider an orientation of the pseudomanifold and replace congruences by equalities.

4. PROOF OF THEOREM 1

Let D^n be a set in $\mathfrak{R}^n$ bounded by $1 + 2p + q$ $(n-1)$-spheres S_j^{n-1} $(0 \le j \le 2p + q)$, and let A_i, A_{-i} $(1 \le i \le m)$ be $2m$ closed subsets of D^n satisfying (2), (3), and (4). Let λ be the Lebesgue number for the closed covering $\{A_i: i = \pm 1, \pm 2, \ldots, \pm m\}$ of D^n. Make D^n into an n-pseudomanifold M^n by a barycentric derived octahedral triangulation of D^n such that the diameter of each simplex of M^n is less than λ. For each vertex v of M^n, choose an integer $\phi(v) \in \{\pm 1, \pm 2, \ldots, \pm m\}$ such that

$$v \in A_{\phi(v)} \tag{16}$$

and conditions (a), (b), (c) of Lemma 3 are satisfied. Such an assignment ϕ is possible on account of hypothesis (2), (3), (4).

By Lemma 3, we have $m \geq n + 1$ and congruence (12). This congruence implies the existence of integers $1 \leq k_1 < k_2 < \cdots < k_{n+1} \leq m$ such that

$$\alpha\left(k_1, -k_2, k_3, \ldots, (-1)^n k_{n+1}\right) + \alpha\left(-k_1, k_2, -k_3, \ldots, (-1)^{n+1} k_{n+1}\right) \geq 1.$$

For these $n + 1$ integers $1 \leq k_1 < k_2 < \cdots < k_{n+1} \leq m$, there is an n-simplex of M^n with vertices $v_1, v_2, \ldots, v_{n+1}$ such that either

$$\phi(v_i) = (-1)^{i-1} k_i \quad (1 \leq i \leq n + 1)$$

or

$$\phi(v_i) = (-1)^i k_i \quad (1 \leq i \leq n + 1).$$

In view of (16), we have either

$$v_i \in A_{(-1)^{i-1} k_i} \quad (1 \leq i \leq n + 1) \tag{17}$$

or

$$v_i \in A_{(-1)^i k_i} \quad (1 \leq i \leq n + 1). \tag{18}$$

Since each simplex of M^n has a diameter less than the Lebesgue number λ, (17) or (18) implies

$$\bigcap_{i=1}^{n+1} A_{(-1)^{i-1} k_i} \neq \varnothing \quad \text{or} \quad \bigcap_{i=1}^{n+1} A_{(-1)^i k_i} \neq \varnothing. \tag{19}$$

Thus we have proved the existence of $n + 1$ indices $1 \leq k_1 < k_2 < \cdots < k_{n+1} \leq m$ such that at least one of the two intersections in (19) is nonempty.

With γ_j defined by (14), we have seen in the proof of Lemma 3 (or directly from Lemma 2) that γ_j is odd if $0 \leq j \leq 2p$. Let us fix an index j such that $0 \leq j \leq 2p$. Since γ_j is odd, there exist n integers $1 \leq h_1 < h_2 < \cdots < h_n \leq m$ such that

$$\beta_j\left(h_1, -h_2, h_3, -h_4, \ldots, (-1)^{n-1} h_n\right) \geq 1,$$

where β_j counts the number of relevant $(n-1)$-simplexes on S_j^{n-1}. This means the existence of an $(n-1)$-simplex on S_j^{n-1} with vertices $w_1, w_2, \ldots, w_n$ such that $\phi(w_i) = (-1)^{i-1} h_i$ $(1 \leq i \leq n)$. By (16), we have

$$w_i \in S_j^{n-1} \cap A_{(-1)^{i-1} h_i} \quad (1 \leq i \leq n),$$

and, because each simplex of M^n has a diameter less than the Lebesgue number λ, we have

$$S_j^{n-1} \cap \bigcap_{i=1}^{n} A_{(-1)^{i-1}h_i} \neq \emptyset. \tag{20}$$

This completes the proof of Theorem 1. ∎

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