JOURNAL OF COMBINATORIAL THEORY, Series A 30, 326–330 (1981)

Note

On the Number of Disjoint Mendelsohn Triple Systems*

C. C. LINDNER

Department of Mathematics, Auburn University, Auburn, Alabama 36830

Communicated by the Managing Editors

Received May 9, 1980

1. INTRODUCTION

In what follows an ordered pair will *always* be an ordered pair (x, y), where $x \neq y$. A *cyclic triple* is a collection t of three ordered pairs such that an element occurs as a first coordinate of an ordered pair in t if and only if it occurs as a second coordinate of an ordered pair in t. We will denote the cyclic triple $\{(a, b), (b, c), (c, a)\}$ by (a, b, c), (b, c, a), or (c, a, b). A *Mendelsohn triple system* (MTS) is a pair (S, T) where S is a set containing v elements and T is a collection of cyclic triples of elements of S such that every ordered pair of distinct elements of S belongs to *exactly one* cyclic triple of T. The number $|S| = v$ is called the *order* of the MTS (S, T) and in 1971 N. S. Mendelsohn proved that the spectrum for MTSs is the set of all $v \equiv 0$ or $1 \pmod 3$, *except* $v = 6$ [4]. Mendelsohn himself called such systems *cyclic triple systems*. This vernacular, however, can be a bit confusing since Steiner triple systems admitting a cyclic automorphism (see Peltesohn [5]) are also called cyclic triple systems. The terminology "Mendelsohn triple system" is due to Mathon and Rosa [3]. It is well taken since it not only eliminates some ambiguity but recognizes, as well, the fact that Mendelsohn was the first to determine the spectrum for such systems. (It is well known (of course) that a MTS is equivalent to a quasigroup satisfying the identities $x^2 = x$ and $x(yx) = y$. However, in what follows, we will use design vernacular exclusively.)

For example, the pairs (S, T) and (Q, B) defined as follows are MTSs:

$$S = \{1, 2, 3\}, \qquad T = \{(1, 2, 3), (2, 1, 3)\};$$

and

$$Q = \{1, 2, 3, 4\}, \qquad B = \{(1, 2, 3), (1, 3, 4), (1, 4, 2), (2, 4, 3)\}.$$

* Research supported by NSF Grant MCS 77-03464 A01.

0097-3165/81/030326-05$02.00/0

It is a trivial exercise to see that if (S, T) is a MTS of order v that $|T| = v(v-1)/3$. Now, if S is a set of size v and $C(S)$ is the set of all cyclic triples of S, then $|C(S)| = v(v-1)(v-2)/3$. In view of these last remarks, the following problem arises quite naturally: Given a set S of size $v \equiv 0$ or $1 \pmod 3$, is it always possible to partition $C(S)$ (the set of all cyclic triples of S) into $v-2$ subsets $T_1, T_2, \ldots, T_{v-2}$ so that each of (S, T_1), $(S, T_2), \ldots, (S, T_{v-2})$ is a MTS? Such a collection of MTSs is called a *large set* of pairwise disjoint MTSs. (Two MTSs (S, T_1) and (S, T_2) are said to be *disjoint* provided that $T_1 \cap T_2 = \varnothing$.) Barring the existence of a large set of pairwise disjoint MTSs of order v, we can ask for the largest positive integer $D(v)$ for which $D(v)$ pairwise disjoint MTSs of order v exist. This paper is the first attack on this problem. In particular we prove that $D(3v) \geqslant 2v + D(v)$ (except possibly for $3v = 18$) and $D(3v+1) \geqslant 2v + D(v+1)$ for all $v \geqslant 3$, where, of course, $D(v)$ and $D(v+1)$ are zero whenever v or $v+1$ is $\not\equiv 0$ or $1 \pmod 3$.

It is worth remarking that an extensive amount of work has been done on the similar problem for Steiner triple systems; i.e., the construction of large sets of pairwise disjoint Steiner triple systems. (See, for example, [1, 2, 6, 8, 9, 10].) This problem remains far from settled however, and the reader is referred to the excellent survey article by Rosa [7] for an up-to-date account of constructing large sets of pairwise disjoint Steiner triple systems.

2. $D(3v) \geqslant 2v + D(v)$

Let $v \geqslant 3$, $v \neq 6$, and let Q be a latin square of order v having v disjoint transversals $T_1, T_2, \ldots, T_v$ (equivalent to a pair of orthogonal latin squares). We write

$$T_k = \begin{pmatrix} 1 & 2 & 3 & \cdots & v \\ x_{1k} & x_{2k} & x_{3k} & \cdots & x_{vk} \\ y_{1k} & y_{2k} & y_{3k} & \cdots & y_{vk} \end{pmatrix}$$

to indicate that the cell (i, x_{ik}) belongs to T_k *and* is occupied by y_{ik}. Now let $\alpha = (1\ 2\ 3 \cdots v)$ and for each $k = 1, 2, \ldots, v$ define six permutations $\alpha_k, \beta_k, \gamma_k, \alpha_k^*, \beta_k^*$, and γ_k^* on Q by

$$\alpha_k = \begin{pmatrix} 1 & 2 & 3 & \cdots & v \\ x_{1k} & x_{2k} & x_{3k} & \cdots & x_{vk} \end{pmatrix},$$

$$\alpha_k^* = \begin{pmatrix} 1 & 2 & 3 & \cdots & v \\ y_{1k}^* & y_{2k}^* & y_{3k}^* & \cdots & y_{vk}^* \end{pmatrix},$$

$$\beta_k = \begin{pmatrix} x_{1k} & x_{2k} & x_{3k} & \cdots & x_{vk} \\ y_{1k} & y_{2k} & y_{3k} & \cdots & y_{vk} \end{pmatrix},$$

$$\beta_k^* = \begin{pmatrix} y_{1k}^* & y_{2k}^* & y_{3k}^* & \cdots & y_{vk}^* \\ x_{1k} & x_{2k} & x_{3k} & \cdots & x_{vk} \end{pmatrix},$$

$$\gamma_k = \begin{pmatrix} y_{1k} & y_{2k} & y_{3k} & \cdots & y_{vk} \\ 1 & 2 & 3 & \cdots & v \end{pmatrix},$$

$$\gamma_k^* = \begin{pmatrix} x_{1k} & x_{2k} & x_{3k} & \cdots & x_{vk} \\ 1 & 2 & 3 & \cdots & v \end{pmatrix},$$

where $y_{ik}^* = y_{ik}\alpha$.

Set $S = Q \times \{1, 2, 3\}$, let $(Q, \circ)$ be any idempotent quasigroup (not necessarily related in any way to the latin square Q), and define $2v + D(v)$ MTSs as follows:

(1) For each $k = 1, 2, \ldots, v$, define a collection of cyclic triples t_k by

(i) $((i, 1), (x_{ik}, 2), (y_{ik}, 3))$ and $((x_{ik}, 2), (i, 1), (y_{ik}, 3))$ belong to t_k for every $i \in Q$; and

(ii) if $i \neq j \in Q$, the six cyclic triples $((i, 1), (j, 1), ((i \circ j)\alpha_k, 2))$, $((j, 1), (i, 1), ((j \circ i)\alpha_k, 2))$, $((i, 2), (j, 2), ((i \circ j)\beta_k, 3))$, $((j, 2), (i, 2), ((j \circ i)\beta_k, 3))$, $((i, 3), (j, 3), ((i \circ j)\gamma_k, 1))$ and $((j, 3), (i, 3), ((j \circ i)\gamma_k, 1))$ belong to t_k.

(2) For each $k = 1, 2, \ldots, v$, define a collection of cyclic triples t_k^* by

(i) $((i, 1), (x_{ik}, 2), (y_{ik}^*, 3))$ and $((x_{ik}, 2), (i, 1), (y_{ik}^*, 3))$ belong to t_k^* for every $i \in Q$; and

(ii) if $i \neq j \in Q$, the six cyclic triples $((i, 1), (j, 1), ((i \circ j)\alpha_k^*, 3))$, $((j, 1), (i, 1), ((j \circ i)\alpha_k^*, 3))$, $((i, 3), (j, 3), ((i \circ j)\beta_k^*, 2))$, $((j, 3), (i, 3), ((j \circ i)\beta_k^*, 2))$, $((i, 2), (j, 2), ((i \circ j)\gamma_k^*, 1))$ and $((j, 2), (i, 2), ((j \circ i)\gamma_k^*, 1))$ belong to t_k^*.

(3) Let (Q, q_1), $(Q, q_2), \ldots, (Q, q_u)$ be any collection of $u = D(v)$ pairwise disjoint MTSs and for each $k = 1, 2, \ldots, u$, define a collection of cyclic triples d_k by

(i) For each $i = 1, 2, 3$, $((x, i), (y, i), (z, i)) \in d_k$ if and only if $(x, y, z) \in q_k$; and

(ii) if $i \neq j \in Q$, $((i, 1), (j, 2), ((i \circ j)\alpha^{k+1}, 3))$ and $((j, 2), (i, 1), ((i \circ j)\alpha^{k+1}, 3)) \in d_k$.

It is straightforward to see that the $2v + D(v)$ MTSs constructed in (1), (2), and (3) are pairwise disjoint giving the following theorem.

THEOREM 2.1. *$D(3v) \geqslant 2v + D(v)$ for all $v \geqslant 3$, except possibly $v = 6$.*

3. $D(3v+1) \geqslant 2v + D(v+1)$

Let $F = \{\infty, 1, 2, 3\}$ and denote by f_1 and f_2 the collections of cyclic triples given by

$$f_1 = \{(\infty, 1, 2), (\infty, 2, 3), (\infty, 3, 1), (1, 3, 2)\},$$

and

$$f_2 = \{(\infty, 1, 3), (\infty, 2, 1), (\infty, 3, 2), (1, 2, 3)\}.$$

Then, of course, (F, f_1) and (F, f_2) are *disjoint* MTSs of order 4. Now let Q be as in Section 2, set $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$, and define $2v + D(v+1)$ MTSs of order $3v+1$ by modifying the constructions (1), (2), and (3) in Section 2 as follows:

(1′) For each $k = 1, 2, \ldots, v$, define a collection of cyclic triples t'_k by replacing (i) in (1) with: $(\infty, (i, 1), (x_{ik}, 2))$, $(\infty, (x_{ik}, 2), (y_{ik}, 3))$, $(\infty, (y_{ik}, 3), (i, 1))$, and $((i, 1), (y_{ik}, 3), (x_{ik}, 2))$ belong to t'_k for every $i \in Q$. (*Note* that these four cyclic triples are a copy of (F, f_1).)

(2′) For each $k = 1, 2, \ldots, v$, define a collection of cyclic triples $t^{*\prime}_k$ by replacing (i) in (2) with: $(\infty, (i, 1), (y^*_{ik}, 3))$, $(\infty, (x_{ik}, 2), (i, 1))$, $(\infty, (y^*_{ik}, 3), (x_{ik}, 2))$, and $((i, 1), (x_{ik}, 2), (y^*_{ik}, 3))$ belong to $t^{*\prime}_k$ for every $i \in Q$. (*Note* that these four cyclic triples are a copy of (F, f_2).)

(3′) Let $(\{\infty\} \cup Q, q_1)$, $(\{\infty\} \cup Q, q_2), \ldots, (\{\infty\} \cup Q, q_u)$ be any collection of $u = D(v+1)$ pairwise disjoint MTSs of order $v+1$ and for each $k = 1, 2, \ldots, u$, define a collection of cyclic triples d'_k by

(i′) For each $i = 1, 2, 3$, define a copy of q_k on $\{\infty\} \cup (Q \times \{i\})$ and place these cyclic triples in d'_k;

(ii′) if $i \neq j \in Q$ *and* $2 \leqslant k \leqslant v-2$, $((i, 1), (j, 2), ((i \circ j)\,\alpha^{k+1}, 3))$ and $((j, 2), (i, 1), ((i \circ j)\,\alpha^{k+1}, 3)) \in d'_k$; and

(iii′) If $i \neq j$ *and* $k = v-1$, set $T = \{((x, 1), (y, 2), (z, 3)) \mid \text{all } x, y, z \in Q\}$ and place the cyclic triples $T \backslash (t'_1 \cup t'_2 \cup \cdots \cup t'_v \cup t^{*\prime}_1 \cup t^{*\prime}_2 \cup \cdots \cup t^{*\prime}_v \cup d'_1 \cup d'_2 \cup \cdots \cup d'_{v-2})$ in d'_{v-1}.

There is no difficulty in showing that the $2v + D(v+1)$ MTSs constructed in (1′), (2′), and (3′) are pairwise disjoint MTSs.

THEOREM 3.1. *$D(3v+1) \geqslant 2v + D(v+1)$ for all $v \geqslant 3$.*

Proof. The case where $3v+1 = 19$ is handled as follows. In [1] Denniston has shown the existence of a large set of pairwise disjoint Steiner triple systems of order 19 (i.e., 17 pairwise disjoint Steiner triple systems of order 19). In each Steiner triple system of this collection replace each triple

$\{a, b, c\}$ with the two cyclic triples (a, b, c) and (b, a, c). This, of course, gives a large set of pairwise disjoint MTSs of order 19.

Combining Theorems 2.1 and 3.1 gives the following result.

THEOREM 3.2. *$D(u) \geqslant [2u/3]$ for all $u \equiv 0$ or $1 \pmod 3$ and $u \geqslant 9$, except possibly $u = 18$.*

4. REMARKS

Trivially $D(3) = 1$ and $D(4) = 2$, and C. Colbourn (unpublished) has shown that $D(7) = 5$. Beginning with $u = 9$, Theorems 2.1 and 3.1 coupled with the fact that a large set of pairwise disjoint Steiner triple systems of order u exists for all $u \equiv 1$ or $3 \pmod 6$ such that $9 \leqslant u \leqslant 99$ (except possibly $u = 37$, 85, and 97) give a large set of pairwise disjoint MTSs of *every* order $u \equiv 0$ or $1 \pmod 3$ such that $9 \leqslant u \leqslant 100$, except possibly for $u = 16$, 18, 22, 24, 40, 42, 46, 48, 52, 54, 58, 60, 64, 66, 70, 72, 76, 78, 85, 94, and 96. *Note* that Theorems 2.1 and 3.1 produce large sets of pairwise disjoint MTSs of orders 37 and 97 even though the similar problem for Steiner triple systems is unsettled.

REFERENCES

1. R. H. F. DENNISTON, Some packings with Steiner triple systems, *Discrete Math.* **9** (1974), 213–227.
2. J. DOYEN, Constructions of disjoint Steiner triple systems, *Proc. Amer. Math. Soc.* **32** (1972), 409–416.
3. R. MATHON AND A. ROSA, A census of Mendelsohn triple systems of order nine, *Ars Combin.* **4** (1977), 309–315.
4. N. S. MENDELSOHN, A natural generalization of Steiner triple systems, *in* "Computers in Number Theory," pp. 323–338, Academic Press, New York, 1971.
5. R. PELTESOHN, Eine Lösung der beiden Heffterschen Differenzenprobleme, *Compositio Math.* **6** (1939), 251–257.
6. A. ROSA, A theorem on the maximum number of disjoint Steiner triple systems, *J. Combin. Theory Ser. A* **18** (1975), 305–312.
7. A. ROSA, Intersection properties of Steiner systems, *Ann. Discrete Math.* **7** (1980), 115–128.
8. S. SCHREIBER, Covering all triples on n marks by disjoint Steiner systems, *J. Combin. Theory Ser. A* **15** (1973), 347–350.
9. L. TEIRLINCK, On the maximum number of disjoint Steiner triple systems, *Discrete Math.* **9** (1973), 299–300.
10. R. M. WILSON, Some partitions of all triples into Steiner triple systems, *in* "Hypergraph Seminar (Ohio State University, 1972)," pp. 267–277, Lecture Notes in Math. No. 411, Springer-Verlag, Berlin, 1974.