Note

A combinatorial proof of a result of Gessel and Greene

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Abstract

A combinatorial proof is given of a result of Gessel and Greene relating the sizes of two classes of permutations. A natural map from one class to the other is described in terms of a shared recursive structure.

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1. Introduction

A permutation \( \pi \) on \( \{1, 2, \ldots, n\} \) symbols has length \( \ell(\pi) = n \). When \( \pi \) is written in one line form, the symbol in position \( i \) is called a maximum (respectively, minimum) if it is greater than (respectively, less than) both the symbol in position \( i - 1 \) and the symbol in position \( i + 1 \). Both terms are well defined for all even positions of a permutation of odd length.

In an analysis of skew tableaux, Gessel and Greene considered permutations of odd length with no maxima or minima in even positions \[2\]. Gessel had previously considered permutations with no maxima in even positions \[1\]. By comparing exponential generating series, they obtained (6), and concluded

\[ a_n = 2^{n-1} b_n, \tag{1} \]

where \( a_n \) is the number of permutations of length \( 2n - 1 \) with no maxima in even positions, and \( b_n \) is the number of permutations of length \( 2n - 1 \) with no maxima or minima in even positions. The result can also be obtained using the pattern algebra of Goulden and Jackson \[4\]. Gessel and Greene asked \[3\] for a combinatorial proof of (1), which this paper provides. The result is strengthened to apply to a finer partition of the permutations involved, and a natural map is given from one class of permutations to the other.

2. Notation

Let \( \mathcal{A} \) (respectively, \( \mathcal{B} \)) denote the sets of permutations of odd length with no maxima (respectively, no maxima or minima) in even positions. So \( a_n \) (respectively, \( b_n \)) is the number of elements of \( \mathcal{A} \) (respectively, \( \mathcal{B} \)) of length \( 2n - 1 \).
Denote the subset of $\mathcal{A}$ (respectively, $\mathcal{B}$) in which the largest symbol occurs in position $2i - 1$ by $\mathcal{A}_i$ (respectively, $\mathcal{B}_i$), and use $a_{n,i}$ (respectively, $b_{n,i}$) to denote the number of such permutations of length $2n - 1$. In particular,

\[
a_n = \sum_{i=1}^{n} a_{n,i} \quad \text{and} \quad b_n = \sum_{i=1}^{n} b_{n,i}.
\]

(2)

For $\pi = (p_1, p_2, \ldots, p_n) \in \mathcal{S}_n$, the complement of $\pi$, denoted by $\pi^c$, is

\[
\pi^c = (n - p_1 + 1, n - p_2 + 1, \ldots, n - p_n + 1).
\]

The complement operation converts minima to maxima, and maxima to minima, and is a length-preserving involution of $\mathcal{B}$. The reverse of $\pi$, denoted by $\pi^r$, is defined by

\[
\pi^r = (p_n, p_{n-1}, \ldots, p_1).
\]

The reverse operation is an involution on both $\mathcal{A}$ and $\mathcal{B}$, and it follows that

\[
a_{n,i} = a_{n,n-i+1} \quad \text{and} \quad b_{n,i} = b_{n,n-i+1}.
\]

(3)

Example 1. $(5, 3, 1, 2, 4)^c = (1, 3, 5, 4, 2)$ and $(5, 3, 1, 2, 4)^r = (4, 2, 1, 3, 5)$.

A permutation $\pi$ of length $n$ on a well-ordered set of symbols other than $\{1, 2, \ldots, n\}$ can be canonically decomposed into a pattern and a support. The pattern of $\pi$, denoted by $\pi^i$, is obtained by replacing the $i$th smallest symbol of $\pi$ by $i$ for each $1 \leq i \leq n$. The support of $\pi$, denoted by $\text{Supp}(\pi)$, is the set on which $\pi$ is a permutation. Together, a pattern and support uniquely encode a permutation.

Example 2. The permutation $(9, 4, 1, 6, 7)$ has pattern $(5, 2, 1, 3, 4)$ and support $\{1, 4, 6, 7, 9\}$.

A permutation $\pi = (p_1, p_2, \ldots, p_n) \in \mathcal{S}_n$ with $p_i = n$, is uniquely decomposed into a prefix, denoted by $\pi_L$, and a suffix, denoted by $\pi_R$, through

\[
\pi_L = (p_1, p_2, \ldots, p_i), \quad \pi_R = (p_i, p_{i+1}, \ldots, p_n).
\]

Let $\Omega$ be the map defined by

\[
\Omega(\pi) = (\pi^i_L, \pi^i_R, \text{Supp}(\pi_L) \setminus \{n\}).
\]

Since $\pi$ is a permutation on $\{1, 2, \ldots, n\}$, the supports of both the prefix and suffix can be recovered from $\text{Supp}(\pi_L) \setminus \{n\}$, and $\Omega$ is injective.

Example 3. For $\pi = (7, 6, 1, 4, 9, 8, 5, 3, 2)$, $\pi_L = (7, 6, 1, 4, 9)$ and $\pi_R = (9, 8, 5, 3, 2)$, so

\[
\Omega(\pi) = ((7, 6, 1, 4, 9)^i, (9, 8, 5, 3, 2)^i, \text{Supp}(7, 6, 1, 4, 9) \setminus \{9\})
\]

\[
= ((5, 2, 1, 3, 4), (5, 4, 3, 2, 1), \{1, 4, 6, 7\}).
\]

A permutation of length $n \geq 2$ beginning with the symbol $n$ is uniquely encoded by an ordered pair specifying its second symbol and the pattern of the trailing $n - 2$ symbols. Thus the map $\omega$ defined by

\[
\omega((n, p_2, p_3, \ldots, p_n)) = (p_2, (p_3, p_4, \ldots, p_n)^i)
\]

is also injective.

Example 4. $\omega(5, 2, 1, 3, 4) = (2, (1, 3, 4)^i) = (2, (1, 2, 3))$. 

3. Proof of the main theorem

We shall need the following elementary lemmas.

Lemma 5. For all \( n \geq 1 \),
\[
a_{n+1,1} = 2na_n.
\]

Proof. The left side of this equation is the number of length \( 2n + 1 \) elements of \( \mathcal{A}_1 \). The right side is obtained by enumerating the \( \omega \)-images of these permutations, noting that the ordered pair \((k, \pi)\) is the \( \omega \)-image of a length \( 2n + 1 \) element of \( \mathcal{A}_1 \) precisely when \( k \in \{1, 2, \ldots, 2n\} \) and \( \pi \) is a length \( 2n - 1 \) element of \( \mathcal{A} \). \( \square \)

Lemma 6. For \( n \geq 1 \),
\[
a_{n+1,1} = 2na_n.
\]

Proof. The ordered pair \((k, \pi)\) is the \( \omega \)-image of an element of \( \mathcal{B}_1 \) precisely when \( \pi \in \mathcal{B} \) and the second symbol of \( \omega^{-1}(k, \pi) \) is not a minimum. So, for \( \pi \in \mathcal{B} \) of length \( 2n - 1 \), \((k, \pi)\) is an element of \( \omega(\mathcal{B}_1) \) if and only if \( k \) is greater than the first symbols of \( \pi \). Thus, exactly one of \((k, \pi)\) and \((2n + 1 - k, \pi^c)\) is the \( \omega \)-image an element of \( \mathcal{B}_1 \), and the number of length \( 2n + 1 \) elements of \( \mathcal{B}_1 \) is \( \frac{1}{2}2nb_n \), completing the proof. \( \square \)

We now prove the main theorem.

Theorem 8. For all \( i \leq n \) and for all \( n \),
\[
a_{n,i} = 2^{n-1}b_{n,i} \quad \text{and} \quad a_n = 2^{n-1}b_n.
\]

Proof. As a base case, the unique permutation on one symbol is an element of \( \mathcal{A} \) and \( \mathcal{B} \). Since it starts with its largest symbol,
\[
a_1 = a_{1,1} = b_1 = b_{1,1} = 1,
\]
and the result holds for \( n = 1 \).

We now proceed by induction and assume that the result holds for all \( n \leq k \). By (3), Lemmas 6 and 7, and the induction hypothesis
\[
a_{k+1,k+1} = a_{k+1,1} = 2ka_k = 2k2^{k-1}b_k = 2^k b_{k+1,1} = 2^k b_{k+1,k+1}.
\]
This leaves the case \( 2 \leq i \leq k \), for which we have, by Lemma 5 and the induction hypothesis
\[
a_{k+1,i} = \left(\frac{2k}{2i - 2}\right)a_{i,1}a_{k-i+2,1}
\]
\[
= 2^k \left(\frac{2k}{2i - 2}\right)b_{i,1}b_{k-i+2,1}
\]
\[
= 2^k b_{k+1,i}.
\]
Thus \( a_{k+1,i} = 2^k b_{k+1,i} \) for all \( 1 \leq i \leq k + 1 \). The final step of the induction follows from (2). For the second statement, we have, with the aid of the first statement,
\[
a_{k+1} = \sum_{i=1}^{k+1} a_{k+1,i} = 2^k \sum_{i=1}^{k+1} b_{k+1,i} = 2^k b_{k+1}. \quad \square
\]

4. Element-wise action

There are \( 2^{n-1} \) ways to decorate a permutation of length \( 2n - 1 \) by marking a subset of even positions with dots. Use \( \mathcal{B} \) (respectively, \( \mathcal{B}_1 \)), to denote the set of permutations in \( \mathcal{B} \) (respectively, \( \mathcal{B}_1 \)) with all possible decorations. Then Theorem 8 asserts that there is a length preserving bijection from \( \mathcal{B} \) to \( \mathcal{A} \). Explicitly constructing this bijection gives a slightly stronger result.

The operations \( \cdot^r, \cdot_L, \cdot_R, \cdot^\downarrow, \) and \( \cdot^c \) are extended to act on \( \mathcal{B} \), with \( \cdot^r, \cdot_L, \) and \( \cdot_R \) treating dots as part of the associated symbols, and \( \cdot^\downarrow \) and \( \cdot^c \) preserving the position of dots. The maps \( \Omega \) and \( \omega \) are extended to \( \mathcal{B} \) in terms of these elementary operations.

Example 9. \((4, \hat{3}, 1, 2, 5)^r = (5, 2, 1, 3, 4), (4, \hat{2}, 1, 3, 5)^c = (2, \hat{4}, 5, 3, 1), \) and \((4, \hat{3}, 1)^\downarrow = (3, \hat{2}, 1).\)

Corollary 10. The map \( \psi: \mathcal{B} \to \mathcal{A} \), defined recursively by
\[
\psi(\pi) = \begin{cases} 
(1) & \text{if } \pi = (1), \\
\omega^{-1} \psi_1 \omega(\pi), & \pi \in \mathcal{B}_1 \setminus \{(1)\}, \\
\Omega^{-1} \psi_2 \Omega(\pi) & \text{otherwise},
\end{cases}
\]

where \( \psi_1: \omega(\mathcal{B}_1 \setminus \{(1)\}) \to \omega(\mathcal{A}_1 \setminus \{(1)\}) \) is defined by
\[
\psi_1(k, \pi) = (k, \psi(\pi)),
\]
\[
\psi_1(\hat{k}, \pi) = (\ell(\pi) + 2 - k, \psi(\pi^c)),
\]

and \( \psi_2: \Omega(\mathcal{B} \setminus \mathcal{B}_1) \to \Omega(\mathcal{A} \setminus \mathcal{A}_1) \) is defined by
\[
\psi_2(\pi_L, \pi_R, A) = (\psi(\pi_L), \psi(\pi_R), A),
\]

is a bijection from \( \mathcal{B} \) to \( \mathcal{A} \) that preserves the length of a permutation, the position of its maximum element, and the partitioning of the symbols of the permutation into those occurring before and those occurring after the maximum element.

Proof. Induction on the length of permutations verifies that \( \psi, \psi_1, \) and \( \psi_2 \) are well defined. The map \( \psi \) is seen to preserve the length of a permutation, the position of its maximum element, and the partitioning of the symbols into those occurring before and those occurring after the maximum element. It remains only to show that \( \psi \) is bijective.

Consider the map \( \phi: \mathcal{A} \to \mathcal{B} \) defined by
\[
\phi(\pi) = \begin{cases} 
(1) & \text{if } \pi = (1), \\
\omega^{-1} \phi_1 \omega(\pi), & \pi \in \mathcal{A}_1 \setminus \{(1)\}, \\
\Omega^{-1} \phi_2 \Omega(\pi) & \text{otherwise},
\end{cases}
\]

where \( \phi_1: \omega(\mathcal{A}_1 \setminus \{(1)\}) \to \omega(\mathcal{B}_1 \setminus \{(1)\}) \) is defined by
\[
\phi_1(k, \pi) = \begin{cases} 
(k, \phi(\pi)) & \text{if } (k, \phi(\pi)) \in \omega(\mathcal{B}_1), \\
((\ell(\pi) + 2 - k, \phi(\pi)^c) & \text{otherwise},
\end{cases}
\]
and \( \phi_2: \Omega(\mathcal{A} \setminus \mathcal{A}_1) \to \Omega(\mathcal{B} \setminus \mathcal{B}_1) \) is defined by
\[
\phi_2(\pi_L, \pi_R, A) = (\phi(\pi_L), \phi(\pi_R), A).
\]
It is easily verified that \( \psi \) and \( \phi \) are mutually inverse, completing the proof. \( \square \)

**Example 11.** For \( \pi = (7, 6, 1, 4, 9, 8, 5, 3, 2) \),
\[
\psi(\pi) = \Omega^{-1}(\psi(5, 2, 1, 3, 4), \psi(5, 4, 3, 2, 1), \{1, 4, 6, 7\})
\]
\[
= \Omega^{-1}(\omega^{-1}(3, \psi((1, 2, 3), \psi(5, 3, 2, 1)), \{1, 4, 6, 7\}))
\]
\[
= \Omega^{-1}(\omega^{-1}(3, 2, 1), \omega^{-1}(4, 3, 1, 2), \{1, 4, 6, 7\})
\]
\[
= \Omega^{-1}(5, 3, 4, 2, 1), (5, 4, 3, 1, 2), \{1, 4, 6, 7\})
\]
\[
= (1, 4, 7, 6, 9, 8, 5, 2, 3).
\]

5. Generating series

The observations of Section 3 can be used to explicitly determine \( A(x) \) and \( B(x) \), the exponential generating series for \( \mathcal{A} \) and \( \mathcal{B} \) with respect to length.

**Theorem 12.**
\[
A(x) = \sum_{n=1}^{\infty} a_n x^{2n-1} \frac{(2n-1)!}{(2n-1)!} = \frac{\tanh x}{1 - x \tanh x}.
\]

**Proof.** For notational convenience, we denote the exponential generating series for elements of \( \mathcal{A}_1 \) with respect to length by
\[
C(x) = \sum_{i=1}^{\infty} a_{n,1} x^{2n-1} \frac{(2n-1)!}{(2n-1)!}.
\]
When interpreted in terms of generating series, Lemma 5 and (2) give
\[
\frac{d}{dx} A(x) = \left( \frac{d}{dx} C(x) \right)^2.	ag{4}
\]
By Lemma 6, \( A(x) \) and \( C(x) \) also satisfy
\[
\frac{d}{dx} C(x) = x A(x) + 1.	ag{5}
\]
Combining (4) and (5) we see that \( A(x) \) must satisfy the differential equation
\[
\frac{d}{dx} A(x) = (x A(x) + 1)^2,
\]
subject to the initial condition \( A(0) = 0 \), since there are no permutations of odd length on 0 symbols. Making the substitution \( A(x) = P(x)/(1 - x P(x)) \), produces
\[
\frac{(d/dx) P(x) + (P(x))^2}{(1 - x P(x))^2} = \left( \frac{1}{1 - x P(x)} \right)^2,
\]
which further simplifies to
\[
\frac{d}{dx} P(x) = 1 - P(x)^2,
\]
subject to the initial condition \( P(0) = 0 \). This has the unique solution \( P(x) = \tanh(x) \), and the unique solution to the original differential equation is thus

\[
A(x) = \frac{\tanh(x)}{1 - x \tanh(x)}. \quad \square
\]

The appearance of \( \tanh(x) \) in this generating series is explained in terms of alternating permutations. Starting with \( \tan(x) \), the exponential generating series for odd-length alternating permutations, an application of the principle of inclusion and exclusion provides an alternate proof of Theorem 12.

**Corollary 13.**

\[
B(x) = \sum_{n=1}^{\infty} b_n \frac{x^{2n-1}}{(2n-1)!} = \frac{\sqrt{2} \tanh(x/\sqrt{2})}{1 - (x/\sqrt{2}) \tanh(x/\sqrt{2})}.
\]

**Proof.** By Theorem 8 we see that \( A(x) \) and \( B(x) \) satisfy the relation

\[
B(x) = \sqrt{2} A \left( \frac{x}{\sqrt{2}} \right).
\]

The result follows from Theorem 12. \( \square \)

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**References**