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Weak Type Classes

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Let (A_0, A_1) be an interpolation couple of normed Abelian groups. The approximation functional is defined for each $a \in A_0 + A_1$ and $0 < t$ by

$$E(t, a; A_0, A_1) = \inf\{|a - a_0|_{A_1} / |a_0|_{A_0} \leqslant t\}. \tag{1}$$

For the properties of E, see [3]. The interpolation groups $(A_0, A_1)_{\alpha,q;E}$ are defined by

$$|a|_{(A_0,A_1)_{\alpha,q;E}} = \left(\int_0^\infty (t^\alpha E(t, a; A_0, A_1))^q \frac{dt}{t}\right)^{1/q}, \tag{2}$$

where $0 < \alpha < \infty$, $0 < q \leqslant \infty$.

An operator $T: A_0 + A_1 \to B_0 + B_1$ is defined to be E-quasilinear [4], iff constants c, d exist such that

$$\begin{aligned} E(t_0 + t_1, T(a_0 + a_1); B_0, B_1) \leqslant c\{&E(t_0 d, Ta_0; B_0, B_1) \\ &+ E(t_1 d, Ta_1; B_0, B_1)\}. \end{aligned} \tag{3}$$

For operators mapping into $L(p, q)$ spaces, pointwise quasilinearity implies E-quasilinearity.

Theorem 1 is proved in [2].

THEOREM 1. *Let T: $A_0 + A_1 \to B_0 + B_1$ be E-quasilinear and*

$$\frac{1}{t}\int_0^t E(s, Ta; B_0, B_1)\, ds - E(t, Ta; B_0, B_1) \leqslant c_1 |a|_{A_1}, \tag{4}$$

$$|Ta|_{B_0} \leqslant c_0 |a|_{A_0}, \tag{5}$$

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for all $a \in A_0 \cap A_1$. *Then, for* $0 < \alpha < \infty$, $0 < q \leqslant \infty$

$$|Ta|_{(B_0,B_1)_{\alpha,q;E}} \leqslant c\,|a|_{(A_0,A_1)_{\alpha,q;E}}. \tag{6}$$

Theorem 1 generalizes the well-known interpolation theorem, where (4) is replaced by $|Ta|_{B_1} \leqslant c_1\,|a|_{A_1}$.

Let us recall some notation (see [2]).

DEFINITION 1. Let f be defined on $(0, \infty)$, and integrable on each interval $(0, t)$. We define

$$f_{\#}(t) = \frac{1}{t}\int_0^t f(s)\,ds - f(t) \tag{7}$$

$$|f|_W = |f_{\#}|_\infty. \tag{8}$$

Condition (4) is therefore $|E(s, Ta; B_0, B_1)|_W \leqslant c_1\,|a|_{A_1}$. The calculation of $|E(s, a; A_0, A_1)|_W$ for given couples (A_0, A_1) becomes therefore a matter of considerable interest. We begin by giving some results along these lines. L_0 is defined by

$$|f|_{L_0} = \int_{\{x/|f(x)|>0\}} d\mu(x) \tag{9}$$

and clearly the best approximation $f - f_0$ to f in L_∞ norm for all choices of $f_0 : |f_0|_{L_0} \leqslant t$ is achieved by removing from f its piece supported on a set of which f is largest. Hence

$$E(t, f; L_0, L_\infty) = f^*(t) \tag{10}$$

where f^* is the nonincreasing rearrangement of f. See [3] for this and related results. Denote by $W(A_0, A_1)$ the set of all a for which $|E(s, a; A_0, A_1)|_W < \infty$. Then $W(L_0, L_\infty)$ is the class of functions f satisfying

$$\frac{1}{t}\int_0^t f^*(s)\,ds - f^*(t) \leqslant c. \tag{11}$$

On R^n this class was shown by Bennett–DeVore–Sharpley [1] to be the rearrangement invariant hull of $BMO(Q)$, where Q is any fixed cube. They called this class weak-L_∞. We shall prove in the sequel that for $0 < p < \infty$, $W(L_0, L(p, 1)) \supset L(p, \infty)$, and for $0 < p \leqslant 1$, $W(L_0, L(p, 1)) = L(p, \infty)$. Another result we shall prove is that for $1 < p < \infty$, $W(L_0, L_p) \supsetneqq L(p, \infty)$. The result raises an interesting question: What is an intrinsic characterization of functions in $W(L_0, L_p)$? This class interpolates with $L(r, q)$ $(r \neq p)$ in much the same way as $L(p, \infty)$ does and a characterization of its elements might make verification of condition (4) for particular operators, easier. Another result proved in the sequel is that we can replace the difference between E and its average by differences of higher averages, and still get the interpolation theorem. This shows for example that L_∞ is only one in a whole scale of function classes, each interpolating with L_0 as L_∞ does, and furthermore the class corresponding to the nth averages contain certain functions growing as fast as $(\log^+ 1/t)^n$.

The reader should recognize of course that statements such as $W(L_0, L_p) \supsetneqq L(p, \infty)$ involve an abuse of notation. $E(t, a; A_0, A_1)$ is defined for $a \in A_0 + A_1$. and so $|a|_{W(A_0, A_1)} \leqslant c$ is defined for $a \in A_0 + A_1$, and the statement $W(L_0, L_p) \supsetneqq L(p, \infty)$ means that for $f \in L_0 + L_p$

$$|f|_{W(L_0, L_p)} \leqslant c\,|f|_{L(p,\infty)} \tag{12}$$

and that the reverse of the inequality does not hold.

The classes $W(A_0, A_1)$ do not lend themselves to sublinear conditions. Our final result is an attempt to generalize them to classes which will be more tractable. This involves the introduction of functions $g_a(t)$ which compensate for sudden descents of $E(t, Ta; B_0, B_1)$.

THEOREM 2. *$W(L_0, L(p, 1)) \supset L(p, \infty)$, for all $0 < p < \infty$, and $W(L_0, L(p, 1)) = L(p, \infty)$ for $0 < p \leqslant 1$.*

Proof. The same considerations which show $E(t, f; L_0, L_\infty) = f^*(t)$, show that

$$E(t, f; L_0, L(p, 1)) = \int_t^\infty f^*(u)(u-t)^{1/p-1}\,du. \tag{13}$$

Therefore

$$\begin{aligned}
E(\cdot, f)_\#(t) &= \frac{1}{t}\int_0^t E(s, f)\,ds - E(t, f)\\
&= \frac{1}{t}\int_0^t \int_s^\infty f^*(u)(u-s)^{1/p-1}\,du\,ds - \int_t^\infty f^*(u)(u-t)^{1/p-1}\,du\\
&= \frac{1}{t}\int_0^t \int_s^t f^*(u)(u-s)^{1/p-1}\,du\,ds\\
&\quad + \frac{1}{t}\int_0^t \int_t^\infty f^*(u)[(u-s)^{1/p-1} - (u-t)^{1/p-1}]\,du\,ds\\
&= \frac{p}{t}\int_0^t f^*(u)\,u^{1/p}\,du + \int_t^\infty f^*(u)\frac{1}{t}\int_0^t [(u-s)^{1/p-1} - (u-t)^{1/p-1}]\,ds\,du.
\end{aligned}$$

If $0 < p \leqslant 1$, both terms are nonnegative so that if $f \in W(L_0, L(p, 1))$, then $(1/t)\int_0^t f^*(u)\, u^{1/p}\, du \leqslant c$. Since f^* is nonincreasing this is equivalent to $f^*(u)\, u^{1/p} \leqslant c$, ($c$ does not have the same value at each occurrence), and so $f \in L(p, \infty)$. Consider the reverse inclusion $L(p, \infty) \subset W(L_0, L(p, 1))$. Clearly, it suffices to show that the second integral is bounded. Denote

$$\begin{aligned} \varphi(x) &= (1-x)^{1/p-1}, \qquad 0 \leqslant x \leqslant 1, \\ &= 0, \qquad\qquad\qquad 1 < x. \end{aligned} \tag{14}$$

If $u^{1/p} f^*(u) \leqslant c$,

$$\begin{aligned} &\int_t^\infty f^*(u) \frac{1}{t} \int_0^t \left[(u-s)^{1/p-1} - (u-t)^{1/p-1}\right] ds\, du \\ &\leqslant c \int_t^\infty \frac{1}{t} \int_0^t \left[\left(1 - \frac{s}{u}\right)^{1/p-1} - \left(1 - \frac{t}{u}\right)^{1/p-1}\right] ds\, \frac{du}{u} \\ &= c \int_t^\infty \frac{u}{t} \int_0^{t/u} \left[(1-x)^{1/p-1} - \left(1 - \frac{t}{u}\right)^{1/p-1}\right] dx\, \frac{du}{u} \\ &= c \int_t^\infty \varphi_\#(t/u)\, \frac{du}{u} = c \int_0^1 \varphi_\#(u)\, \frac{du}{u}. \end{aligned}$$

Since

$$\frac{1}{t} \int_0^t g(u)\, du - \frac{1}{s} \int_0^s g(u)\, du = \int_t^s g_\#(u)\, \frac{du}{u} \tag{15}$$

(see [1]) and since $\varphi_\#$ does not change sign, we have

$$\begin{aligned} \int_0^1 |\varphi_\#(u)|\, \frac{du}{u} &= \lim_{t \to 0+} \int_t^1 |\varphi_\#(u)|\, \frac{du}{u} \\ &= \lim_{t \to 0+} \left| \frac{1}{t} \int_0^t \varphi(u)\, du - \int_0^1 \varphi(u)\, du \right| = |1 - p|. \end{aligned}$$

The proof is complete.

We next consider $W(L_0, L_p)$.

THEOREM 3. *If* $1 < p < \infty$, $W(L_0, L_p) \supsetneqq L(p, \infty)$.

Proof. $E(t, f; L_0, L_p) = (\int_t^\infty [f^*(u)]^p\, du)^{1/p}$, so that

$$\begin{aligned}
E(\cdot, f)_{\#}(t) &= \frac{1}{t}\int_0^t E(s, f)\, ds - E(t, f) \\
&= \frac{1}{t}\int_0^t \left(\int_s^\infty (f^*(u))^p\, du\right)^{1/p} ds - \left(\int_t^\infty (f^*(u))^p\, du\right)^{1/p} \\
&\leqslant \frac{1}{t}\int_0^t \left[\int_s^\infty (f^*(u))^p\, du - \int_t^\infty (f^*(u))^p\, du\right]^{1/p} ds \\
&= \frac{1}{t}\int_0^t \left(\int_s^t (f^*(u))^p\, du\right)^{1/p} ds \\
&\leqslant \left(\frac{1}{t}\int_0^t \int_s^t (f^*(u))^p\, du\, ds\right)^{1/p} \\
&= \left(\frac{1}{t}\int_0^t (f^*(u))^p\, u\, du\right)^{1/p}.
\end{aligned}$$

If $f^*(u)\, u^{1/p} \leqslant c$ we get therefore $|f|_{W(L_0, L_p)} \leqslant c$, and $L(p, \infty) \subset W(L_0, L_p)$. Take on $[0, 1]$, $f(s) = f^*(s) = s^{-1/p}(\log(1/s))^{1/p'}$, where $(1/p) + (1/p') = 1$. Clearly, $f \notin L(p, \infty)$, while $f \in W(L_0, L_p)$,

$$\left(\int_s^1 f^p(u)\, du\right)^{1/p} = \left(\int_s^1 \left(\log\frac{1}{u}\right)^{p-1} \frac{du}{u}\right)^{1/p} = \left(\frac{1}{p}\right)^{1/p} \log\frac{1}{s}.$$

Therefore

$$|f|_{W(L_0, L_p)} = \sup_{0<t} \left(\frac{1}{p}\right)^{1/p} \frac{1}{t}\int_0^t \left(\log^+\frac{1}{s} - \log^+\frac{1}{t}\right) ds = \left(\frac{1}{p}\right)^{1/p}.$$

We should emphasize, however, that $W(L_0, L_p)$ is not a lattice and the condition $(Ta)^*(t)\, t^{1/p} \leqslant c$ cannot be replaced by $(Ta)^*(t)\, t^{1/p}(\log^+ 1/t)^{-1/p'} \leqslant c$. The functions in the range whose rearrangements are larger than would be permitted by $L(p, \infty)$ conditions, have to compensate for this by having smoother rearrangements. This smoothness, as expressed by

$$\frac{1}{t}\int_0^t \left(\int_s^\infty (f^*(u))^p\, du\right)^{1/p} ds - \left(\int_t^\infty (f^*(u))^p\, du\right)^{1/p} \leqslant c \qquad (16)$$

seems hard to verify in practice. The results we have, while interesting from the point of view of interpolation theory, will probably be be useful in applications only after condition (16) is replaced by conditions involving f directly.

We next consider various extensions of Theorem 1. One extension involves higher averages. If both f and $f_{\#}$ are in $L_1(0,t)$ $\forall t>0$, we can consider $f_{\#2}=(f_{\#})_{\#}$. Note that if $f\in W$, $f_{\#}\in L_{\infty}$ and $f_{\#2}$ is defined. In the same way we make

DEFINITION 2. $f_{\#n}=(f_{\#(n-1)})_{\#}$ and $|f|_{W_n}=|f_{\#n}|_{L_\infty}$.

The spaces $W_n=\{f/|f|_{W_n}<\infty\}$ increase with n. We check that $(\log^+(1/t))^n\in W_n$, $(\log^+(1/t))^n\notin W_{n-1}$. Note also that there is an abuse of notation in referring to functions in W_n. The elements of W_n are equivalence classes after factoring out polynomials of degree $(n-1)$ in $\log t$. A final comment: we have $|f_{\#n}|_\infty=|f_{\#(n-1)}|_W=|f_{\#}|_{W_{n-1}}$.

For the application of Wolff's theorem in the sequel, we shall need

THEOREM 4. *W_n is complete.*

Proof. We first show $f\in W_n\Rightarrow |f|\leqslant c[1+|\log t|^n]$. This will show that the following operator is well defined on each W_n:

$$b\colon h\to h_b=-h(t)-\int_1^t h(u)\frac{du}{u}. \tag{17}$$

From (15) we have

$$\left|\frac{1}{t}\int_0^t f(u)\,du\right|\leqslant\left|\int_0^1 f(u)\,du\right|+\left|\int_1^t |f_{\#}(u)|\frac{du}{u}\right|,$$

$f\in W_n$ implies $f_{\#}\in W_{n-1}$, and so by induction, $|f_{\#}(u)|\leqslant c(1+|\log u|^{n-1})$, and

$$\left|\frac{1}{t}\int_0^t f(u)\,du\right|\leqslant c(1+|\log u|^n).$$

Also since $|f(t)|\leqslant|f_{\#}(t)|+|1/t\int_0^t f(u)\,du|$ we have the desired inequality, and b is well defined. We can verify $f_{b\#}=f$, and $f_{\#b}(t)=f(t)-\int_0^1 f(u)\,du$. Since the elements of W_n are defined modulo polynomials in $\log t$, we have $f_{\#b}\equiv f$. Therefore $b\colon W_{n-1}\to W_n$ is 1–1 isometry onto W_n and this proves inductively that W_n is complete.

THEOREM 5. $(L_0, W_n)_{1/p,q;E}=L(p,q)$, *where* $0<p<\infty$, $0<q\leqslant\infty$.

Proof. The proof, using induction, is similar to that of [2, Theorem 4], and we shall be brief. It is shown in [2] that for $1<p$

$$|f|_{L(p,q)}\sim|f_{\#}|_{L(p,q)}. \tag{18}$$

Also, our theorem for $n=1$ is proved there. We can assume the theorems holds therefore for W_{n-1}, and prove it for W_n. From the validity of the theorem for W_{n-1} follows, using standard reiteration theorems $(L(1,\infty), W_{n-1})_{\theta,q;K} = L(p,q)$, where $(1/p) = 1-\theta$, $0<\theta<1$. Since clearly $\#\colon W_n \to W_{n-1}$ and $\#\colon L_1 \to L(1,\infty)$ we have $\#\colon (L_1, W_n)_{\theta,q;K} \to (L(1,\infty), W_{n-1})_{\theta,q;K} = L(p,q)$. Using this and (18), we have for $(1/p) = 1-\theta$, $0<q\leqslant\infty$,

$$\begin{aligned} |f|_{L(p,q)} &\leqslant c\,|f_{\#}|_{L(p,q)} \leqslant c\,|f|_{(L_1,W_n)_{\theta,q;K}} \\ &\leqslant c\,|f|_{(L_1,L_\infty)_{\theta,q;K}} = c\,|f|_{L(p,q)}. \end{aligned}$$

(The last inequality simply since $L_\infty \subset W_n$.) We therefore have $(L_1, W_n)_{\theta,q;K} = L(p,q)$, $(1/p) = 1-\theta$, $0<q\leqslant\infty$. The extension to the full range, using Wolff's theorem follows as in [2], and the theorem is proved. Wolff's theorem is in [5]. Its restatement in [2] is easier to apply here: A_1, A_2, A_3, A_4 are quasi-Banach Abelian groups and $A_1 \cap A_4 \subset A_2 \cap A_3$. Assume

$$(A_1, A_3)_{\beta,q;E} = A_2, \qquad 0<\beta<\infty, \quad 0<q\leqslant\infty, \tag{19}$$

$$(A_2, A_4)_{\psi,r;K} = A_3, \qquad 0<\psi<1, \quad 0<r\leqslant\infty. \tag{20}$$

Then

$$(A_1, A_4)_{\alpha_2,q;E} = A_2, \qquad \alpha_2 = \beta/\psi, \tag{21}$$

$$(A_1, A_4)_{\alpha_3,r;E} = A_3, \qquad \alpha_3 = \beta(1-\psi)/\psi. \tag{22}$$

THEOREM 6. *Let* $T\colon A_0 + A_1 \to B_0 + B_1$ *be E-quasilinear and satisfy* $\exists p>0$, $\delta>0$ *such that for every* $a \in A_0 \cap A_1$, $\exists g_a(t)\colon R^+ \to R^+$ *measurable, and* $\delta \leqslant g_a(t)\ \forall t$, *and*

$$|\,g_a(t)\,E^p(t, Ta; B_0, B_1)|_{W_n} \leqslant c_1\,|a|_{A_1}^p. \tag{23}$$

$$|Ta|_{B_0} \leqslant c_0\,|a|_{A_0}. \tag{24}$$

Then $T\colon (A_0, A_1)_{\alpha,q;E} \to (B_0, B_1)_{\alpha,q;E}$ *for* $0<\alpha<\infty$, $0<q\leqslant\infty$.

Proof. Consider, as in [2], $E_T\colon A_0 + A_1 \to L_0 + W_n$, defined

$$E_T(a)(s) = E(s, Ta; B_0, B_1). \tag{25}$$

Since T is E-quasilinear, $|Ta|_{(B_0,B_1)_{\alpha,q;E}}$ is a semi-quasinorm on $A_0 \cap A_1$, which satisfies

$$\begin{aligned} |Ta|^p_{(B_0,B_1)_{\alpha,q;E}} &= |E^p_T(a)|_{L(1/\alpha p, q/p)} \\ &\leqslant \frac{1}{\delta} |g_a(t)\, E^p_T(a)|_{L(1/\alpha p, q/p)} \\ &\sim \frac{1}{\delta} |g_a(t)\, E^p_T(a)|_{(L_0, W_n)_{\alpha p, q/p;E}} \\ &\leqslant \frac{1}{\delta} |g_a(t)\, E^p_T(a)|^{\alpha p}_{L_0} |g_a(t)\, E^p_T(a)|_{W_n} \\ &= \frac{1}{\delta} |E^p_T(a)|^{\alpha p}_{L_0} |g_a(t)\, E^p_T(a)|_{W_n} \\ &\leqslant \frac{c}{\delta} |a|^{\alpha p}_{A_0} |a|^p_{A_1}. \end{aligned}$$

Taking pth roots we get $|Ta|_{(B_0,B_1)_{\alpha,q;E}} \leqslant c/\delta^{1/p} |a|^{\alpha}_{A_0} |a|_{A_1}$ and reiteration between different values of α gives $|Ta|_{(B_0,B_1)_{\alpha,q;E}} \leqslant c\,|a|_{(A_0,A_1)_{\alpha,q;E}}$. The theorem is proved.

Theorem 6 generalizes [2, Theorem 5] in three respects: the introduction of p, of $g_a(t)$, and taking W_n norms instead of W norms. $|E^p|_W < \infty$, leads to still more interpolation classes, sometimes larger than those defined by $|E|_W$. The analysis in each case is straightforward.

The introduction of $g_a(t)$ is motivated by the following considerations: the W classes are very rigid, because of the subtraction which occurs in the definition of $\#$. The introduction of $g_a(t)$ helps in this respect; for example, take $f(t)$ which is positive on $(0, 1)$, 0 for $1 < t$ and $f(t) \leqslant \log^+(1/t)$. An f satisfying these conditions and which is not in W is easy to construct. Taking $g(t) = \log^+(1/t)/f(t) \geqslant 1$ gives $|g(t)f(t)|_W = 1$.

The generalization involving the replacement of $|\ |_W$ by $|\ |_{W_n}$ is clear.

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