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Generalized Kuhn–Tucker Conditions and Duality for Continuous Nonlinear Programming Problems

THOMAS W. REILAND* AND MORGAN A. HANSON†

* *Department of Statistics and Graduate Program in Operations Research, North Carolina State University, Raleigh, North Carolina 27650; and*
† *Department of Statistics, Florida State University, Tallahassee, Florida 32306*

Submitted by M. Aoki

1. INTRODUCTION

Bellman [1] first introduced continuous time programming in the treatment of production and inventory "bottleneck" problems. Tyndall [21] extended Bellman's theory and obtained existence and duality theorems for a class of continuous linear programming problems. The theory was again extended by Levinson [13] who dealt with problems of the following form:

Primal Linear Problem

Maximize

$$\int_0^T a'(t)\, z(t)\, dt$$

subject to

$$z(t) \geqslant 0, \qquad 0 \leqslant t \leqslant T,$$

and

$$B(t)\, z(t) \leqslant c(t) + \int_0^t K(t, s)\, z(s)\, ds, \qquad 0 \leqslant t \leqslant T,$$

where $z(\cdot)$ is a bounded and measurable n-dimensional function on $[0, T]$.

Dual Linear Problem

Minimize

$$\int_0^T c'(t)\, w(t)\, dt$$

subject to

$$w(t) \geqslant 0, \qquad 0 \leqslant t \leqslant T,$$

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and

$$B'(t)\, w(t) \geqslant a(t) + \int_t^T K'(s, t)\, w(s)\, ds, \qquad 0 \leqslant t \leqslant T.$$

In this generalization Levinson required that each element of the time-dependent matrices a, B, c, and K be piecewise continuous and that the latter three matrices satisfy the following positivity conditions:

(i) $B(t)$, $c(t)$, and $K(t, s)$ are nonnegative for $0 \leqslant s \leqslant t \leqslant T$,

(ii) $\{x \in E^n : x \geqslant 0, B(t)\, x \leqslant 0, 0 \leqslant t \leqslant T\} = \{0\}$,

(iii) $K(t, s) = 0$ for $s > t$.

These assumptions enabled Levinson to construct uniform bounds for the primal and dual solution sets and apply weak convergence in the normed linear space $L^2[0, T]$ to obtain existence and duality theorems.

The assumptions made by Levinson drew considerable attention in the subsequent literature. Hanson and Mond [9] relaxed the assumption of piecewise continuity for the vector functions a and c and required only that their components be bounded and measurable. Grinold [7] further extended these results by similarly relaxing the assumption of piecewise continuity for the entries of the matrices B and K. Later Schecter [19] investigated the effects of allowing the components of c to be in $L^1[0, T]$ and eliminated the requirement that $K(t, s) = 0$ for $s > t$. In each case existence and duality theorems were obtained.

In summary, this sequence of works established that the assumptions (i) and (ii), together with the assumptions of boundedness and measurability, provide sufficient conditions for the existence of optimal solutions and duality for problems of the form stated above.

Hanson and Mond [9] further generalized the problem by introducing an objective function of the form

$$\int_0^T \phi_2(z(t))\, dt,$$

where ϕ is a twice differentiable concave function, and obtained similar existence and duality results. The introduction of concavity to the objective function was considered an important development from an economic perspective since it allowed the concept of diminishing returns to be represented in the continuous time frame. This work also presented a continuous time analogue to the Kuhn–Tucker Theorem [11] for nonlinear programming in finite dimensions. Farr and Hanson [4] introduced nonlinearity to the constraints and considered the problem

Maximize

$$\int_0^T \phi(z(t))\, dt$$

subject to

$$z(t) \geqslant 0, \qquad 0 \leqslant t \leqslant T,$$

and

$$f(z(t)) \leqslant c(t) + \int_0^t K(t, s)\, g(z(s))\, ds, \qquad 0 \leqslant t \leqslant T,$$

where $z(\cdot)$ is a bounded and measurable n-dimensional function on $[0, T]$, ϕ is a continuously twice differentiable scalar function, $K(t, s)$ has nonnegative entries with $K(t, s) = 0$ for $s > t$, $c(\cdot) \geqslant 0$, and each component of $-f$ and g is concave and differentiable. Under positivity conditions analogous to those of Levinson [13], they obtained existence and duality theorems for this class of problems as well as a theorem which establishes the necessity and sufficiency of a set of Kuhn–Tucker conditions.

The technique used by Farr and Hanson to prove duality entailed a linearization of the constraints by expansion about the optimal solution. This technique allowed application of Grinold's Duality Theorem [7] and provided duality for a linearized form of the problem. To establish duality for the nonlinear problem Farr and Hanson assumed that, for each $t \in [0, T]$, the components of the gradient vector of ϕ, evaluated at the optimal solution $\bar{z}(\cdot)$, be either all negative or all nonnegative. Modification to eliminate the need for such an assumption is desirable since the assumption relates to properties of the objective function beyond the typical regularity conditions (e.g. whether the function is continuous or differentiable). The only possible finite-dimensional analogue to a requirement of this nature is the constraint qualification proposed by Geoffrion [5, pp. 6–7].

In this paper we extend the results obtained by Farr and Hanson [4] by considering a more general form of the constraints and removing the abovementioned assumption on the objective function. We develop a constraint qualification analogous to that presented by Zangwill [23] which hopefully will allow for direct extensions of other basic concepts underlying finite-dimensional programming. An example is presented wherein these results are applied to a version of the oil terminal model considered by Christofides, Martello, and Toth [2].

2. Primal Problem A

The problem to be considered is:

Maximize

$$P(z) = \int_0^T \Psi(g(z(t), t), t)\, dt \tag{1}$$

subject to the constraints

$$z(t) \geqslant 0, \qquad 0 \leqslant t \leqslant T, \tag{2}$$

and

$$f(z(t), t) \leqslant h(y(z, t), t), \qquad 0 \leqslant t \leqslant T, \tag{3}$$

where $z \in L_n^\infty[0, T]$, i.e. z is a bounded and measurable n-dimensional function; y is a mapping from $L_n^\infty[0, T] \times [0, T]$ into E^p defined by

$$y(z, t) = \int_0^t g(z(s), s)\, ds; \tag{4}$$

$f(z(t), t)$, $h(y(z, t), t) \in E^m$ and $g(z(t), t) \in E^p$; and $\Psi(g(\cdot, t), t)$ is a scalar function, continuously differentiable in its first argument throughout $[0, T]$ and concave in z. It is further assumed that each component of $-f$, g, and h is a scalar function, concave and differentiable in its first argument throughout $[0, T]$ with each component of h also concave in z, that there exists $\delta > 0$ such that either

$$\nabla_k f_i(\eta, t) = 0 \qquad \text{or} \qquad \nabla_k f_i(\eta, t) \geqslant \delta, \tag{5}$$

where

$$\nabla_k f_i(\eta, t) = \partial f_i(\eta, t)/\partial \eta_k, \qquad i = 1,..., m, \quad k = 1,..., n,$$

for

$$\eta \in E^n, \qquad \eta \geqslant 0, \qquad \text{and} \qquad t \in [0, T],$$

$$\{x \in E^n: [\nabla f(\eta, t)]\, x \leqslant 0,\, x \geqslant 0,\, 0 \leqslant t \leqslant T\} = \{0\} \qquad \text{for } \eta \in E^n, \quad \eta \geqslant 0, \tag{6}$$

where

$$[\nabla f(\eta, t)] = \{\nabla_k f_i(\eta, t)\}_{m \times n};$$

and

$$\nabla_j h_i(\nu, t) = \partial h_i(\nu, t)/\partial \nu_j \geqslant 0, \qquad \text{for } \nu \in E^p \quad \text{and} \quad t \in [0, T]. \tag{7}$$

The scalar functions $h_i(0, t)$ and the vectors $\nabla h_i(0, t)$, $i = 1,..., m$, are continuous on $[0, T]$; similarly for $g_j(0, t)$ and $\nabla g_j(0, t)$, $j = 1,..., p$.

Note that if each entry of the matrix $[\nabla f(\eta, t)]$ is nonnegative, then assumption (6) is equivalent to the statement that each column of $[\nabla f(\eta, t)]$ has at least one positive element. Hence assumptions (5) and (6) are nonlinear extensions of assumptions imposed initially by Levinson ([13], (1.7) and (1.8)) to prove the existence of an optimal solution to the linear problem. Assumptions (5) and (6) are implemented below in the proof of Theorem 1 to establish the existence of an optimal solution to the nonlinear problem.

A function $z \in L_n^\infty[0, T]$ is termed *feasible* for Primal Problem A if it satisfies the constraints (2) and (3). The primal problem is itself said to be feasible if a feasible z exists.

THEOREM 1 (Existence). *If Primal Problem A is feasible then it has an optimal solution that is, there exists a feasible* $\bar{z}$ *for which*

$$P(\bar{z}) = \sup P(z),$$

where the supremum is taken over all feasible z.

We preface the proof of this theorem with two lemmas which are provided by Levinson [13].

LEMMA 1. *If q is a nonnegative integrable function for which there exists scalar constants* $\theta_1 \geqslant 0$ *and* $\theta_2 > 0$, *such that*

$$q(t) \leqslant \theta_1 + \theta_2 \int_0^t q(s)\, ds, \qquad 0 \leqslant t \leqslant T,$$

then $q(t) \leqslant \theta_1 e^{\theta_2 t}$, $0 \leqslant t \leqslant T$.

LEMMA 2. *If* $\{q_d\}$, $d = 1, 2,\ldots,$ *is a uniformly bounded sequence of measurable functions which converges weakly on* $[0, T]$ *to* q_0, *then*

$$q_0(t) \leqslant \limsup_{d\to\infty} q_d(t), \qquad a.e. \quad in \quad [0, T],$$

that is, the inequality holds for all $t \in [0, T]$ *except possibly on a set of Lebesgue measure zero.*

Proof of Theorem 1. Let z be feasible for Primal Problem A and multiply the constraint (3) by the m-dimensional vector $(1,\ldots, 1)$ to obtain the inequality

$$\sum_{i=1}^{m} f_i(z(t), t) \leqslant \sum_{i=1}^{m} h_i(y(z, t), t), \qquad 0 \leqslant t \leqslant T. \tag{8}$$

From the convexity of each f_i in its first argument it follows from [17, p. 242] that

$$\sum_{i=1}^{m} f_i(z(t), t) \geqslant \sum_{i=1}^{m} f_i(0, t) + \sum_{k=1}^{n} a_k(t)\, z_k(t),$$

where

$$a_k(t) = \sum_{i=1}^{m} \nabla_k f_i(0, t).$$

Set $\theta_0 = \max\{0; -\sum_{i=1}^{m} f_i(0, t), 0 \leqslant t \leqslant T\}$ and observe that by assumptions (5) and (6)

$$\inf_t \min_k a_k(t) > 0.$$

Since z is feasible and therefore satisfies constraint (2), it then follows that there exists a positive scalar Δ for which

$$\Delta \sum_{k=1}^{n} z_k(t) \leqslant \theta_0 + \sum_{i=1}^{m} h_i(y(z, t), t), \qquad 0 \leqslant t \leqslant T. \tag{9}$$

Define

$$[\nabla g(\eta, s)] = \{\nabla_k g_j(\eta, s)\}_{p\times n}, \qquad \text{for } \eta \in E^n, \quad s \in [0, T],$$

and

$$[\nabla h(\nu, t)] = \{\nabla_j h_i(\nu, t)\}_{m\times p}, \qquad \text{for } \nu \in E^p, \quad t \in [0, T],$$

and set

$$G(z, t, s) = [\nabla h(y(z, t), t)]\, g(z(s), s)$$

and

$$H(z, t, s) = [\nabla h(y(z, t), t)]\, [\nabla g(z(s), s)].$$

By application of the chain rule for differentiation, the concavity of g and h, [17] and assumption (7), it follows that

$$h(y(z, t), t) \leqslant h(0, t) + \int_0^t G(0, t, s)\, ds + \int_0^t H(0, t, s)\, z(s)\, ds, \qquad 0 \leqslant t \leqslant T.$$

Select $\theta_1 \geqslant 0$ and $\theta_2 > 0$ such that

$$\sup_t \left\{ \sum_{i=1}^{m} h_i(0, t) + \sum_{i=1}^{m} \int_0^t G_i(0, t, s)\, ds \right\} \leqslant \theta_1$$

and

$$\sup_t \max_k \left\{ \sum_{i=1}^{m} H_{ik}(0, t, s) \right\} \leqslant \theta_2$$

From (9) we have that $\theta_1^* = (\theta_0 + \theta_1)/\Delta$ and $\theta_2^* = \theta_2/\Delta$ are nonnegative and positive constants respectively for which

$$\sum_{k=1}^{n} z_k(t) \leqslant \theta_1^* + \theta_2^* \int_0^t \sum_{k=1}^{n} z_k(s)\, ds, \qquad 0 \leqslant t \leqslant T.$$

Hence by Lemma 1, it is concluded that the set of feasible solutions for Primal Problem A is uniformly bounded on $[0, T]$.

Since the composite function $\phi(\cdot, t) = \Psi(g(\cdot, t), t)$ is concave and differentiable in its first argument throughout $[0, T]$, it follows from [17] and the uniform boundedness property that, for any feasible solutions z and z^0,

$$P(z) - P(z^0) \leqslant T \sum_{k=1}^{n} \sup_t (z_k(t) - z_k{}^0(t)) \sup_t \nabla_k \phi(z^0(t), t) < \infty$$

where

$$\nabla_k \phi(z^0(t), t) = \sum_{j=1}^{p} \nabla_j \Psi(g(z^0(t), t), t) \nabla_k g_j(z^0(t), t),$$

and hence P is bounded above for all feasible z.

Let $\bar{P} = \sup P(z)$, where the supremum is taken over all feasible z. Then there exists a sequence of feasible solutions $\{z^d\}$ such that

$$\lim_{d \to \infty} P(z^d) = \bar{P}.$$

Since $\{z^d\}$ is uniformly bounded, it follows from [20] that there exists a $\tilde{z}$ to which a subsequence of $\{z^d\}$ converges weakly in $L_n{}^2[0, T]$. The application of Lemma 2 to each component of z^d then provides uniform boundedness for $\tilde{z}$ except possibly on a set of measure zero where, as will be shown later, it can be assumed to be zero.

From (7), [17] and the concavity of g and h we have

$$h(y(z^d, t), t) \leqslant h(y(\tilde{z}, t), t) + \int_0^t H(\tilde{z}, t, s)\,(z^d(s) - \tilde{z}(s))\, ds.$$

Since each entry of the $m \times n$ matrix $H(\tilde{z}, t, s)$ is bounded and measurable, it follows that $H_i(\tilde{z}, t, \cdot) \in L_n{}^\infty[0, T] \subset L_n{}^2[0, T]$ and so by weak convergence

$$\int_0^t H(\tilde{z}, t, s)\,(z^d(s) - \tilde{z}(s))\, ds \to 0, \qquad \text{as} \qquad d \to \infty.$$

Thus by constraint (3)

$$\limsup_{d \to \infty} f(z^d(t), t) \leqslant h(y(\tilde{z}, t), t). \tag{10}$$

By the convexity of f

$$f(z^d(t), t) \geqslant f(\tilde{z}(t), t) + [\nabla f(\tilde{z}(t), t)]'\,(z^d(t) - \tilde{z}(t)).$$

Therefore, from (10)

$$f(\tilde{z}(t), t) \leqslant h(y(\tilde{z}, t), t) \tag{11}$$

except on a set of measure zero, since by [17], assumption (5) and Lemma 2

$$\limsup_{d\to\infty}[\nabla f(\tilde{z}(t), t)]' (z^d(t) - \tilde{z}(t)) \geq 0$$

except on such a set.

A second application of Lemma 2 to each component of z^d provides

$$-\tilde{z}(t) \leq \limsup_{d\to\infty}(-z^d(t)) \leq 0, \qquad \text{a.e.} \quad \text{in} \quad [0, T],$$

and consequently $\tilde{z}$ is nonnegative except on a set of measure zero. From this result and expression (11), it is observed that $\tilde{z}$ can violate the constraints of Primal Problem A on, at most, a set of measure zero in $[0, T]$. We define $\bar{z}$ to be zero on this set of measure zero and equal to $\tilde{z}$ on the complement. The feasibility of $\bar{z}$ is then established by noting that

$$y(\bar{z}, t) = y(\tilde{z}, t), \qquad 0 \leq t \leq T,$$

and that

$$\limsup_{d\to\infty} f(z^d(t), t) \geq f(0, t), \qquad 0 \leq t \leq T,$$

by the convexity of f, constraint (2), and assumption (5).

By the concavity and differentiability of ϕ

$$\int_0^T \phi(z^d(t), t)\, dt \leq \int_0^T \phi(\bar{z}(t), t)\, dt + \int_0^T (z^d(t) - \bar{z}(t))' \nabla\phi(\bar{z}(t), t)\, dt.$$

Therefore by weak convergence

$$\bar{P} = \lim_{d\to\infty} \int_0^T \phi(z^d(t), t)\, dt \leq \int_0^T \phi(\bar{z}(t), t)\, dt = P(\bar{z}).$$

By the definition of $\bar{P}$ and the feasibility of $\bar{z}$, $P(\bar{z}) \leq \bar{P}$, thus $P(\bar{z}) = \bar{P}$ and $\bar{z}$ is an optimal solution for Primal Problem A. Q.E.D.

3. Dual Problem A

Before the dual to Primal Problem A is formally stated, a continuous time Lagrangian function and its Fréchet differential will be introduced. This introduction serves to further unify the theory with that of finite dimensional programming as well as allowing brevity in the notation.

For $u \in L_n^\infty[0, T]$ and $w \in L_m^\infty[0, T]$, define

$$L(u, w) = \int_0^T [\phi(u(t), t) + w'(t) F(u, t)]\, dt, \tag{12}$$

where

$$F(u, t) = h(y(u, t), t) - f(u(t), t), \qquad 0 \leqslant t \leqslant T,$$

and let $\delta_1 L(u, w; \gamma)$ denote the Fréchet differential [15] of L with respect to its first argument, evaluated at u with the increment $\gamma \in L_n^{\infty}[0, T]$. The differentiability of each of the functions involved in L ensures that the Fréchet differential exists and allows $\delta_1 L(u, w; \gamma)$ to be determined by the simple differentiation

$$\delta_1 L(u, w; \gamma) = \frac{d}{d\alpha} L(u + \alpha\gamma, w) \Big|_{\alpha=0}. \tag{13}$$

The Fréchet differential has two additional properties which will be used extensively in the ensuing discussion, namely the linearity of $\delta_1 L(u, w; \gamma)$ in its increment γ and the continuity of $\delta_1 L(u, w; \gamma)$ in γ under the norm

$$\| \gamma \|_n^{\infty} = \max_k \| \gamma_k \|^{\infty}.$$

Here $\| \cdot \|^{\infty}$ denotes the essential supremum [18].

From (13) it is observed that

$$\begin{aligned} \delta_1 L(u, w; \gamma) = \int_0^T \Big\{ & [\nabla\phi(u(t), t)]' \gamma(t) \\ & - \int_0^t w'(t) H(u, t, s) \gamma(s) \, ds - w'(t) [\nabla f(u(t), t)] \gamma(t) \Big\} \, dt \end{aligned} \tag{14}$$

which, through application of Fubini's Theorem [18] to interchange the limits of integration, can be expressed as

$$\delta_1 L(u, w; \gamma) = \delta P(u, \gamma) + \int_0^T \gamma'(t) F^*(u, w, t) \, dt, \tag{15}$$

where

$$F^*(u, w, t) = \int_t^T H'(u, s, t) w(s) \, ds - [\nabla f(u(t), t)]' w(t), \qquad 0 \leqslant t \leqslant T. \tag{16}$$

Under this notation the dual of Primal Problem A will be shown to be:

Minimize

$$G(u, w) = L(u, w) - \delta_1 L(u, w; u) \tag{17}$$

subject to the constraints

$$w(t) \geqslant 0, \qquad 0 \leqslant t \leqslant T, \tag{18}$$

and

$$F^*(u, w, t) + [\nabla\phi(u(t), t)] \leqslant 0, \qquad 0 \leqslant t \leqslant T. \tag{19}$$

THEOREM 2 (Weak Duality). *If z and (u, w) are feasible solutions for Primal and Dual Problems A respectively, then*

$$P(z) \leqslant G(u, w).$$

Proof. By the concavity of ϕ, $-f$, g, and h and by assumption (7) it follows that L is concave in its first argument and

$$L(z, w) - L(u, w) \leqslant \delta_1 L(u, w; z - u).$$

Thus

$$\begin{aligned} P(z) - G(u, w) &= L(z, w) - \int_0^T w'(t) F(z, t)\, dt - L(u, w) + \delta_1 L(u, w; u) \\ &\leqslant \delta_1 L(u, w; z - u) + \delta_1 L(u, w; u) - \int_0^T w'(t) F(z, t)\, dt \\ &= \delta_1 L(u, w; z) - \int_0^T w'(t) F(z, t)\, dt \end{aligned}$$

by the linearity of the Fréchet differential in its increment,

$$P(z) - G(u, w) = \int_0^T z'(t) \{[\nabla\phi(u(t), t)] - F^*(u, w, t)\}\, dt - \int_0^T w'(t) F(z, t)\, dt$$

by (15),

$$\leqslant 0$$

by constraints (2), (3), (18) and (19). Q.E.D.

From Theorem 2 it is observed that if there exist feasible solutions, $\hat{z}$ and $(\hat{u}, \hat{w})$, for the primal and dual problems and if the corresponding primal and dual objective function values, $P(\hat{z})$ and $G(\hat{u}, \hat{w})$, are equal, then these solutions are optimal for their respective problems.

4. The Constraint Qualification and Duality

The content of this section is in general analogous to concepts and results that are well-known in finite-dimensional mathematical programming; see, for example, Theorem 3 (strongly duality), Lemma 3, and Lemma 4. In addition, the constraint qualification introduced here is motivated by the constraint qualification presented by Zangwill [23]. The basic theory surrounding this qualification is established to provide a framework for the remaining theorems.

LEMMA 3. *If*

$$\delta P(z; \gamma) = \int_0^T \gamma'(t) [\nabla\phi(z(t), t)]\, dt > 0 \tag{20}$$

where $z, \gamma \in L_n^\infty[0, T]$, *then there exists a scalar* $\sigma > 0$ *such that*

$$P(z + \tau\gamma) > P(z), \qquad \text{for } 0 < \tau \leqslant \sigma.$$

Proof. By (13)

$$\lim_{\tau \downarrow 0} [P(z + \tau\gamma) - P(z)]/\tau = \delta P(z; \gamma) > 0,$$

thus a positive σ can be chosen which is sufficiently small so that

$$P(z + \tau\gamma) > P(z), \qquad \text{for } 0 < \tau \leqslant \sigma. \qquad \text{Q.E.D.}$$

DEFINITION 1. For each z which is feasible for Primal Problem A define $D(z)$ to be the set of n-vector functions γ for which

(i) $\gamma \in L_n^\infty[0, T]$

(ii) there exists a scalar $\sigma > 0$ such that

$$z(t) + \tau\gamma(t) \geqslant 0, \qquad \text{a.e. in } [0, T],$$

and

$$F(z + \tau\gamma, t) \geqslant 0, \qquad \text{a.e. in } [0, T],$$

for $0 \leqslant \tau \leqslant \sigma$.

DEFINITION 2. Define $\bar{D}(z)$ to be the closure of $D(z)$ under the norm $\|\cdot\|_n^\infty$, that is, if a sequence $\{\gamma^d\} \subset D(z)$ and $\|\gamma^d - \gamma\|_n^\infty \to 0$, as $d \to \infty$, then $\gamma \in \bar{D}(z)$.

Henceforth the Fréchet differential of the mapping $F(\cdot, t): L_n^\infty[0, T] \to E^m$, evaluated at z and with increment γ, will be denoted by $\delta F(z; \gamma)_t$. It should be observed that, for any specified value of $t \in [0, T]$, the existence of $\delta F(z; \gamma)_t$ is ensured by the differentiability of f, g, and h and that

$$\delta F(z; \gamma)_t = \int_0^t H(z, t, s)\, \gamma(s)\, ds - [\nabla f(z(t), t)]\, \gamma(t). \tag{21}$$

Similarly, the Fréchet differential of a component $F_i(\cdot, t)$ of $F(\cdot, t)$, evaluated at z and with increment γ, will be denoted by $\delta F_i(z; \gamma)_t$.

DEFINITION 3. For each z which is feasible for Primal Problem A define $\mathscr{D}(z)$ to be the set of n-vector functions γ for which

(i) $\gamma \in L_n^\infty[0, T]$,

(ii) $\gamma_k(t) \geqslant 0$ a.e. in $T_{1k}(z)$, $k = 1, \ldots, n$,

(iii) $\delta F_i(z; \gamma)_t \geqslant 0$ a.e. in $T_{2i}(z)$, $i = 1, \ldots, m$, where

$$T_{1k}(z) = [t \in [0, T]: z_k(t) = 0], \qquad k = 1,..., n,$$

and

$$T_{2i}(z) = [t \in [0, T]: F_i(z, t) = 0], \qquad i = 1,..., m.$$

In a comparison of the sets $D(z)$ and $\mathscr{D}(z)$ with their finite dimensional counterparts presented in Zangwill [23], it is observed that $D(z)$ is analogous to the set of "feasible directions" at z and $\mathscr{D}(z)$ is analogous to that set of directions for which the directional derivatives of each of the active constraints at z are nonnegative. Furthermore the form of the constraint qualification for continuous time problems given below is identical to that for finite-dimensional problems.

DEFINITION 4 (Constraint Qualification). Primal Problem A will be said to satisfy the Constraint Qualification if the problem is feasible and if

$$\bar{D}(\bar{z}) = \mathscr{D}(\bar{z}),$$

where $\bar{z}$ is an optimal solution to the problem.

In problems such as Primal Problem A where convexity and concavity properties are assumed, violations of the Constraint Qualification can be shown to arise when the constraints take the form of equalities on some set of positive measure. For example, consider the constraints

$$z_1(t) \geqslant 0, \qquad z_2(t) \geqslant 0, \qquad 0 \leqslant t \leqslant T,$$

and

$$(z_1(t) + z_2(t) - t)^2 \leqslant 1 - I_E(t), \qquad 0 \leqslant t \leqslant T,$$

where E is a set of positive measure in $[0, T]$ and $I_E(\cdot)$ is its indicator function. For $z(t) = (t/2, t/2)'$, $0 \leqslant t \leqslant T$, the function $\gamma(t) = (t, t)'$, $0 \leqslant t \leqslant T$, is an element of $\mathscr{D}(z)$ but not an element of $\bar{D}(z)$; thus the Constraint Qualification is violated.

To establish strong duality two additional assumptions are required. These are:

$$H(\bar{z}, t, s) \geqslant 0, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{22}$$

and

$$F(\bar{z}, t) - \delta F(\bar{z}; \bar{z})_t \geqslant 0, \qquad 0 \leqslant t \leqslant T, \tag{23}$$

where $\bar{z}$ is an optimal solution for Primal Problem A.

THEOREM 3 (Strong Duality). *Under the Constraint Qualification and assumptions* (22) *and* (23), *there exists an optimal solution* $(\bar{u}, \bar{w})$ *for Dual Problem* A *such that* $\bar{u} = \bar{z}$ *and* $G(\bar{z}, \bar{w}) = P(\bar{z})$.

Before proving Theorem 3 the following linearized problem, called *Primal Problem* A′, will be considered:

Maximize

$$\delta P(\bar{z}; z - \bar{z})$$

subject to the constraints

$$z(t) \geqslant 0, \qquad 0 \leqslant t \leqslant T, \tag{24}$$

and

$$F(\bar{z}, t) + \delta F(\bar{z}; z - \bar{z})_t \geqslant 0, \qquad 0 \leqslant t \leqslant T. \tag{25}$$

LEMMA 4. *Under the Constraint Qualification, $\bar{z}$ is an optimal solution for Primal Problem* A′.

Proof. Assume that there exists a $\hat{z}$ which satisfies (24) and (25) and for which

$$\delta P(\bar{z}; \hat{z} - \bar{z}) > 0.$$

Then there exists a positive scalar σ^* such that

$$[P(\bar{z} + \tau(\hat{z} - \bar{z})) - P(\bar{z})]/\tau > 0, \qquad \text{for } 0 < \tau \leqslant \sigma^*.$$

Observe that $\hat{\gamma} = (\hat{z} - \bar{z}) \in \mathscr{D}(\bar{z})$ since

$$\hat{\gamma}_k(t) \geqslant 0, \qquad \text{for } t \in T_{1k}(\bar{z}), \quad k = 1,..., n,$$

and

$$\delta F_i(\bar{z}; \hat{\gamma})_t \geqslant 0, \qquad \text{for } t \in T_{2i}(\bar{z}), \quad i = 1,..., m,$$

and therefore, under the Constraint Qualification, $\hat{\gamma} \in \bar{D}(\bar{z})$.

If $\hat{\gamma} \in D(\bar{z})$, then a positive scalar $\hat{\sigma} \in (0, \sigma^*]$ can be chosen such that $\bar{z} + \hat{\sigma}\hat{\gamma}$ is feasible for Primal Problem A except on a set of Lebesgue measure zero where, as in the proof of Theorem 1, it can be assumed to be zero. The feasibility of the adjusted $\bar{z} + \hat{\sigma}\hat{\gamma}$ and the inequality

$$P(\bar{z} + \hat{\sigma}\hat{\gamma}) > P(\bar{z})$$

contradict the optimality of $\bar{z}$ for Primal Problem A and it is therefore concluded that

$$\delta P(\bar{z}; \hat{\gamma}) \leqslant 0, \qquad \text{if} \qquad \hat{\gamma} \in D(\bar{z}). \tag{26}$$

If $\{\gamma^d\}$ is a sequence of functions in $D(\bar{z})$ which converges to $\hat{\gamma}$ in the norm $\|\cdot\|_n^\infty$, then by (26)

$$\limsup_{d \to \infty} \delta P(\bar{z}; \gamma^d) \leqslant 0.$$

This result and the assumption that $\delta P(\bar{z}; \hat{\gamma}) > 0$ contradict the continuity of $\delta P(\bar{z}; \cdot)$ and it is therefore concluded that

$$\delta P(\bar{z}; z - \bar{z}) \leqslant 0$$

for all z satisfying (24) and (25). The optimality of $\bar{z}$ follows since $\bar{z}$ is feasible for Primal Problem A′ and since $\delta P(\bar{z}; 0) = 0$. Q.E.D.

Proof of Theorem 3. To apply the results obtained in the sequence of works by Tyndall [21], Levinson [13], and Hanson and Mond [9], Grinold [7], and Schechter [19], Primal Problem A′ is rewritten in the form

Maximize

$$\int_0^T a'(t)\, z(t)\, dt$$

subject to the constraints

$$z(t) \geqslant 0, \qquad 0 \leqslant t \leqslant T.$$

and

$$B(t)\, z(t) \leqslant c(t) + \int_0^t K(t, s)\, z(s)\, ds, \qquad 0 \leqslant t \leqslant T,$$

where $a(t) = [\nabla\phi(\bar{z}(t), t)]$, $B(t) = [\nabla f(\bar{z}(t), t)]$, $c(t) = F(\bar{z}, t) - \delta F(\bar{z}; \bar{z})_t$, and $K(t, s) = H(\bar{z}, t, s)$. From assumptions (5), (22) and (23) it is observed that $B(t)$, $c(t)$, and $K(t, s)$ are nonnegative for $0 \leqslant s \leqslant t \leqslant T$ and from assumption (6),

$$\{x \in E^n : x \geqslant 0, B(t)\, x \leqslant 0, 0 \leqslant t \leqslant T\} = \{0\}.$$

Thus Primal Problem A′ satisfies the requirements for duality summarized by Schechter and there exists an m-vector function $\bar{w}$ satisfying

$$\bar{w}(t) \geqslant 0, \qquad 0 \leqslant t \leqslant T, \tag{27}$$

and

$$B'(t)\, \bar{w}(t) \geqslant a(t) + \int_t^T K'(s, t)\, w(s)\, ds, \qquad 0 \leqslant t \leqslant T, \tag{28}$$

for which

$$\int_0^T \bar{w}'(t)\, c(t)\, dt = \int_0^T a'(t)\, \bar{z}(t)\, dt. \tag{29}$$

With the identities (12), (14), and (16), the expressions (28) and (29) can be expressed as

$$F^*(\bar{z}, \bar{w}, t) + [\nabla\phi(\bar{z}(t), t)] \leqslant 0, \qquad 0 \leqslant t \leqslant T, \tag{28′}$$

and

$$L(\bar{z}, \bar{w}) - \delta_1 L(\bar{z}, \bar{w}; \bar{z}) = P(\bar{z}), \tag{29'}$$

respectively. From (27) and (28′) it then follows that $(\bar{z}, \bar{w})$ is feasible for Dual Problem A, and from (17) and (29′)

$$G(\bar{z}, \bar{w}) = P(\bar{z}). \tag{30}$$

Finally, by the weak duality established in Theorem 2, it is concluded from (30) that $(\bar{z}, \bar{w})$ is an optimal solution for Dual Problem A. Q.E.D.

In order to apply Theorem 3 in practice, it is desirable to be able to verify conditions (22) and (23), without prior knowledge of the optimal solution $\bar{z}$. The following corollary provides this capability.

COROLLARY 1. *If*

$$\nabla_k g_j(\eta, t) = \partial g_j(\eta, t)/\partial \eta_k \geqslant 0, \tag{31}$$

$$j = 1,..., p, \qquad k = 1,..., n, \qquad \textit{for } \eta \in E^n$$

$$\eta \geqslant 0, \qquad \textit{and} \qquad t \in [0, T],$$

$$F(0, t) \geqslant 0, \qquad 0 \leqslant t \leqslant T, \tag{32}$$

then under the Constraint Qualification there exists an optimal solution $(\bar{u}, \bar{w})$ *for Dual Problem* A *such that* $\bar{u} = \bar{z}$ *and* $G(\bar{z}, \bar{w}) = P(\bar{z})$.

Proof. We have from (7) and (31) that

$$H(\bar{z}, t, s) = [\nabla h(y(\bar{z}, t), t)] [\nabla g(\bar{z}(s), s)] \geqslant 0, \qquad 0 \leqslant s \leqslant t \leqslant T,$$

and by (32) and the concavity of F that

$$F(\bar{z}, t) - \delta F(\bar{z}; \bar{z})_t \geqslant F(0, t) \geqslant 0, \qquad 0 \leqslant t \leqslant T.$$

From these results it follows that the conditions of Theorem 3 are satisfied. Q.E.D.

5. KUHN–TUCKER THEORY

THEOREM 4 (Complementary Slackness Principle). *If* $\bar{z}$ *and* $(\bar{z}, \bar{w})$ *are optimal solutions for the Primal and Dual Problems* A, *then*

$$\int_0^T \bar{w}'(t) F(\bar{z}, t)\, dt = 0 \tag{33}$$

and

$$\int_0^T \bar{z}'(t) \{F^*(\bar{z}, \bar{w}, t) + [\nabla\phi(\bar{z}(t), t)]\}\, dt = 0. \tag{34}$$

Proof. Since $\bar{z}(t) \geqslant 0$ and $F^*(\bar{z}, \bar{w}, t) + [\nabla\phi(\bar{z}(t), t)] \leqslant 0,\ 0 \leqslant t \leqslant T$, it follows from the identity (15) that

$$\int_0^T \bar{z}'(t)\{F^*(\bar{z}, \bar{w}, t) + [\nabla\phi(\bar{z}(t), t)]\}\,dt = \delta_1 L(\bar{z}, \bar{w}; \bar{z}) \leqslant 0,$$

and therefore, by (29′)

$$L(\bar{z}, \bar{w}) - P(\bar{z}) = \int_0^T \bar{w}'(t)F(\bar{z}, t)\,dt \leqslant 0.$$

Since $\bar{w}(t) \geqslant 0$ and $F(\bar{z}, t) \geqslant 0,\ 0 \leqslant t \leqslant T$, it also follows that

$$\int_0^T \bar{w}'(t)F(\bar{z}, t)\,dt \geqslant 0, \tag{35}$$

thus the equality in (33) is established.

Similarly, (29′) and (35) imply that

$$\delta_1 L(\bar{z}, \bar{w}; \bar{z}) \geqslant 0$$

and therefore, by (15)

$$\int_0^T \bar{z}'(t)\{F^*(\bar{z}, \bar{w}, t) + [\nabla\phi(\bar{z}(t), t)]\}\,dt \geqslant 0.$$

The equality in (34) is then established since $\bar{z}(t) \geqslant 0$ and

$$F^*(\bar{z}, \bar{w}, t) + [\nabla\phi(\bar{z}(t), t)] \leqslant 0, \qquad 0 \leqslant t \leqslant T. \qquad \text{Q.E.D.}$$

THEOREM 5 (Kuhn–Tucker Conditions). *Assume that* (31) *and* (32) *are satisfied for Primal Problem* A. *Then under the Constraint Qualification* $\bar{z}$ *is an optimal solution if and only if there exists an m-vector function* $\hat{w}$ *such that*

$$\hat{w}(t) \geqslant 0, \qquad 0 \leqslant t \leqslant T, \tag{36}$$

$$F^*(\bar{z}, \hat{w}, t) + [\nabla\phi(\bar{z}(t), t)] \leqslant 0, \qquad 0 \leqslant t \leqslant T, \tag{37}$$

$$\int_0^T \bar{z}'(t)\{F^*(\bar{z}, \hat{w}, t) + [\nabla\phi(\bar{z}(t), t)]\}\,dt = 0, \tag{38}$$

$$\int_0^T \hat{w}'(t)F(\bar{z}, t)\,dt = 0. \tag{39}$$

Proof. Necessity: The necessity of the conditions follows from Corollary 1 and Theorem 4, since the m-vector function $\bar{w}$ of the optimal solution $(\bar{z}, \bar{w})$ to Dual Problem A satisfies conditions (36)–(39).

Sufficiency: Let z be feasible for Primal Problem A. Then since P is concave

$$P(z) - P(\bar{z}) \leqslant \delta P(\bar{z}; z - \bar{z})$$
$$= \int_0^T [z(t) - \bar{z}(t)]' [\nabla\phi(\bar{z}(t), t)] \, dt$$
$$\leqslant - \int_0^T [z(t) - \bar{z}(t)]' F^*(\bar{z}, \hat{w}, t) \, dt$$

by (37) and (38) since $z(t) \geqslant 0$, $0 \leqslant t \leqslant T$,

$$P(z) - P(\bar{z}) = - \int_0^T \hat{w}'(t) \, \delta F(\bar{z}; z - \bar{z})_t \, dt$$

by (16), (21), and Fubini's Theorem [18],

$$P(z) - P(\bar{z}) \leqslant - \int_0^T \hat{w}'(t) [F(z, t) - F(\bar{z}, t)] \, dt$$

by (36) and the concavity of F,

$$P(z) - P(\bar{z}) = - \int_0^T \hat{w}'(t) F(z, t) \, dt$$

by (39),

$$P(z) - P(\bar{z}) \leqslant 0$$

since $\hat{w}(t) \geqslant 0$ and $F(z, t) \geqslant 0$, $0 \leqslant t \leqslant T$. Thus $P(z) \leqslant P(\bar{z})$ and $\bar{z}$ is an optimal solution for Primal Problem A. Q.E.D.

Nonlinear programming in infinite-dimensional spaces has been previously considered by Luenberger [14], Varaiya [22], Guignard [8], and others [see, e.g. [3], [6], [10] and [12] and the references therein].

Luenberger [14, p. 247] considers the problem

$$\text{minimize } f(x)$$

subject to

$$G(x) \leqslant \theta,$$

where f is a real-valued function on a vector space X and G is a mapping from X into a normed space Z having positive cone P. Note that in contrast to this problem the domain $L_n^\infty[0, T] \times [0, T]$ of our constraint function $F(z, t) \equiv h(y(z, t), t) - f(z(t), t)$ differs from the domain $L_n^\infty[0, T]$ of our objective function $P(z)$. This difference between the domain of the constraint and objective functions generates associated Kuhn–Tucker conditions that are more general than those of Luenberger [14, p. 249]. In particular, if X has a positive cone then Luenberger's Kuhn–Tucker conditions can be modified to allow for the condition "$x \geq \theta$". The Kuhn–Tucker conditions (36) thru (39) are an extension of Luenberger's modified Kuhn–Tucker conditions applied to the Banach space $X = L_n^\infty[0, T]$. In addition, Luenberger's constraint qualification [14, p. 248] in the finite-dimensional setting is more stringent than the finite-dimensional version of our constraint qualification [16, p. 647].

Guignard [8] addresses nonlinear programming in Banach spaces and generalizes the Kuhn–Tucker conditions obtained by Varaiya [22]. Both papers consider problems of the form:

$$\text{maximize}\{\psi(x): x \in C,\ a(x) \in B\},$$

where $\psi(\cdot)$ is a real-valued function on a real Banach space X, $a(\cdot)$ is a map from X into Y, also a real Banach space, and B and C are nonempty subsets of Y and X, respectively. Varaiya assumes differentiability of $\psi(\cdot)$ and $a(\cdot)$ over X while Guignard limits differentiability to a particular point $\bar{x}$ under consideration; the authors express their results in terms of cones of tangents [22, Def. 2.2] and pseudotangents [8, Def. 5]. As in the comparison in the preceding paragraph, the difference between the domains of our constraint and objective functions result in Kuhn–Tucker conditions that are natural extensions of those derived by Varaiya [22, Th. 4.1] and Guignard [8, Th. 2]. For example, conditions (36) and (39) are, for every $t \in [0, T]$, equivalent to Guignard's "$u \in P^-(B, a(\bar{x}))$" [8, Th. 2] by setting $u = w(t)$, $B = E_+^m$, and $a(\bar{x}) = F(\bar{z}, t)$. Similarly, conditions (37) and (38) are, for every $t \in [0, T]$, equivalent to Guignard's "$\nabla\psi(\bar{x}) + u \cdot \nabla a(\bar{x}) \in G^-$" [8, Th. 2] by setting $G^- = P^-(L_{n+}^\infty[0, T], \bar{z})$, where $L_{n+}^\infty[0, T] = \{y \in L_n^\infty[0, T]: y(t) \geqslant 0, 0 \leqslant t \leqslant T\}$. We note that the constraint qualifications of Varaiya [22, Def. 3.4] and Guignard [8, Th. 2] are equivalent to our constraint qualification in finite-dimensional spaces [16, p. 647] and that an existence theorem comparable to Theorem 1 is not established in any of the three above-mentioned papers.

One of the factors motivating the extension of mathematical programming to infinite-dimensional spaces was to derive results that are applicable to the theory of optimal control; hopefully, our formulation will facilitate such applications.

6. Example—The Operation of an Oil Terminal

We consider a generalization of the oil terminal model considered by Christofides [2]. Crude oil arrives on ships and is to be unloaded to a "tank farm" which consists of several storage tanks. Each ship carries several grades of crude and there is a separate storage tank for each grade. The storage tanks in turn supply a refinery where the crudes are mixed and processed into refined grades. Given that a group of ships has arrived at the terminal, we desire a policy for unloading the ships and supplying the refinery in the time period $[0, T]$. The policy should be optimal in the sense that, subject to resource and production constraints, the net gain realized from production in the interval $[0, T]$ is maximized.

For the dynamics of the problem, we assume that each storage tank is capable of simultaneously receiving oil from the ships and pumping oil to the refinery. Let Ω_i and θ_i, $i = 1,..., r$ denote the initial store of crude oil in tank i and the capacity of tank i, respectively. Let $\alpha_i(t)$ denote the combined flow rate of crude i from all the ships to storage tank i, $\beta_i(t)$ the flow rate from tank i to the refinery, and $\gamma_{jk}(t), j, k = 1,..., p, j \neq k$, the rate at which refined grade j is recycled back into the production process for use in producing refined grade k. The store of crude oil in tank i at time t can then be expressed as

$$C_i(t) = \Omega_i + \int_0^t (\alpha_i(s) - \beta_i(s))\, ds, \qquad i = 1,..., r.$$

Let $z = (\alpha, \beta, \gamma)'$ be the n-vector of flow rates, where $n = 2r + p(p - 1)$; the resource constraints can thus be formulated as:

$$0 \leqslant \alpha_i(t) \leqslant A_i, \qquad 0 \leqslant t \leqslant T, \tag{40}$$

$$0 \leqslant \beta_i(t) \leqslant B_i, \qquad 0 \leqslant t \leqslant T, \tag{41}$$

$$0 \leqslant C_i(t) \leqslant \theta_i, \qquad 0 \leqslant t \leqslant T, \tag{42}$$

$$L_i(t) \leqslant \int_0^t \alpha_i(s)\, ds \leqslant U_i(t), \qquad 0 \leqslant t \leqslant T \tag{43}$$

for $i = 1,..., r$ and

$$\sum_{\substack{k=1 \\ k \neq j}}^{p} \gamma_{jk}(t) \leqslant \Pi_j(z(t), t), \qquad 0 \leqslant t \leqslant T, \tag{44}$$

for $j = 1,..., p$, where A_i, B_i are the maximum allowable flow rates into and out of tank i, respectively; $\Pi_j(z(t), t)$ is the rate of production of refined grade j when operating at a flow rate level $z(t)$ at time t, where $\Pi_j(\cdot, t)$ is concave and differentable in its first argument throughout $[0, T]$. The functions $L_i(t)$, $U_i(t)$ are lower and upper bounds, respectively, for the cummulative flow of crude i; these bounds typically are determined by the total amount of crude i to be unloaded from the ships and by the desire to maintain uniform utilization of the facilities over the planning period $[0, T]$ and avoid penalties in the form of overhead costs that result if the ships are in port for an excessive length of time.

The production constraints can be expressed as

$$H(t) \leqslant \int_0^t \Pi(z(s), s)\, ds \leqslant V(t), \qquad 0 \leqslant t \leqslant T, \tag{45}$$

where $\Pi(\cdot, t) = (\Pi_1(\cdot, t),..., \Pi_p(\cdot, t))'$; the p-dimensional functions $H(t)$ and $V(t)$ are, respectively, the lower and upper bounds for the production of the p refined grades of oil that must take place in the portion of the planning period from 0 to t. These bounds are determined by estimates of the demand function during the planning period $[0, T]$ and by the desire to control inventory costs.

Assuming the role of the oil company, we want to select an oil flow policy to maximize the net profit realized by production during the planning period; the net profit for the period $[0, T]$ is

$$\int_0^T \{m(t)' \Pi(z(t), t) - c(z(t), t\} \, dt$$

where the jth component of the function $m(t) = (m_1(t),..., m_p(t))'$ is the market value of one unit of the jth refined grade of oil at time t and $c(z(t), t)$ is the cost of operating at the rate $z(t)$ at time t; $c(\cdot, t)$ is a scalar function, convex and differentiable in its first argument throughout $[0, T]$. The problem is to maximize the above integral by proper choice of the functions $\alpha_i(\cdot)$, $\beta_i(\cdot)$, $\gamma_{jk}(\cdot)$ subject to constraints (40)–(50).

Through proper association of the terms of this model with those of Primal Problem A, it can be shown by application of Theorem 1 that feasibility ensures the existence of an optimal oil flow policy; necessary and sufficient conditions for a solution are given by the Kuhn–Tucker Conditions in Section 5.

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