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Closed Fredholm and Semi-Fredholm Operators, Essential Spectra and Perturbations

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In these notes we provide rather extensive characterizations of closed densely defined Fredholm and semi-Fredholm operators on a Banach space, and their perturbation classes. We make use of the notions of measure of noncompactness, special norm equalities, and certain "pseudo" Banach algebra concepts as they pertain to closed operators. Classes of perturbations of closed Fredholm and semi-Fredholm operators are effectively identified, respectively, with classes of perturbations of the Wolf, Schechter, and Gustafson–Weidman essential spectra for closed operators. A by-product of this identification is a generalization of the celebrated Weyl theorem which characterizes the essential spectrum of a compact self-adjoint operator on a Hilbert space. We obtain spectral mapping theorems for some particular Wolf and Schechter essential spectra.

INTRODUCTION

Fredholm operator theory has played an ever increasing role in investigations of different classes of singular integral equations, in the theory of the perturbation of Hermitian operators by Hermitian and non-Hermitian summands, in the theory of operators in spaces with indefinite metric, and in obtaining a priori estimates in determining properties of certain differential operators.

Indeed, linear operators which arise from ordinary differential expressions of the form

$$T = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 ,$$

where $D = d/dt$ and the coefficients a_k are complex valued functions of a real variable, provide significant examples of *closed densely defined unbounded Fredholm operators* on familar Banach spaces of functions. In these notes we provide rather extensive characterizations of closed densely defined Fredholm and semi-Fredholm operators on a Banach space, and their perturbation classes. We utilize the notions of measure of noncompactness, special norm equalities, and certain "pseudo" Banach algebra concepts as they pertain to closed operators. Classes of perturbations of closed Fredholm and semi-Fredholm operators are effectively identified, respectively, with the classes of perturbations of the Schechter, Wolf, and Gustafson–Weidman essential spectra for closed operators. A by-product of this identification is a generalization of the celebrated Weyl theorem which characterizes the essential spectrum of a compact self-adjoint operator on a Hilbert space. We obtain spectral mapping theorems for some particular Wolf and Schechter essential spectra.

Let X be a Banach space. By an operator A on X we mean a linear operator with domain $D(A) \subset X$ and range $R(A) \subset X$. The null space of an operator A on X is denoted by $N(A)$. The nullity of A, that is, the dimension of $N(A)$, is designated by $a(A)$, whereas $b(A)$ represents the defect of A, that is, the codimension of $R(A)$ in X. When either $a(A)$ or $b(A)$ is finite, the index $i(A)$ is given by $i(A) = a(A) - b(A)$.

A densely defined closed operator A on X is said to be a *Fredholm* operator if it satisfies the following.

(i) $a(A)$ is finite,

(ii) $b(A)$ is finite,

(iii) $R(A)$ is closed in X.

Densely defined closed operators on X satisfying (i) and (iii) above are said to be *upper semi-Fredholm* operators, and those satisfying (ii) and (iii) are called *lower semi-Fredholm* operators. Let $\Phi(X)$ denote the set of Fredholm operators, $\Phi_+(X)$ the set of upper semi-Fredholm operators, and $\Phi_-(X)$ the set of lower semi-Fredholm operators. Observe that if X is finite dimensional, each densely defined operator on X is trivially a member of each of the above classes, so hereafter we assume that X is infinite dimensional.

Given a subset Q of X, the *measure of noncompactness* $q(Q)$ is defined by

$$q(Q) = \begin{cases} \infty, & \text{if } Q \text{ is unbounded,} \\ \inf\{r \mid Q \text{ can be covered by a finite number of spheres with radius } r\} \\ \text{if } Q \text{ is bounded.} \end{cases}$$

We observe that a subset Q of X is totally bounded if and only if $q(Q) = 0$, and thus the closure $\bar{Q}$ of Q is compact if and only if $q(Q) = 0$. It is also clear that given subsets Q_1, Q_2 of X, $q(Q_1 + Q_2) \leqslant q(Q_1) + q(Q_2)$. We notice further that if Q is a subset of X and $A \in B(X)$, that is, A is a bounded linear operator on X with $D(A) = X$, then $q(A(Q)) \leqslant \| A \| \, q(Q)$.

Let A be an operator on X. Then a *singular sequence* in X relative to A is sequence $\{x_k\}$, $k = 1, 2,\ldots$ in $D(A)$ such that

(i) $\| x_k \| = 1$, $k = 1, 2,\ldots$;

(ii) $\{x_k\}$, $k = 1, 2,\ldots$, has no convergent subsequence in X;

(iii) Ax_k converges to 0 as k approaches infinity.

Let A be an operator on X. Let U be the collection of all closed subspaces W of X with finite codimension such that $W \cap D(A) \neq \{0\}$. Define $v(A)$ by

$$v(A) = \sup_{W \in U} \{ \inf_{\substack{x \in W \cap D(A) \\ \|x\| = 1}} \| Ax \| \}$$

LEMMA 1 (Kato [1, 3]). *Let $K(X)$ denote the set of compact operators on X with domains the whole space X. Suppose $A \in \Phi_+(X) \cup \Phi_-(X)$ and $K \in K(X)$. Then*

(i) *$A + K$ has closed range in X, $(D(A + K) = D(A))$.*

(ii) *$i(A + K) = i(A)$.*

(iii) *For scalars c, $a(A + cK)$ and $b(A + cK)$ have constant values m, n, respectively, except perhaps at isolated points. At the isolated points $a(A + cK) > m$ and $b(A + cK) > n$.*

An immediate consequence of the above lemma is the following.

(i) For $A \in \Phi(X)$, and $K \in K(X)$, $A + K \in \Phi(X)$ with $i(A + K) = i(A)$.

(ii) For $A \in \Phi_+(X)$ and $K \in K(X)$, $A + K \in \Phi_+(X)$ with $i(A + K) = i(A)$.

(iii) For $A \in \Phi_-(X)$ and $K \in K(X)$, $A + K \in \Phi_-(X)$ with $i(A + K) = i(A)$.

LEMMA 2 (Yood [2, 4]). *Suppose $A \in \Phi(X)$. Then there exists $A_0 \in B(X)$ and finite rank operators F_1, F_2 in $K(X)$ such that*

(i) *$A_0 A = I - F_1$ on $D(A)$,*

(ii) *$AA_0 = I - F_2$ on X.*

Moreover, if $A \in \Phi(X) \cap B(X)$, then $A_0 \in \Phi(X)$ with $i(A) = -i(A_0)$. Conversely if A is a densely defined closed operator on X and there exist A_0 in $B(X)$ and K_1, K_2 in $K(X)$ satisfying $A_0A = I - K_1$ on $D(A)$, and $AA_0 = I - K_2$ on X, then $A \in \Phi(X)$.

LEMMA 3. *Let A be a densely defined closed operator on X. Then $A \in \Phi_+(X)$ with $i(A) \leqslant 0$ if and only if $A = A_0 + F$ on $D(A)$, where $A_0 \in \Phi_+(X)$ with $a(A_0) = 0$ and F is a finite rank operator on X.*

Proof. Suppose $A = A_0 + F$ with A_0 and F as above. Then it is clear from Lemma 1 that $A \in \Phi_+(X)$ with $i(A) = i(A_0 + F) = i(A_0) \leqslant 0$. Assume now that $A \in \Phi_+(X)$ and $i(A) \leqslant 0$. Let $a(A) = n$. If $n = 0$, the result is clear with F taken to be the zero operator. Suppose that $n > 0$. $i(A) \leqslant 0$. Therefore $b(A) > n$. Hence there exist n elements $y_1, y_2, \ldots, y_n$ in X which are linearly independent modulo $R(A)$. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for $N(A)$. Let x_i' in the dual space X' of X, $i = 1, 2, \ldots, n$ be chosen so that

$$x_i'(x_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \qquad (i, j = 1, 2, \ldots, n).$$

Define $F \in B(X)$ by

$$Fx = \sum_{i=1}^{n} x_i'(x)\, y_i \qquad \text{for each} \quad x \in X.$$

F is clearly a finite rank operator. Define A_0 on $D(A)$ by $A_0x = Ax - Fx$ for $x \in D(A)$, $A_0 \in \Phi_+(X)$ by Lemma 1. We claim further that $a(A_0) = 0$. For suppose $x \in N(A_0)$. Then

$$Ax = Fx = \sum_{i=1}^{n} x_i'(x)\, y_i\,.$$

The y_i are linearly independent modulo $R(A)$. Therefore, $x_i'(x) = 0$ for $i = 1, 2, \ldots, n$. Hence, $Ax = 0$. Therefore, $x \in N(A)$. Thus

$$x = \sum_{i=1}^{n} c_i x_i\,,$$

where the c_i are scalars. Now $x_i'(x) = 0$ for all i and $x_i'(x_j) = \delta_{ij}$. Thus $c_i = 0$ for all i. Hence $x = 0$.

THEOREM I. *Let A be a densely defined closed operator on X. Then the following are equivalent.*

(i) $A \in \Phi_+(X)$.

(ii) *There exists positive constant* C *such that for each bounded subset* Q *of* $D(A)$, $q(Q) \leqslant Cq(A(Q))$.

(iii) A *does not have a singular sequence.*

(iv) $v(A) \neq 0$.

(v) *For each* $K \in K(X)$, $a(A - K)$ *is finite.*

Proof. We show first that (i) implies (ii) implies (iii) implies (i) and subsequently (i) if and only if (iv) and (i) if and only if (v).

(i) *implies* (ii).

Let $A \in \Phi_+(X)$ and suppose Q is a bounded subset of $D(A)$. Either $A \in \Phi(X)$ or $i(A) \leqslant 0$. Suppose $A \in \Phi(X)$. Then by Lemma 2 there exist $F \in K(X)$ and $A_0 \in B(X)$ such that $A_0A = I - F$ on $D(A)$. Thus,

$$\begin{aligned} q(Q) &= q((A_0A + F)Q) \leqslant q(A_0A(Q)) + q(F(Q)) \\ &= q(A_0A(Q)) \leqslant \| A_0 \| \, q(A(Q)). \end{aligned}$$

Now suppose that $i(A) \leqslant 0$. Let Q be a bounded subset of $D(A)$. If $q(A(Q)) = \infty$, the result is clear. Suppose then that $q(A(Q)) < \infty$. By Lemma 3 there exists $A_0 \in \Phi_+(X)$ with $a(A_0) = 0$ and $F \in K(X)$ such that $A_0 = A - F$ on $D(A)$. Thus,

$$q(A_0(Q)) = q((A - F)Q) \leqslant q(A(Q)) < \infty.$$

As a consequence of the closed range theorem for one-to-one closed operators on a Banach space, there exists $M > 0$ such that for each $x \in D(A)$,

$$\| x \| \leqslant M \| A_0 x \|. \tag{1}$$

Let $y_1, y_2, \ldots, y_n$ be an ϵ-net for $A_0(Q)$. For each y_k whose ϵ-sphere has nonempty intersection with $A_0(Q)$, consider the set of x in Q such that $\| A_0x - y_k \| < \epsilon$. Choose one and label it x_k. Now let x be any point of Q and let y_k be such that $\| A_0x - y_k \| < \epsilon$. Then there is an x_k from above mentioned set such that $\| A_0x_k - y_k \| < \epsilon$. Thus

$$\| A_0(x - x_k) \| \leqslant \| A_0x - y_k \| + \| y_k - A_0x_k \| < 2\epsilon.$$

By inequality (1), $\| x - x_k \| < 2M\epsilon$. Thus the x_k form a $2M\epsilon$-net for Q. Hence

$$q(Q) \leqslant 2Mq(A_0(Q)) = 2Mq((A - F)Q) \leqslant 2Mq(A(Q)).$$

(ii) *implies* (iii). Suppose A has a singular sequence x_k, $k =$

$1, 2, \ldots$. Let Q be the set of points of the sequence. Then Q is a bounded subset of $D(A)$ satisfying $\bar{Q}$ is not compact. However, by (ii), $q(Q) \leqslant Mq(A(Q))$ for some $M > 0$. Clearly $A(Q) = \{0\}$. Thus $q(Q) = 0$. Hence $\bar{Q}$ is compact. (contrad.)

(iii) *implies* (i). Suppose A is not a member of $\Phi_+(X)$. Then either $a(A) = \infty$ or $a(A) < \infty$ and $R(A)$ is not closed. Suppose $a(A) = \infty$. By repeated use of Riesz's lemma we easily obtain a sequence $\{x_k\}$, $k = 1, 2, \ldots$ in $N(A)$ such that $\| x_k \| = 1$ for $k = 1, 2, \ldots$, and $\| x_i - x_j \| \geqslant \frac{1}{2}$ for $i \neq j$. Clearly the sequence $\{Ax_k\}$, $k = 1, 2, \ldots$, converges to 0 as k approaches infinity. Hence $\{x_k\}$, $k = 1, 2, \ldots$, is a singular sequence for A and by (iii) we have a contradiction. Suppose now that $a(A) < \infty$ and $R(A)$ is not closed. $N(A)$ has finite dimension; therefore there exists closed subspace X_0 of X such that $X = N(A) \oplus X_0$. The restriction of A to the Banach space X_0 is a one-to-one closed operator from X_0 to X with range $R(A)$. Thus by the closed range theorem $R(A)$ is closed in X if and only if there exists $C > 0$ such that for each $x \in X_0 \cap D(A)$, $\| x \| \leqslant C \| Ax \|$. Hence for each positive integer k there exists $x_k \in D(A)$ such that

$$\| x_k \| = 1 \qquad \text{and} \qquad \| x_k \| > k \| Ax_k \|.$$

Clearly $\{Ax_k\}$ converges to 0 as k approaches infinity. Moreover, $\{x_k\}$ has no convergent subsequence. For suppose that $\{x_n\}$ is a subsequence of $\{x_k\}$ which converges to x as n approaches infinity. Now $\{Ax_n\}$ converges to 0 as n approaches infinity and A is a closed operator on X_0. Hence $x \in D(A) \cap X_0$ and $Ax = 0$. Thus $x \in N(A) \cap X_0$ which is a contradiction since $\| x \| = 1$ and $N(A) \cap X_0 = \{0\}$. Thus $\{x_k\}$ is a singular sequence for A which contradicts our assumption (iii).

(i) *implies* (iv). Let A be a member of $\Phi_+(X)$. Then the dimension of $N(A)$ is finite. Hence there exists a closed subspace W of X such that $X = N(A) \oplus W$. The operator A has dense domain in the infinite dimensional Banach space X. Thus $W \cap D(A) \neq 0$. The restriction of A to the Banach space W is a one-to-one closed operator from W to X with closed range $R(A)$. Thus there exists $M > 0$ such that for each $x \in D(A) \cap W$, $\| x \| \leqslant M \| Ax \|$. In particular, for $x \in D(A) \cap W$ satisfying $\| x \| = 1$, $\| Ax \| \geqslant 1/M > 0$. Thus,

$$\inf_{\substack{x \in W \cap D(A) \\ \|x\| = 1}} \| Ax \| \geqslant 1/M.$$

Consequently $v(A) > 0$.

(iv) *implies* (i). Suppose A is not a member of $\Phi_+(X)$. Then either the dimension of $N(A)$ is finite and $R(A)$ is not closed or the dimension of $N(A)$ is infinite. Suppose $a(A)$ is finite and $R(A)$ is not closed. Let W be a closed subspace of X with finite codimension in X such that $W \cap D(A) \neq 0$. Then there exists finite dimensional subspace X_1 of X such that $X = X_1 \oplus W$. There also exists closed subspace X_0 of X such that $X = X_0 \oplus N(A)$. Clearly,

$$W \cap D(A) = (W \cap N(A)) \oplus ((W \cap (X_0 \cap D(A)))$$

Hence,

$$A(W \cap D(A)) = A(W \cap X_0 \cap D(A))$$

and

$$R(A) = A(X_1 \cap D(A)) + A(W \cap X_0 \cap D(A)).$$

Now $R(A)$ is not closed in X and $A(X_1 \cap D(A))$ is finite dimensional; hence $A(W \cap X_0 \cap D(A))$ is not a closed subspace of X. The operator A restricted to the Banach space $W \cap X_0$ is a one-to-one closed operator from $W \cap X_0$ to X with range $A(W \cap X_0 \cap D(A))$ not closed in X. Hence we conclude as earlier in similar situations that there exists a sequence $\{x_k\}$, $k = 1, 2, \ldots$, in $W \cap X_0 \cap D(A)$ such that $\| x_k \| = 1$, $k = 1, 2, \ldots$, and $\{Ax_k\}$ converges to 0 as k approaches infinity. Thus,

$$\inf_{\substack{x \in W \cap D(A) \\ \|x\|=1}} \| Ax \| = 0.$$

W was arbitrary. Therefore, $v(A) = 0$.

Suppose now that the dimension of $N(A)$ is infinite. Let W be given as above. W has finite codimension; hence the dimension of X/W is finite. We proceed to show that $W \cap N(A) \neq \{0\}$, and it follows as above that $v(A) = 0$. Suppose to the contrary that $W \cap N(A) = \{0\}$. Let $\{x_1, x_2, \ldots\}$ be a denumerable linearly independent subset of $N(A)$. Suppose

$$\sum_{i=1}^{n} c_i \bar{x}_i = \bar{0} \qquad (\text{where } \bar{x} = x/W).$$

Then

$$\sum_{i=1}^{n} c_i x_i \text{ is a member of } W.$$

Now $W \cap N(A) = \{0\}$. Thus,

$$\sum_{i=1}^{n} c_i x_i = 0.$$

Thus $c_i = 0$ for $i = 1, 2, \ldots, n$. Hence X/W is infinite dimensional (contradiction).

(i) *implies* (v). Given that A is a member of $\Phi_+(X)$ and K is a member of $K(X)$, then $A - K$ is a member of $\Phi_+(X)$ by Lemma 1. Thus $a(A - K)$ is finite.

(v) *implies* (i). Suppose A is not a member of $\Phi_+(X)$. (iii) implies (i). Therefore A has a singular sequence. Thus it is clear that there exists x_1 in X such that $\| x_1 \| = 1$ and $\| Ax_1 \| < 2^{-1}$. By the Hahn–Banach theorem there exists x_1' in X' such that $\| x_1' \| = 1$ and $x_1'(x_1) = 1$. Suppose now that sets $\{x_1, x_2, \ldots, x_k, \ldots, x_{n-1}\}$, $x_k \in X$, and $\{x_1', x_2', \ldots, x_k', \ldots, x_{n-1}\}$, $x_k' \in X'$, with $k = 1, 2, \ldots, n-1$, have been constructed such that $\| x_k \| = 1$, $\| Ax_k \| < 2^{1-2k}$, $\| x_k' \| \leqslant 2^{k-1}$, and $x_i'(x_j) = \delta_{ij}$, $1 = i, j = n - 1$. Let M_k be the null space of x_k' for $k = 1, 2, \ldots, n - 1$. Let

$$M = \bigcap_{k=1}^{n-1} M_k .$$

Each M_k has codimension one in X; thus, M has finite codimension in X. A has a singular sequence in X. Thus, there exists x_n in M such that $\| x_n \| = 1$ and $\| Ax_n \| < 2^{1-2n}$. Let x' in X' be such that $\| x' \| = 1$ and $x'(x_n) = 1$. Then the linear functional

$$x_n' = x' - \sum_{k=1}^{n-1} x'(x_k)\, x_k'$$

satisfies $x_n'(x_k) = \delta_{nk}$, $k = 1, 2, \ldots, n$, and $\| x_n' \| \leqslant 2^{n-1}$. Thus by induction it is possible to construct sets

$$\{x_1, x_2, \ldots, x_k, \ldots\} \qquad \text{and} \qquad \{x_1', x_2', \ldots, x_k', \ldots\},$$

where $\| x_k \| = 1$, $\| x_k' \| \leqslant 2^{k-1}$, $\| Ax_k \| < 2^{1-2k}$, and $x_i'(x_j) = \delta_{ij}$. Define K_n in $K(X)$ by

$$K_n x = \sum_{k=1}^{n} x_k'(x)\, Ax_k \qquad \text{for each} \quad x \text{ in } X.$$

For m, n positive integers (say $m < n$),

$$\|(K_n - K_m)x\| \leqslant \Big(\sum_{k=m+1}^{n} 2^{k-1}2^{1-2k}\Big) \|x\| \leqslant 2^{-m}\|x\|.$$

Thus there exists K in $K(X)$ such that K_n converges to K in $B(X)$ as n approaches infinity. The denumerable set $\{x_1, x_2, \ldots, x_k, \ldots\}$ is clearly linearly independent and satisfies $K(x_k) = Ax_k$, $k = 1, 2, \ldots$. Thus $a(A - K)$ is infinite.

Definition. Let A be an operator on X. Let V be the collection of infinite dimensional subspaces W of X such that $W \cap D(A) \neq \{0\}$. Let $t(A)$ be defined by

$$t(A) = \sup_{W \in V} \{ \inf_{\substack{x \in W \cap D(A) \\ \|x\|=1}} \|Ax\| \}.$$

Lemma 4 (Lacey [5, 3]). *Suppose B is a member of $B(X)$. Then B is precompact if and only if for every $\epsilon > 0$ there exists a subspace N having finite deficiency in X such that B restricted to N has norm not exceeding ϵ.*

Theorem II. *Let A be a member of $\Phi_+(X)$. Suppose E is an operator on X such that*

(i) $D(A) \subset D(E)$.

(ii) *$A + E$ is a closed operator on X.*

(iii) $t(E) < v(A)$.

Then $A + E$ is a member of $\Phi_+(X)$.

Proof. Clearly $A + E$ is a densely defined closed operator on X. Hence by Theorem I it is sufficient to show that $a(A + E + K)$ is finite for each K in $K(X)$. Indeed, suppose there exists K in $K(X)$ such that $a(A + E + K)$ is infinite. $A + E + K$ is a closed operator on X. Thus $N(A + E + K)$ is a closed infinite dimensional subspace of X. Let W be any closed subspace of X with finite codimension such that $W \cap D(A) \neq \{0\}$. Let $\epsilon > 0$ be given. By Lemma 4 above there exists infinite dimensional subspace M of X with finite codimension such that K restricted to M has norm not exceeding ϵ. Let $Q = W \cap M \cap N(A + E + K)$. Clearly Q is a infinite dimensional subspace of X satisfying $Q \cap D(A) \neq \{0\} \neq Q \cap D(E)$.

For each x in Q, such that $\| x \| = 1$, $\| Ex \| = \|(A + K)x\| \geqslant \| Ax \| - \epsilon$. Thus,

$$\begin{aligned} \inf_{\substack{x \in Q \cap D(E) \\ \|x\|=1}} \| Ex \| &\geqslant \inf_{\substack{x \in Q \cap D(A) \\ \|x\|=1}} (\| Ax \| - \epsilon) \\ &= \{ \inf_{\substack{x \in Q \cap D(A) \\ \|x\|=1}} \| Ax \| \} - \epsilon \\ &\geqslant \{ \inf_{\substack{x \in W \cap D(A) \\ \|x\|=1}} \| Ax \| \} - \epsilon. \end{aligned}$$

Now W was an arbitrary closed subspace of X with finite codimension such that $W \cap D(A) \neq \{0\}$. Thus

$$\inf_{\substack{x \in Q \cap D(E) \\ \|x\|=1}} \| Ex \| \geqslant v(A) - \epsilon.$$

Hence $t(E) \geqslant v(A) - \epsilon$. Thus, $t(E) \geqslant v(A)$, since ϵ was arbitrary (contradiction).

Definition. Let A be an operator on X. Suppose $N(A)$ is closed. Then the minimum modulus of A, $y(A)$ is defined by

$$y(A) = \inf_{x \in D(A)} \| Ax \| / d(x, N(A)),$$

where $0/0$ is defined to be ∞.

Lemma 5 (Kato and Sz.-Nagy [1, 3, 6]). *Suppose A is a member of $\Phi_+(X) \cup \Phi_-(X)$. Let E be an operator on X with $D(A) \subset D(E)$ for which there exist nonnegative numbers a, b satisfying*

(i) $\| Ex \| \leqslant a \| x \| + b \| Ax \|$ *for each x in $D(A)$,*

(ii) $a + by(A) < y(A)$.

Then

(i) *$A + E$ is a closed operator with closed range in X.*

(ii) *$a(A + E) \leqslant a(A)$ and $b(A + E) \leqslant b(A)$.*

(iii) *$i(A + E) = i(A)$.*

Corollary. *Let A be a member of $\Phi_+(X)$. Let E be a member of $B(X)$ and suppose $t(E)$ is less than $v(A)$. Then $A + E$ is a member of $\Phi_+(X)$ and $i(A + E) = i(A)$.*

Proof. Clearly $D(A) \subset D(E)$ and $A + E$ is closed. Thus by Theorem II, $A + E$ is a member of $\Phi_+(X)$. We proceed to show that $i(A + E) = i(A)$. It is an immediate consequence of the definition of $t(E)$ that for c in the closed interval from 0 to 1, $t(cE) = ct(E)$. Thus for each such c, $t(cE) < v(A)$ and $A + cE \in \Phi_+(X)$. The mapping f of the closed interval from 0 to 1 into the extended set of integers defined by $f(c) = i(A + cE)$ is continuous with respect to the usual topology on [0, 1] and the discrete topology on the extended set of integers. For suppose c_0 is in [0, 1]. Then $A + c_0E$ is in $\Phi_+(X)$. Hence $y(A + c_0E) > 0$. Now suppose c in [0, 1] satisfies $|c - c_0| \|E\| < y(A + c_0E)$. Then by Lemma 5 we conclude that $(c - c_0)E + A + c_0E$ is a member of $\Phi_+(X)$. Moreover, $i((c - c_0)E + (A + c_0E)) = i(A + c_0E)$. Thus

$$f(c) = i(A + cE) = i((c - c_0)E + (A + c_0E)) = i(A + c_0E) = f(c_0).$$

Continuity of f is hence established. Therefore f is constant. Hence $i(A) = f(0) = f(1) = i(A + E)$.

DEFINITION. A Banach space X is called *subprojective* if for each infinite dimensional subspace V of X there is a closed infinite dimensional subspace $W \subset V$ and a bounded projection T of X onto W.

Remark. Clearly every Hilbert space is subprojective. Moreover, the spaces c_0, 1_p, $1 \leqslant p < \infty$, and $L_p(0, 1)$, $2 \leqslant p < \infty$, are also subprojective [7].

DEFINITION. A closed operator A on X is a *left divisor of zero modulo* $K(X)$ if there exists a noncompact operator T in $B(X)$ such that $R(T) \subset D(A)$ and $AT = K$ on X for some K in $K(X)$.

THEOREM III. *Let X be subprojective. Let A be a densely defined closed operator on X. Then if A is not a left zero divisor modulo $K(X)$, A is a member of $\Phi_+(X)$.*

Proof. Suppose A is not a member of $\Phi_+(X)$. Then by Theorem I there exists K in $K(X)$ such that the dimension of $N(A - K)$ is infinite. X is subprojective. Thus there exists a closed infinite dimensional subspace W of $N(A - K)$ and a bounded projection T of X onto W. Clearly $T(X) \subset D(A)$. T is not compact, for such an assumption clearly implies in particular that the closed unit sphere in W is compact and hence W is finite dimensional. We observe further that $(A - K)T = 0$ on X. Thus $AT = KT$. KT is of course compact. Thus A is a left divisor of zero modulo $K(X)$.

Given a densely defined closed operator A on X, we denote by

A' the adjoint or conjugate of A defined on X'. Given $M \subset D(A)$, ${}^0M$ denotes the set of x' in X' satisfying $x'(x) = 0$ for each x in M. Given $S \subset D(A')$, S^0 denotes the set of x in $D(A)$ such that $x'(x) = 0$ for each x' in S.

LEMMA 6 (standard result [4]). *Let A be a densely defined closed linear operator on X. Then*

(i) *If $R(A)$ is closed in X, $R(A') = {}^0N(A)$ and hence is closed in X'.*

(ii) *If $R(A')$ is closed X', then $R(A) = N(A')^0$ and is therefore closed in X.*

THEOREM IV. *A is a member of $\Phi_-(X)$ if and only if $b(A - K)$ is finite for each K in $K(X)$.*

Proof. Suppose A is a member of $\Phi_-(X)$. Then $b(A - K)$ is finite for each K in $K(X)$ by Lemma 1. Suppose now that A is not a member of $\Phi_-(X)$. Then either $b(A)$ is infinite or $b(A)$ is finite and $R(A)$ is not closed in X. If $b(A)$ is infinite we are finished, since $K = 0$ is compact. Suppose $b(A)$ is finite and $R(A)$ is not closed in X. We show first that there exists a sequence $\{x_k'\}$ with x_k' a member of $D(A')$, $k = 1, 2, \ldots$, and a sequence $\{x_k\}$ with x_k in X, $k = 1, 2, \ldots$, such that

(i) $\| x_k \| \leqslant (k + 1)^k$, $k = 1, 2, \ldots$;

(ii) $\| x_k' \| = 1$;

(iii) $\| A'x_k' \| < 1/2^k(k + 1)^k$;

(iv) $x_i'(x_j) = \delta_{ij}$, $i, j = 1, 2, \ldots$.

By assumption, $R(A)$ is not closed in X. Thus by Lemma 6, $R(A')$ is not closed in X'. Hence by the closed range theorem there exists x_1' in $D(A')$ such that $\| x_1' \| = 1$ and $\| A'x_1' \| < \frac{1}{4}$. By definition of norm there exists x_1 in X such that $\| x_1 \| < 2$ and $x_1'(x_1) = 1$. Assume now that $x_1, x_2, \ldots, x_{n-1}$ and $x_1', x_2', \ldots, x_{n-1}'$ have been found satisfying (i) through (iv). Observe that $R(A')$ not closed in X' implies that $A'(D(A') \cap {}^0\{x_1, x_2, \ldots, x_{n-1}\})$ is not closed in X'. Thus there exists x_n' in $D(A') \cap {}^0\{x_1, x_2, \ldots, x_{n-1}\}$ such that $\| x_n' \| = 1$ and $\| A'x_n' \| < 1/2^n(n + 1)^n$. There also exists x in X such that $x_n'(x) = 1$ and $\| x \| < 2$. Set

$$x_n = x - \sum_{k=1}^{n-1} x_k'(x)\, x_k .$$

Then

$$\| x_n \| \leqslant \| x \| \left(1 + \sum_{k=1}^{n-1} \| x_k \|\right) \leqslant 2 \left(1 + \sum_{k=1}^{n-1} (k+1)^k\right) \leqslant (n+1)^n.$$

We observe further that $x_n'(x_n) = 1$ and $x_n'(x_k) = 0$ for $k = 1, 2,..., n-1$. Furthermore, $x_k'(x_n) = x_k'(x) - x_k'(x) = 0$ for $k = 1, 2,..., n-1$. Thus the existence of our pair of biorthogonal sequences is established by induction. We now define finite rank operators Kn by

$$K_n x = \sum_{k=1}^{n} A' x_k'(x)\, x_k \qquad \text{for} \quad x \text{ in } X,\ n = 1, 2,... .$$

Let n, m be arbitrary positive integers with $m < n$. Then

$$\begin{aligned} \| K_n x - K_m x \| &\leqslant \sum_{m+1}^{n} \| A' x_k' \| \| x \| \| x_k \| \\ &\leqslant \left(\sum_{m+1}^{n} 2^{-k}\right) \| x \| \\ &\leqslant (1/2^m) \| x \|, \qquad \text{for} \quad x \text{ in } X. \end{aligned}$$

Thus the sequence $\{K_n\}$ converge in $B(X)$ to a compact operator K where K is given by

$$Kx = \sum_{k=1}^{\infty} A' x_k'(x)\, x_k .$$

Clearly for x in $D(A)$ and for each k, $x_k'(K(x)) = A' x_k'(x) = x_k'(Ax)$. Thus each of the x_k' annihilates $R(A - K)$. The x_k' are linearly independent. Consequently $b(A - K)$ is infinite.

THEOREM V. *Let X be a reflexive Banach space. Let A be a densely defined closed operator on X. Then*

(i) *A is a member of $\Phi_-(X)$ if and only if A' is a member of $\Phi_+(X')$.*

(ii) *A is a member of $\Phi_+(X)$ if and only if A' is a member of $\Phi_-(X)$.*

Proof. It is immediate that A' is a well-defined closed operator on X'. Moreover, $D(A')$ is dense in X'. For suppose y'' in X'' satisfies $y''(x') = 0$ for each x' in $D(A')$. Then $y'' = 0$. Suppose not. X is reflexive. Thus there exists x_0 in X such that $y''(x') = x'(x_0)$ for each x' in X'. Clearly $x_0 \neq 0$. A is a closed operator. Hence its graph $G_A = \{(x, Ax) \mid x \in D(A)\}$ is a closed subspace of $X \times X$. $(0, x_0)$ is not a member of G_A. Thus there exists z' in $(X \times X)'$ such that $z'(G_A) = \{0\}$ and $z'(0, x_0) \neq 0$. Define x' in X' and y' in X' by

$$x'(x) = z'(x, 0) \qquad \text{for each } x \text{ in } X,$$

$$y'(x) = z'(0, x) \qquad \text{for each } x \text{ in } X.$$

Now $z'(x, Ax) = 0$ for each x in $D(A)$. Thus for each x in $D(A)$, $y'(Ax) = -x'(x)$. Hence y' is a member of $D(A')$. Therefore $y''(y') = 0$. But $y'(x_0) \neq 0$. Hence $y''(y') \neq 0$ (contradiction).

Suppose now that either A is a member of $\Phi_+(X) \cup \Phi_-(X)$ or A' is a member of $\Phi_+(X') \cup \Phi_-(X')$. Then by Lemma 6,

(i) $R(A)$ is closed in X,

(ii) $R(A')$ is closed in X',

(iii) $R(A) = N(A')^0$,

(iv) $R(A') = {}^0N(A)$.

Thus if A is a member of $\Phi_-(X)$ or A' is a member of $\Phi_+(X')$, $b(A)$ = dimension $X/R(A)$ = dimension $X/N(A')^0$ = dimension $(X/N(A')^0)'$ = dimension ${}^0(N(A')^0)$ = dimension $N(A') = a(A')$. Hence A is a member of $\Phi_-(X)$ if and only if A' is a member of $\Phi_+(X')$. Now if A is a member of $\Phi_+(X)$ or A' is a member of $\Phi_-(X')$, $a(A)$ = dimension $N(A)$ = dimension $N(A)'$ = dimension $X'/{}^0N(A)$ = dimension $X'/R(A') = b(A')$. Hence A is a member of $\Phi_+(X)$ if and only if A' is a member of $\Phi_-(X')$.

Corollary 1. *Let X be reflexive and let A be a densely defined closed operator on X. Then the following are equivalent.*

(i) *A is a member of $\Phi_-(X)$.*

(ii) *There exists $C > 0$ such that $q(Q) \leqslant C(A'(Q))$ for each bounded subset Q of $D(A')$.*

(iii) $v(A') \neq 0$.

(iv) *A' does not have a singular sequence in X'.*

(v) *$a(A' - K)$ is finite for each K in $K(X')$.*

Proof. This is an immediate consequence of Theorems I and V.

COROLLARY 2. *Let X be reflexive. Then A is a member of $\Phi_+(X)$ if and only if $b(A' - K)$ is finite for each K in $K(X')$.*

Proof. Immediate consequence of Theorems IV and V.

DEFINITIONS. Let A be a closed operator on X. A is a *right divisor of zero modulo* $K(X)$ if there is a noncompact operator T in $B(X)$ and compact operator K in $K(X)$ such that $TA = K$ on $D(A)$.

The operator A is *invertible modulo* $K(X)$ if there exists T in $B(X)$ and K_1, K_2 in $K(X)$ such that $AT = I + K_1$ on X and $TA = I + K_2$ on $D(A)$.

A Banach space X is *superprojective* if every subspace V having infinite codimension in X is contained in a closed subspace W having infinite codimension in X and there is a bounded projection P from X onto W.

Examples of superprojective spaces are 1_p, $1 < p < \infty$, and $L_p(0, 1)$, $1 < p \leqslant 2$ [9].

THEOREM VI. *Let X be superprojective and suppose A is a densely defined closed operator on X which is not a right divisor of zero modulo $K(X)$. Then A is a member of $\Phi_-(X)$.*

Proof. Suppose A is not a member of $\Phi_-(X)$. Then by Theorem IV there exists K in $K(X)$ such that $b(A - K)$ is infinite. X is superprojective. Thus there exists closed subspace W of X which contains $R(A - K)$, and has infinite codimension in X. In addition there exists a bounded projection P of X onto W. Now $I - P$ is a noncompact member of $B(X)$ satisfying $(I - P)(A - K)x = 0$ for each x in $D(A)$. Thus $(I - P)A = (I - P)K$ on $D(A)$. Hence A is a right zero divisor modulo $K(X)$.

THEOREM VII. *Let A be a closed densely defined operator on X. Then*

(i) *A is a member of $\Phi(X)$ if and only if $a(A - K)$ is finite and $b(A - K)$ is finite for each K in $K(X)$.*

(ii) *A is a member of $\Phi(X)$ if and only if A is invertible modulo $K(X)$.*

(iii) *If X is reflexive, A is a member of $\Phi(X)$ if and only if A' is a member of $\Phi(X')$, in which case $i(A) = -i(A')$.*

(iv) *If X is a Hilbert space and A is self-adjoint, then A is a* *member of $\Phi(X)$ if and only if A is a member of $\Phi_+(X) \cup \Phi_-(X)$, in which case $i(A) = 0$.*

Proof. (i) is an immediate consequence of Theorems I and IV. (ii) is an immediate consequence of Lemma 2. (iii) is an immediate consequence of Theorem V and its proof. Since a Hilbert space is reflexive and in addition may be identified with its dual space, (iv) is a consequence of (iii).

COROLLARY. *If X is both subprojective and superprojective and A is a densely defined closed operator on X which is neither a right divisor of zero modulo $K(X)$ nor a left divisor of zero modulo $K(X)$, then A is invertible modulo $K(X)$.*

Proof. Immediate consequence of Theorems III, VI, and part (ii) of Theorem VII.

LEMMA 7 (Gokhberg and Krein [10, 4]). *Suppose both A and B are members of $\Phi(X)$. Then*

(i) *AB is a member of $\Phi(X)$. (with $D(AB)$ defined in the usual way).*

(ii) $i(AB) = i(A) + i(B)$.

Remark. An equivalence relation is defined on $\Phi(X)$ by the relation, for A, B in $\Phi(X)$ A is equivalent to B if $i(A) = i(B)$. Now the set $\overline{\Phi(X)}$ of equivalence classes relative to this relation is as a result of Lemma 7 above, an abelian semigroup with identity under the binary operation $\bar{A} \cdot \bar{B} = \overline{AB}$.

DEFINITION. An infinite dimensional Banach space is a *shift* space if the following conditions are satisfied.

(i) It has a Schauder basis. That is, there exists a denumerable set $\{x_1, x_2, ..., x_n, ...\}$ of elements of X such that for each x in X there is a unique sequence $\{a_k\}$, $k = 1, 2, ...$, of scalars satisfying

$$x = \sum_{k=1}^{\infty} a_k x_k .$$

(ii) There exists a function f mapping the set of sequences of scalars into the set of nonnegative reals which satisfies:

(a) Given sequences of scalars $\{a_k\}\{b_k\}$ such that for each occurrence of a nonzero scalar c in $\{a_k\}$, there is also an occurrence of c in $\{b_k\}$, then $f(\{a_k\}) \leqslant f(\{b_k\})$.

(b) Given x in X, if

$$x = \sum_{k=1}^{\infty} a_k x_k ,$$

then $\| x \| = f(\{a_k\})$.

Remark. Each separable Hilbert space is a shift space as indeed is $l p$, $1 \leqslant p \leqslant \infty$.

THEOREM VIII. *Let X be a shift space. Then $\overline{\Phi(X)}$ is an abelian group which is isomorphic to the additive group Z of integers.*

Proof. It is clear by Lemma 7 that the mapping g of $\overline{\Phi(X)}$ into Z defined by $g(\bar{A}) = i(A)$ for each A in $\Phi(X)$ is a monomorphism. We proceed to show that g is onto and our proof is completed. Recall that X is a shift space. Hence X has a Schauder basis $\{x_1 , x_2 ,...\}$ relative to which the mapping $x_k \to x_{k+1}$, $k = 1, 2,...$, clearly extends to a densely defined closed operator A in $\Phi(X)$ such that $i(A) = -1$. Let x_0 denote the zero vector in X. Then observe that the mapping $x_k \to x_{k-1}$, $k = 1, 2,...$, extends to an operator B in $\Phi(X)$ satisfying $i(B) = 1$. Let n be an arbitrary positive integer, then by Lemma 7, $\bar{A}^n$ and $\bar{B}^n$ are members of $\overline{\Phi(X)}$ with $g(\bar{A}^n) = i(A^n) = -n$, $g(\bar{B}^n) = i(B^n) = n$ and $g(\bar{A}^n \cdot \bar{B}^n) = i(A^n) + i(B^n) = 0$. Thus g is onto.

DEFINITION. (i) E in $B(X)$ is a *Fredholm perturbation* if $A + E$ is a member of $\Phi(X)$ for each A in $\Phi(X)$. We denote the set of Fredholm perturbations by $F(X)$.

(ii) E in $B(X)$ is an *upper semi-Fredholm perturbation* if $A + E$ is a member of $\Phi_+(X)$ for each A in $\Phi_+(X)$. $F_+(X)$ denotes the collection of upper semi-Fredholm perturbations.

(iii) E in $B(X)$ is a *lower semi-Fredholm perturbation* if $A + E$ is a member of $\Phi_-(X)$ for each A in $\Phi_-(X)$. The set of lower semi-Fredholm perturbations is denoted by $F_-(X)$.

THEOREM IX. *Let E be a member of $B(X)$. Then*

(i) *E is a member of $F_-(X)$ if and only if $b(A - E)$ is finite for each A in $\Phi_-(X)$.*

(ii) *E is a member of $F_+(X)$ if and only if $a(A - E)$ is finite for each A in $\Phi_+(X)$.*

(iii) *E is a member of $F(X)$ if and only if either $b(A - E)$ is finite for each A in $\Phi(X)$ or $a(A - E)$ is finite for each A in $\Phi(X)$.*

Proof. Suppose E is a member of $F_-(X)$ and A is a member of $\Phi_-(X)$. Then clearly $A - E$ is a member of $\Phi_-(X)$. Hence $b(A - E)$ is finite. Now suppose that E is not a member of $F_-(X)$. Then there exists A in $\Phi_-(X)$ such that $A - E$ is not a member of $\Phi_-(X)$. Hence by Theorem IV there exists K in $K(X)$ such that $b(A - E - K)$ is infinite. Now $A - K$ is a member of $\Phi_-(X)$ by Lemma 1; thus $b(A - E - K)$ is finite. Contradiction, and the proof of (i) is completed.

(ii) is proved in a completely analogous manner utilizing Theorem I rather than Theorem IV. The validity of (iii) is seen as follows. Suppose E is a member of $F(X)$. Then clearly for each A in $\Phi(X)$, $A - E$ is also a member of $\Phi(X)$. Thus, $a(A - E)$ is finite and $b(A - E)$ is finite. Now suppose that $b(A - E)$ is finite for each A in $\Phi(X)$. Given A in $\Phi(X)$ and arbitrary scalar $c \neq 0$, then for each K in $K(X)$, $(A - K)/c$ is a member of $\Phi(X)$ by Lemma 1. Hence $b(A - cE - K)$ is finite for all scalars c. Thus $A - cE$ is a member of $\Phi_-(X)$ by Theorem IV. Now as a result of the compactness of the closed interval from 0 to 1, and Lemma 5, $a(A - E) \leqslant a(A)$. $a(A)$ is finite and hence so is $a(A - E)$. $A - E$ is of course a closed densely defined operator on X. Thus $A - E$ is a member of $\Phi(X)$. Hence E is a member of $F(X)$. One shows in an altogether similar manner employing Theorem I and Lemma 5 that $a(A - E)$ is finite for each A in $\Phi(X)$ implies that E is a member of $F(X)$.

COROLLARY. (i) $K(X) \subset F_+(X) \subset F(X)$. (ii) $K(X) \subset F_-(X) \subset F(X)$.

Proof. The first inclusions in (i) and (ii) are immediate consequences of Lemma 1. The second inclusions follow directly from the above theorem.

Remark 1. Let $X = L_1[0, 1]$. Define E mapping X into X by

$$(Ef)t = \int_0^1 k(s, t) f(s)\, ds, \qquad \text{for each } f \text{ in } L_1[0, 1],$$

and $k(s, t)$ is defined on $[0, 1] \times [0, 1]$ by

$$k(s,t) = \begin{cases} 0, & (s,t) \in (1/2^n, 1/2^{n-1}] \\ & \quad \times (2j/2^n, 2j + 1/2^n], \qquad j = 0, 1, \ldots, 2^{n-1} - 1; \\ 2, & (s,t) \in (1/2^n, 1/2^{n-1}] \\ & \quad \times (2j + 1/2^n, 2j + 2/2^n], \qquad j = 0, 1, \ldots, 2^{n-1} - 1; \\ 0, & \text{when } s = 0 \text{ or } t = 0, \qquad n = 1, 2, \ldots . \end{cases}$$

Then E is a member of $F_+(X) \cap F_-(X)$ and E is not a member of $K(X)$. (See [3].)

Remark 2. Let $X = 1_q \times L_p(-1, 1)$ with $1 < p < q < 2$. Then

(i) $F(X) \neq F_+(X)$.

(ii) $F_-(X) \neq K(X)$.

(iii) $F_+(X') \neq F_-(X')$.

(iv) $F(X') \neq F_-(X')$.

(See [11].)

THEOREM X. (i) *Each of* $F_-(X)$, $F_+(X)$, *and* $F(X)$ *is a closed subspace of* $B(X)$. *Moreover*, $F(X)$ *is a two-sided ideal in* $B(X)$.

(ii) *Each of the following is valid.*

(a) *For* A *in* $\Phi(X)$ *and* E *in* $F(X)$, $i(A + E) = i(A)$.

(b) *For* A *in* $\Phi_+(X)$ *and* E *in* $F_+(X)$, $i(A + E) = i(A)$.

(c) *For* A *in* $\Phi_-(X)$ *and* E *in* $F_-(X)$, $i(A + E) = i(A)$.

Proof. $E = 0$ is a member of $F(X)$. Clearly $\Phi(X)$ is closed under multiplication by nonzero scalars; hence $F(X)$ is closed under scalar multiplication. For E_1 and E_2 in $F(X)$ and A in $\Phi(X)$, $A + (E_1 + E_2) = (A + E_1) + E_2$ which is in $\Phi(X)$, thus $E_1 + E_2$ is a member of $F(X)$. Now let $\{E_n\}$ be a sequence in $F(X)$ which converges to E in $B(X)$ as n approaches infinity. Let A be an arbitrary member of $\Phi(X)$. For n sufficiently large, $\| E_n - E \| < \gamma(A)$. Thus by Lemma 5, $A + (E - E_n) + E_n = A + E$ is a member of $\Phi(X)$. Hence $F(X)$ is a closed subspace of $B(X)$. In a completely analogous manner one shows that $F_+(X)$ and $F_-(X)$ are also closed subspaces of $B(X)$. We proceed now to show that $F(X)$ is a two-sided ideal in $B(X)$.

(1) Let E be a member of $F(X)$ and A, A_1 members of $\Phi(X) \cap B(X)$. Then $AE + A_1$ is a member of $\Phi(X)$. We see this as

follows. By Lemma 2 there exists A_0 in $\Phi(X) \cap B(X)$ and K_1, K_2 in $K(X)$ such that $AA_0 = I - K_1$ and $A_0A = I - K_2$. A_0A_1 is a member of $\Phi(X)$ by Lemma 7. Thus $E + A_0A_1$ is a member of $\Phi(X)$. Hence

$$A(E + A_0A_1) = AE + (I - K_1)\,A_1 = AE + A_1 - K_1A_1$$

is a member of $\Phi(X)$. Thus

$$AE + A = (AE + A_1 - K_1A_1) + K_1A_1$$

is a member of $\Phi(X)$, by Lemma 1.

(2) Let E be a member of $F(X)$, B a member of $B(X)$, and A a member of $\Phi(X) \cap B(X)$, then $BE + A$ is a member of $\Phi(X)$. For it is clear that given scalar c sufficiently large in absolute value, each of $B - cI$ and cI is a member of $\Phi(X) \cap B(X)$. Thus

$$BE + A = ((B - cI) + cI) + A = (B - cI) + (cI + A)$$

is a member of $\Phi(X)$ by (1).

Finally, assume that E is a member of $F(X)$, B is a member of $B(X)$, and A is a member of $\Phi(X)$. By Lemma 2 there exists A_0 in $B(X)$, K_1, K_2 in $K(X)$ such that $AA_0 = I - K_1$ on X and $A_0A = I - K_2$ on $D(A)$. $I + A_0BE$ is a member of $\Phi(X)$ by (2). Hence

$$A(I + A_0BE) = A + (I - K_1)\,BE = A + BE - K_1BE$$

is a member of $\Phi(X)$. Thus

$$A + BE = (A + BE - K_1BE) + K_1BE$$

is a member of $\Phi(X)$. Hence BE is a member of $F(X)$ and $F(X)$ is a left ideal in $B(X)$. By making obvious changes in the above development, one shows that $F(X)$ is also a right ideal in $B(X)$.

It is a consequence of Lemma 5 that the mapping f of $[0, 1]$ into $Z \cup \{\infty\} \cup \{-\infty\}$ defined by $f(c) = i(A + cE)$, respectively, for A in $\Phi(X)$, E in $F(X)$; A in $\Phi_+(X)$, E in $F_+(X)$; and A in $\Phi_-(X)$, E in $F_-(X)$ is continuous. Thus $i(A + E) = i(A)$ for A, E taken in each of the above three situations.

DEFINITION. Let A be a densely defined closed operator on X. Let C be the set of complex numbers. Let

$$e_1(A) = C - \{c \mid A - c \in \Phi_+(X)\},$$
$$e_2(A) = C - \{c \mid A - c \in \Phi_-(X)\},$$
$$e_3(A) = C - \{c \mid A - c \in \Phi(X)\},$$
$$e_4(A) = C - \{c \mid A - c \in \Phi(X) \text{ and } i(A - c) = 0\}.$$

$e_1(A)$ and $e_2(A)$ are the Gustafson and Weidman essential spectra of A [8]. $e_3(A)$ is the Wolf essential spectrum of A [12]. $e_4(A)$ is the Schechter essential spectrum of A [13].

The *perturbation* classes $P_i(X)$, $i = 1, 2, 3, 4$, of the above-mentioned essential spectra are defined as follows.

$P_i(X)$ is the set of all E in $B(X)$ such that $e_i(A) = e_i(A + E)$ for each densely defined closed operator A on X.

Remark. In the special case where A is a bounded self-adjoint operator on a Hilbert space, for each i, $e_i(A)$ is the set of limit points of the spectrum of A (with eigenvalues counted according to their multiplicities). A celebrated theorem by Weyl [14] states that $e_i(A) = e_i(A + K)$ for each self-adjoint compact operator K. Recalling that $K(X) \subset F_+(X) \cap F_-(X)$, one observes that the theorem following the lemma stated below is an extension of the Weyl result.

LEMMA 8. *Suppose A is a member of $B(X)$, B is a member of $B(X)$, and BA is a member of $\Phi(X)$. Then A is a member of $\Phi(X)$ if and only if B is a member of $\Phi(X)$* [4].

THEOREM XI. (i) $P_1(X) = F_+(X)$, (ii) $P_2(X) = F_-(X)$, (iii) $P_3(X) = F(X)$, (iv) $P_4(X) = F(X)$.

Proof. (i) Suppose E is a member of $P_1(X)$. Let A be a member of $\Phi_+(X)$. Then 0 is not a member of $e_1(A)$. Therefore 0 is not a member of $e_1(A + E)$. Thus $A + E$ is a member of $\Phi_+(X)$. Hence E is a member of $F_+(X)$. Conversely, suppose that E is a member of $F_+(X)$. Let A be a densely defined closed operator on X. Suppose c is not a member of $e_1(A)$. Then $A - c$ is a member of $\Phi_+(X)$. Thus $(A + E) - c$ is a member of $\Phi_+(X)$. Hence c is not a member of $e_1(A + E)$. Thus $e_1(A + E) \subset e_1(A)$. Since inclusion holds for arbitrary E and A of above type, we conclude that $e_1(A) = e_1(A + E + (-E)) \subset e_1(A + E)$. Thus E is a member of $P_1(X)$.

(ii) and (iii) are established in a completely analogous manner. We proceed to ascertain (iv). Let E be a member of $F(X)$. Let A be a densely defined closed operator on X. Suppose c is not a member of $e_4(A)$. Then $A - c$ is a member of $\Phi(X)$ with $i(A - c) = 0$. Thus $A - c + E$ is a member of $\Phi(X)$ with $i(A - c + E) = 0$ by Theorem X. Hence c is not a member of $e_4(A + E)$. Thus $e_4(A + E) \subset e_4(A)$. This inclusion is valid for arbitrary A and E of above type. Therefore $e_4(A) = e_4(A + E + (-E)) \subset e_4(A + E)$. Thus E is a member of $P_4(X)$. Conversely, suppose E is a member of $P_4(X)$. Let A be a member of $\Phi(X)$. By Lemma 2 there exists $A_0 \in B(X)$ and K_1 in $K(X)$ such that $AA_0 = I - K_1$ on X. For $c > 0$ sufficiently large, it is clear that $A_0 - cI$ is a member of $\Phi(X)$ with $i(A_0 - cI) = 0$. We notice too that cI and $(1/c)I$ are members of $\Phi(X)$ with $i(cI) = 0$ and $i((1/c)I) = 0$. Thus 0 is not a member of $e_4((1/c)I)$. Hence 0 is not a member of $e_4((1/c)I + E)$. Therefore $(1/c)I + E$ is a member of $\Phi(X)$ with $i((1/c)I + E) = 0$. Thus by Lemma 7, $I + cE = cI((1/c)I + E)$ is a member of $\Phi(X) \cap B(X)$ with $i(I + cE) = 0$. $A_0 - cI$ is a member of $B(X) \cap \Phi(X)$. Thus by Lemma 2 there exists B_0 in $\Phi(X) \cap B(X)$ and K_2 in $K(X)$ such that $B_0(A_0 - cI) = I - K_2$ on X with $i(B_0) = -i(A_0 - cI) = 0$. Thus $B_0(I + cE)$ is a member of $\Phi(X)$ with $i(B_0(I + cE)) = 0$. Hence 0 is not a member of $e_4(B_0(I + cE))$. Therefore 0 is not a member of $e_4(B_0(I + cE) + E)$. Hence $B_0(I + cE) + E$ is a member of $\Phi(X)$ with $i(B_0(I+cE)+E) = 0$. Now

$$\begin{aligned} B_0(I + A_0E) &- (B_0(I + cE) + E) \\ &= B_0(I + ((A_0 - cI) + cI)E) - B_0 - cB_0E - E \\ &= B_0 + (I - K_2)\, E + cB_0E - B_0 - cB_0E - E \\ &= -K_2E. \end{aligned}$$

Thus

$$B_0(I + A_0E) = (B_0(I + cE) + E) - K_2E.$$

Thus $B_0(I + A_0E)$ is a member of $\Phi(X)$, by Lemma 1. Recall that each of B_0 and $I + A_0E$ is a member of $B(X)$. Thus, by Lemma 8, $I + A_0E$ is a member of $\Phi(X)$. Hence $A(I + A_0E) = A + (I - K_1)E = (A + E) - K_1E$ is a member of $\Phi(X)$. Therefore, $A + E = ((A + E) - K_1E) + K_1E$ is a member of $\Phi(X)$. Hence E is a member of $F(X)$.

THEOREM XII. *Let X be a complex Banach space. Let A be a* *member of $B(X)$. Let f be a complex-valued function which is analytic in a neighborhood of $\sigma(A)$, the spectrum of A. Then the operator $f(A)$ in $B(X)$ is well defined by the complex functional calculus* [15]. *Suppose now that $e_3(A) = \sigma(A)$. Then*

(i) $f(e_3(A)) = e_3(f(A))$,
(ii) $e_3(f(A)) = \sigma(f(A))$,
(iii) $f(e_4(A)) = e_4(f(A))$,
(iv) $e_4(f(A)) = \sigma(f(A))$.

Proof. Let d be an arbitrary complex number. Suppose that for each c in $e_3(A)$, $f(c) \neq d$. Then $f(z) - d$ is an analytic function in a neighborhood of $\sigma(A)$ which does not vanish on $\sigma(A)$. Thus by the complex functional calculus $f(A) - dI$ has a bounded inverse on X. Thus d is in the resolvent set of A. In particular, d is not a member of $e_3(f(A))$. Thus $e_3(f(A)) \subset f(e_3(A))$. Suppose now that $d = f(c)$, where c is a member of $e_3(A)$. Set

$$g(z) = \begin{cases} (f(z) - d)/(z - c), & \text{if } z \neq c; \\ f'(c), & \text{if } z = c. \end{cases}$$

Then $g(z)$ is analytic in a neighborhood of $\sigma(A)$ and $g(z)(z - c) = f(z) - d$. Thus by the functional calculus, $g(A)(A - cI) = f(A) - dI = (A - cI)g(A)$. Suppose that d is not a member of $e_3(f(A))$. Then $f(A) - d$ is a member of $\Phi(X)$. Observe that as a consequence of the equalities above, $N(A - c) \subset N(f(A) - d)$ and $R(f(A) - d) \subset R(A - c)$. Thus, $a(A - c) \leqslant a(f(A) - d) < \infty$ and $b(A - c) \leqslant b(f(A) - d) < \infty$. $R(f(A) - d)$ is closed in X and $X = R(f(A - d) \oplus N$, where N is a finite dimensional subspace of X. Hence $R(A - c)$ is closed in view of the above inclusion. Thus $A - c$ is a member of $\Phi(X)$. Therefore c is not a member of $e_3(A)$. Hence $f(e_3(A) \subset e_3(f(A)$, and the proof of (i) is completed.

By the standard spectral mapping theorem, $f(\sigma(A)) = \sigma(f(A))$. Observing further that for an arbitrary operator A, $e_3(A) \subset e_4(A) \subset \sigma(A)$, and using (i), we obtain (ii), (iii), and (iv) immediately.

COROLLARY. *Let X and A be as above. Suppose $e_3(A)$ is a* spectral set *of A. That is, there exist disjoint open sets U, V such that $e_3(A) \subset U$ and $\sigma(A) - e_3(A) \subset V$. Let*

$$P = 1/2\pi i \int_\Gamma (z - A)^{-1}\, dz$$

be the spectral projection of A, where Γ consists of a finite number of simple closed curves and is the boundary of an open set W such that $e_3(A) \subset W \subset \overline{W} \subset U$. As is well known, A is completely reduced by $R(P)$ and $N(P)$. Assume that $N(P)$ is finite dimensional and define A_1: $R(P) \to R(P)$ by $A_1 = A/R(P)$. Then for each f analytic in a neighborhood of $e_3(A)$, $f(e_3(A)) = e_3(f(A_1))$.

Proof. As is well known, $e_3(A) = \sigma(A_1)$. We now show that $e_3(A) = e_3(A_1)$. Suppose c is not a member of $e_3(A_1)$. Then $c - A_1$ is a member of $\Phi(R(P))$. Clearly $a(c - A) \leqslant a(c - A_1) +$ dimension $N(P) < \infty$ and $b(c - A) \leqslant b(c - A_1) +$ dimension $N(P) < \infty$. Furthermore, $R(c - A) = (c - A_1)\, R(P) + (c - A)\, N(P)$. Now $(c - A_1)\, R(P)$ is closed in $R(P)$ and $R(P)$ is closed in X, hence $(c - A_1)\, R(P)$ is closed in X. $(c - A)\, N(P)$ is finite dimensional. Thus $R(c - A)$ is closed in X. Hence $c - A$ is a member of $\Phi(X)$ and c is not a member of $e_3(A)$. Suppose now that c is not a member of $e_3(A)$. Then $c - A$ is a member of $\Phi(X)$. Clearly

$$a(c - A_1) \leqslant a(c - A) < \infty$$

and

$$\begin{aligned} b(c - A_1) &= \text{dimension } R(P)/((c - A)\, R(P)) \\ &\leqslant \dim R(P)/((c - A)\, R(P) + (c - A)\, N(P)) + \dim N(P) \\ &\leqslant \dim (R(A) + N(P))/((c - A)\, R(P) + (c - A)\, N(P)) + \dim N(P) \\ &= b(c - A) + \dim N(P) < \infty. \end{aligned}$$

$R(c - A_1)$ is closed in $R(P)$. For let $\{x_n\}$ be a sequence in $R(P)$. Suppose $(c - A_1)x_n \to y$ in $R(P)$ as $n \to \infty$. Now $R(c - A)$ is closed in X. Thus $y = (c - A)(x + z)$ with x in $R(P)$ and z in $N(P)$. Now $(c - A)x$ is a member of $R(P)$ and $(c - A)z$ is a member of $N(P)$. Thus $y - (c - A)x = 0$. Therefore $y = (c - A)x$. Thus y is a member of $R(c - A_1)$, and $R(c - A_1)$ is closed in $R(P)$. Hence $c - A_1$ is a member of $\Phi(R(P))$ and c is not a member of $e_3(A_1)$. The equality $e_3(A) = e_3(A_1)$ is thus established. Recall that $\sigma(A_1) = e_3(A)$; clearly $\sigma(A_1) = e_3(A_1)$ and it is an immediate consequence of the above theorem that $f(e_3(A_1)) = e_3(f(A_1))$. Thus $f(e_3(A)) = e_3(f(A_1))$.

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