Differential Subordinations and Inequalities in the Complex Plane

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Let \( p \) be analytic in the unit disc \( U \) and let \( q \) be univalent in \( U \). In addition, let \( \Omega \) be a set in \( \mathbb{C} \) and let \( h: \mathbb{C} \times U \rightarrow \mathbb{C} \). The authors determine conditions on \( \psi \) so that
\[
\{ \psi(p(z), zp'(z), z^2p''(z); z) \mid z \in U \} \subset \Omega \Rightarrow p(U) \subset q(U).
\]
Applications of this result to differential inequalities, differential subordinations and integral inequalities are presented. © 1987 Academic Press, Inc.

1. Introduction

Let \( f \) and \( F \) be analytic in the unit disc \( U \). The function \( f \) is subordinate to \( F \), written \( f \prec F \) or \( f(z) \prec F(z) \), if \( F \) is univalent, \( f(0) = F(0) \) and \( f(U) \subset F(U) \).

In two previous papers [3, 4] the authors dealt with second order differential subordinations of the form
\[
\psi(p(z), zp'(z), z^2p''(z)) < h(z),
\]
where \( \psi \) is holomorphic on a domain in \( \mathbb{C}^3 \). They found dominants \( q \) of (1), for which \( p \prec q \) for all \( p \) satisfying (1). One of the objects of this paper is to obtain dominants for a more general second order differential subordination of the form
\[
\psi(p(z), zp'(z), z^2p''(z); z) < h(z),
\]
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where \( \psi: C^3 \times U \rightarrow C \). In these differential subordinations we allow functions of \( z \) to be present in addition to the terms \( p(z), zp'(z) \) and \( z^2p''(z) \). This is analogous to generalizing autonomous differential equations to nonautonomous differential equations.

For example, if we let \( B(z) \) be a function defined on \( U \) with \( \Re B(z) \geq 0 \), for \( z \in U \), then a recent paper [5, Lemma 1] proves the following useful lemma: if \( p \) is analytic in \( U \) then

\[
\Re[B(z)zp'(z) + p(z)] > 0, \quad \text{for } z \in U,
\]

\[
\Rightarrow \Re p(z) > 0, \quad \text{for } z \in U.
\]

If we let \( \psi(r, s; z) = B(z)s + r, \ h(z) = (1 + z)/(1 - z) \) and suppose \( B \) is analytic in \( U \) then (3) becomes

\[
\psi(p(z), zp'(z); z) = B(z)zp'(z) + p(z) < (1 + z)/(1 - z) = p(z) < (1 + z)/(1 - z).
\]

Thus, the first part of (3) can be written as a differential subordination of the form (2), and the second part of (3) provides a dominant of this differential subordination.

In addition to finding dominants for (2) we can weaken the holomorphicity condition needed in (2) and prove a more general result: if \( \Omega \) is a set in \( C \), \( q \) is univalent on \( U \) and \( q(U) \) has a “nice boundary,” there exists a class of functions \( \mathcal{D} \), dependent on \( \Omega \) and \( q \), for which

\[
\{ \psi(p(z), zp'(z), z^2p''(z); z) \mid z \in U \} \subset \Omega \Rightarrow p(U) \subset q(U).
\]

Result (3) with \( \Omega = \{ w \mid \Re w > 0 \} \), \( q(z) = (1 + z)/(1 - z) \), and \( \psi(r, s; t; z) = B(z)s + t \) is a special case of (5).

The definition of the class \( \mathcal{D} \) and the fundamental result (5) together with its variations are given in Section 2. In Section 3 we apply the basic result to bounded functions, while in Section 4 we apply the result to functions with a positive real part. Section 5 is concerned with some integral inequalities obtained from the previous sections.

All of the inequalities in this article involving functions of \( z \), such as (3), hold uniformly in the unit disc \( U \). The condition “for all \( z \) in \( U \)” will be omitted in the remainder of this paper, although it is understood to hold.

2. Differential Inequalities and Subordinations

We first need to specify the univalent functions \( q \) (with “nice boundary”) and functions \( \psi \) for which we intend to prove (5).
DEFINITION 1. Let $\Omega$ be set in $C$ and let $q$ be analytic and univalent on $\bar{U}$ except for those $\zeta \in \partial U$ for which $\lim_{z \to \zeta} q(z) = \infty$. We define $\Psi[\Omega, q]$ to be the class of functions $\psi: C^3 \times U \to \bar{C}$ for which

$$\psi(r, s, t; z) \notin \Omega$$

when $r = q(\zeta)$ is finite, $s = m \bar{q}'(\zeta)$, $\Re(1 + t/s) \geq m \Re(1 + \bar{q}''(\zeta)/q'(\zeta))$, and $z \in U$, for $m \geq 1$ and $|\zeta| = 1$.

In the special case where $\Omega$ is a simply connected domain and $h$ is a conformal mapping of $U$ onto $\Omega$ we denote the class by $\Psi[h(U), q]$ or $\Psi[h, q]$.

Note that if $Q_1 \subseteq Q_2$ then $\Psi(Q_2, q) \subseteq \Psi(Q_1, q)$, that is enlarging $\Omega$ decreases the class $\Psi(\Omega, q)$.

For the function $q$ required in the definition of $\Psi[\Omega, q]$, the domain $q(U)$ is simply connected and its boundary consists of either a simple closed analytic curve or the union (possibly infinite) of pairwise disjoint simple analytic curves which converge to $\infty$ in both directions. The function $q(\zeta) = (1 + \zeta)/(1 - \zeta)$ is an example of such a function. The set $\Omega$ need not be a domain nor need its boundary be nicely behaved.

LEMMA 1. [4, p. 158]. Let $q$ be analytic and univalent on $\bar{U}$ except for those points $\zeta \in \partial U$ for which $\lim_{z \to \zeta} q(z) = \infty$. Let $p$ be analytic in $U$ with $p(0) = q(0)$. If $p$ is not subordinate to $q$ then there exist points $z_0 \in U$ and $\zeta_0 \in \partial U$, and an $m \geq 1$ for which

(a) $p(|z| < |z_0|) \subset q(U),$
(b) $p(z_0) = q(\zeta_0),$
(c) $z_0 p'(z_0) = m \bar{q}'(\zeta_0)$ and
(d) $\Re[z_0 p''(z_0)/p'(z_0)^2] \geq m \Re[\bar{q}''(\zeta_0)/q'(\zeta_0) + 1].$

We are now prepared to state and prove our main theorem.

THEOREM 1. Let $\psi \in \Psi[\Omega, q]$, as given in Definition 1. If $p$ is analytic in $U$, with $p(0) = q(0)$, and if $p$ satisfies

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega, \quad z \in U, \quad (6)$$

then $p < q$.

Proof: Assume that $p$ is not subordinate to $q$. By Lemma 1 there exist points $z_0 \in U$ and $\zeta_0 \in \partial U$ that satisfy (a)-(d). Using these conditions with $r = p(z_0)$, $s = z_0 p'(z_0)$, $t = z_0^2 p''(z_0)$ and $z = z_0$ in Definition 1 we obtain

$$\psi(p(z_0), z_0 p'(z_0), z_0^2 p''(z_0); z_0) \notin \Omega.$$ 

Since this contradicts (6) we must have $p < q$. 

Remark 1. On checking the proof of this theorem and Definition 1, we note that if $0 < q < 1$ and if all of the conditions of the theorem hold with the exception of (6) being replaced by
\[ \psi(p(z), zp'(z), z^2p''(z); \eta z) \in \Omega, \quad \text{when} \; z \in U, \quad (6') \]
then we obtain the same conclusion, $p < q$. In fact, the function $\eta z$ in (6') can be replaced by any function $w(z)$ mapping $U$ onto $U$.

Remark 2. For a given $\Psi[\Omega, q]$ there may not exist an analytic function $p$ satisfying (6). As an example, let $\Omega$ be the right-half plane, $q(z) = (1 + z)/(1 - z)$ and $(r, s, t; z) = -r^2s$. A simple computation shows that $r = q(\zeta) = r_2i$ ($r_2$ real) and $s = -(1 + r_2^2)/2 = s_1 < 0$, when $|\zeta| = 1$ and $m \geq 1$. Hence $\psi(r, s) = r_2^2s_1 < 0$, and by Definition 1 $\psi \in \Psi[\Omega, q]$. In this case (6) becomes
\[ \Re[-(p(z))^2zp'(z)] > 0. \]
However, there is no analytic function $p$ that satisfies this inequality at $z = 0$.

Remark 3. Theorem 1 is an improvement of a previous result [3, Theorem 1] of the authors. In that result $\Omega$ was required to be a domain, the set of functions $\Psi[\Omega, q]$ was restricted to those $\phi(r, s, t)$ that were independent of $z$, and $\phi$ was required to be continuous in its domain and to satisfy $\phi(q(0), 0, 0) \in \Omega$. (Note that these conditions imply the existence of a function $p$ satisfying (6).) Examples which could not be handled with the previous result, but which can now be handled, will be presented in Section 3 (Theorem 6) and in Section 4 (Theorem 7).

The definition of $\Psi[\Omega, q]$ requires that the function $q$ behave very nicely on $\partial U$. If this is not the case, or if the behavior of $q$ on $\partial U$ is unknown, it may still be possible to prove $p < q$ by the following limiting procedure.

**Corollary 1.1.** Let $0 < \rho_0 < 1$, let $q$ be univalent in $U$, let $q(\rho z) = q(\rho z)$ for $\rho_0 < \rho < 1$, and suppose that
\[ \psi(r, s, t; z) \in \Psi[\Omega, q_\rho], \quad (7) \]
for all $\rho_0 < \rho < 1$. If $p$ is analytic in $U$, with $p(0) = q(0)$, and if $p$ satisfies $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ when $z \in U$, then $p < q$.

**Proof.** The function $q_\rho$, is univalent on $U$ and hence $\Psi[\Omega, q_\rho]$ is well defined. From (7) and Theorem 1 we obtain $p(z) < q(\rho z)$ for $0 < \rho < 1$. Now letting $\rho \to 1^-$ we obtain $p(z) < q(z)$.

We next consider the subclasses of $\Psi[\Omega, q]$ for which $\Omega$ is a simply connected domain and $\psi(p(z), zp'(z), z^2p''(z); z)$ is an analytic function of $z$. In
this case \( \Omega = h(U) \), where \( h \) is a conformal mapping of \( U \) onto \( \Omega \). The following result is an immediate consequence of Theorem 1.

**Corollary 1.2.** Let \( \psi \in \Psi[h(U), q] \) and let \( p \) and \( \psi(p(z), z'p(z), z^2p''(z); z) \) be analytic in \( U \) with \( p(0) = q(0) \). If \( p \) satisfies
\[
\psi(\rho z), \rho z^p' (\rho z), (\rho z)^2 p''(\rho z); z) < h(\rho z),
\]
then \( p < q \).

An analogue of Corollary 1.1 for \( \Psi[h(U), q] \) can also be given.

**Corollary 1.3.** Let \( 0 < p_0 < 1 \), let \( h \) and \( q \) be univalent in \( U \), and let \( h_\rho(z) = h(\rho z), q_\rho(z) = q(\rho z) \) for \( \rho_0 < \rho < 1 \). Suppose that \( \psi \in \Psi[h_\rho, q_\rho] \) for all \( \rho_0 < \rho < 1 \) and suppose \( p \) and \( \psi(p, zp', z^2p''; z) \) are analytic in \( U \) with \( p(0) = q(0) \). If \( p \) satisfies (8) then \( p < q \).

**Proof.** If \( \rho_0 < \rho < 1 \), then from (8) we obtain
\[
\psi(p(\rho z), \rho z p'(\rho z), (\rho z)^2 p''(\rho z); \rho z) < h(\rho z),
\]
which, by setting \( p_\rho(z) = p(\rho z) \), becomes
\[
\psi(p_\rho(z), z p_\rho'(z), z^2 p_\rho''(z); \rho z) \in h_\rho(U), \text{ for } z \in U.
\]
Since \( \psi \in \Psi[h_\rho(U), q_\rho] \), by using Remark 1 with \( \rho = \eta \), \( h_\rho(U) = \Omega \) and \( p_\rho = p \) we obtain \( p_\rho(z) < h_\rho(z) \). Hence \( p(\rho z) < h(\rho z) \) and by letting \( \rho \to 1^- \) we obtain \( p(z) < q(z) \).

We can apply this last corollary to obtain the following result concerning a linear second order differential subordination.

**Theorem 2.** Let \( h \) be convex in \( U \) with \( h(0) = 0 \), and let \( A \geq 0 \). Suppose that \( B(z) \) and \( C(z) \) are analytic in \( U \) and satisfy
\[
\text{Re}[B(z)] \geq A + |C(z) - 1| - \text{Re}[C(z) - 1],
\]
for \( z \in U \). If \( p \) is analytic in \( U \) with \( p(0) = 0 \), and if \( p \) satisfies
\[
A z^2 p''(z) + B(z) z p'(z) + C(z) p(z) < h(z),
\]
then \( p < h \).

**Proof.** If we let \( \psi(r, s, t; z) = A t + B(z) s + C(z) r \) then \( \psi(p(z), z p'(z), z^2 p''(z); z) \) is analytic in \( U \) and (10) becomes
\[
\psi(\rho z), \rho z^p' (\rho z), (\rho z)^2 p''(\rho z); z) < h(\rho z).
\]
The conclusion of the theorem will follow from Corollary 1.3 if we show that \( \psi \in \Psi[h_\rho, h_\rho] \) for \( 1/2 < \rho < 1 \). According to Definition 1 we only need to show that
\[
\psi_0 = \psi(r, s, t; z) \notin h_\rho(U)
\]
when \( r = h_\rho(\zeta) \), \( s = m\zeta h_\rho'(\zeta) \), \( \text{Re}(1 + i/s) \geq 0 \) and \( z \in U \), for \( |\zeta| = 1 \) and \( m \geq 1 \).

If we set \( \lambda = (\psi_0 - h_\rho(\zeta))/\zeta h_\rho'(\zeta) \) then
\[
\lambda = A\text{mt}/s + mB(z) + (C(z) - 1) h_\rho(\zeta)/\zeta h_\rho'(\zeta)
\]
and
\[
0 = h_\rho(\zeta) + mC(z). \tag{12}
\]

We first show that \( \text{Re}\lambda > 0 \). Since \( h_\rho \) is convex and \( h_\rho(0) = 0 \) we have \( \text{Re}\zeta h_\rho'(\zeta)/h_\rho(\zeta) \geq 1/2 \) for \( |\zeta| = 1 \) \[1, p. 176\], or equivalently
\[
|h_\rho(\zeta)/\zeta h_\rho'(\zeta) - 1| \leq 1. \tag{13}
\]

If \( W \) and \( Z \) are complex numbers and \( |Z - 1| \leq 1 \), then
\[
\text{Re} WZ = \text{Re} W + \text{Re} W(Z - 1) \geq \text{Re} W - |W|.
\]

Using this result with \( W = C(z) - 1 \) and \( Z = h_\rho(\zeta)/\zeta h_\rho'(\zeta) \), from (13) we obtain
\[
\text{Re}[(C(z) - 1) h_\rho(\zeta)/\zeta h_\rho'(\zeta)] \geq \text{Re}(C(z) - 1) - |C(z) - 1|.
\]

Using (9) and this last inequality in (11) we obtain
\[
\text{Re} \lambda \geq (m - 1)[|C(z) - 1| - \text{Re}(C(z) - 1)].
\]

Since \( m \geq 1 \) we obtain \( \text{Re} \lambda \geq 0 \), or equivalently \( |\arg \lambda| \leq \pi/2 \). Applying this in (12) together with the fact that \( h_\rho(U) \) is a convex domain and \( \zeta h_\rho'(\zeta) \) is an outward normal to the boundary of \( h(U) \) we obtain \( \psi_0 \notin h_\rho(U) \), which completes the proof of the theorem.

Note that for \( C(z) = 1 \) the conditions \( p(0) = h(0) = 0 \) are not necessary. In this case we have the following result which is a generalization of (4).

**Corollary 2.1.** Let \( h \) be convex in \( U \) and let \( A \geq 0 \). Suppose \( B(z) \) is analytic in \( U \) with \( \text{Re} B(z) \geq A \). If \( p \) is analytic in \( U \) and \( p(0) = h(0) \) then
\[
Az^2p''(z) + B(z)zp'(z) + p(z) < h(z) \Rightarrow p(z) < h(z).
\]

The case \( A = 0 \) and \( h(z) = (1 + z)/(1 - z) \) corresponds to (4).

As mentioned in the Introduction, the univalent function \( q \) is said to be a
dominant of the differential subordination (8) if \( p < q \) for all \( p \) satisfying (8). If, furthermore, \( \tilde{q} \) is a dominant of (8) and \( \tilde{q} < q \) for all dominants \( q \) of (8), then \( \tilde{q} \) is said to be the best dominant of (8).

From Corollaries 1.2 and 1.3 we see that \( q \) will be a dominant of (8) if \( \psi \in \mathcal{Y}[h(U), q] \) or \( \psi \in \mathcal{Y}[h_{\rho}(U), q_{\rho}] \). If \( q \) is a dominant of (8) and \( q \) also satisfies (8) then \( q \) will be the best dominant. This gives us the following theorem for obtaining the best dominant of (8). The proof follows immediately from Corollaries 1.2 and 1.3, and is omitted.

**Theorem 3.** Let \( h \) be univalent in \( U, \psi : C^{3} \times U \to C \), and suppose that the differential equation

\[
\psi(q(z), zq'(z), z^{2}q''(z); z) = h(z)
\]

has a solution \( q \) that satisfies either:

(i) \( q \) is univalent on \( \overline{U} \) except for those points \( \zeta \in \partial U \) for which

\[
\lim_{z \to \zeta} q(z) = \infty, \quad \text{and} \quad \psi \in \mathcal{Y}[h(U), q],
\]

or

(ii) \( q \) is univalent in \( U \) and \( \psi \in \mathcal{Y}[h_{\rho}(U), q_{\rho}] \) for \( \rho_{0} < \rho < 1 \).

If \( p \) is analytic in \( U \) with \( p(0) = q(0) \), if \( \psi(p(z), zp'(z), z^{2}p''(z); z) \) is analytic in \( U \) and if \( p \) satisfies (8), then \( p < q \) and \( q \) is the best dominant.

3. **Bounded Dominants**

In this section we consider several interesting differential inequalities and subordinations obtained by selecting functions in the class \( \mathcal{Y}[\Omega, z] \).

Substituting \( \Omega = U \) and \( q(z) = z \) in Definition 1 we see that \( q(\zeta) = e^{i\theta} \), \( \zeta q'(\zeta) = e^{i\theta} \) and \( \text{Re}[1 + \zeta q''(\zeta)/q'(\zeta)] = 1 \). Hence the class \( \mathcal{Y}[U, z] \) consists of those \( \psi : C^{3} \times U \to C \) that satisfy

\[
|\psi(e^{i\theta}, me^{i\theta}, t; z)| \geq 1,
\]

when \( m \geq 1 \), \( \text{Re}[te^{-i\theta}] \geq m(m-1) \), and \( z \in U \).

**Theorem 4.** Let \( A \geq 0 \) and let \( B(z) \) be defined on \( U \) with \( \text{Re} B(z) \geq -A \) for \( z \in U \). If \( p \) is analytic in \( U \), \( p(0) = 0 \) and

\[
|Az^{2}p''(z) + B(z) \cdot zp'(z) + [1 - B(z)] \cdot p(z)| < 1,
\]

then \( p(z) < z \).

**Proof.** If we let \( \psi(r, s, t; z) = At + B(z) \cdot s + [1 - B(z)] \cdot r \), then (15) can be rewritten as \( \psi(p(z), zp'(z), z^{2}p''(z); z) \in U \), for \( z \in U \). We obtain our con-
clusion from Theorem 1 if we show that \( \psi \in \mathcal{P}[U, z] \), or equivalently if we show that \( \psi \) satisfies (14). For \( m \geq 1 \), \( \text{Re}[te^{-it}] \geq m(m-1) \) and \( z \in U \) we obtain

\[
|\psi(e^{it}, me^{it}; z)| = |At + me^{it}B(z) + e^{it}[1 - B(z)]| \\
\geq \text{Re}[Ate^{-it} + 1 + (m-1)B(z)] \geq Am(m-1) + 1 + (m-1)(-A) \\
= 1 + A(m-1)^2 \geq 1.
\]

Hence \( \psi \in \mathcal{P}[U, z] \) and we conclude that \( p(z) < z \).

If the expression in (15) is also analytic then

\[
A \cdot z^2 p''(z) + B(z) zp'(z) + [1 - E(z)] p(z) < z
\]

implies \( p(z) < z \), and \( q(z) = z \) is the best dominant.

**Theorem 5.** Let \( B(z) \) and \( C(z) \) be functions defined on \( U \) with \( B(z) \neq 0 \), and for each \( z \in U \) suppose that at least one of the following conditions is satisfied:

(i) \( |B(z) + C(z)| \geq 1 \) and \( \text{Re}[C(z)/B(z)] \geq -1 \), or

(ii) \( |\text{Im}[C(z)/B(z)]| \geq 1/|B(z)| \).

If \( p \) is analytic in \( U \), with \( p(0) = 0 \), and if

\[
|B(z) zp'(z) + C(z) p(z) | < 1, \tag{16}
\]

then \( |p(z)| < 1 \).

**Proof.** Letting \( \psi(r, s; z) = B(z) \cdot s + C(z) \cdot r \), we obtain our conclusion from Theorem 1 by showing that (14) is satisfied. For this particular \( \psi \), condition (14) reduces to

\[
|\psi(e^{it}, me^{it}; z)| = |mB(z) + C(z)| \geq 1,
\]

when \( m \geq 1 \) and \( z \in U \). This is equivalent to showing that

\[
L(m) \equiv m^2 |B|^2 + 2m \text{Re}[\overline{B}C] + |C|^2 - 1 \geq 0, \tag{17}
\]

when \( m \geq 1 \) and \( z \in U \). The conditions in (i) imply that \( L(1) \geq 0 \) and \( L'(m) \geq 0 \) for \( m \geq 1 \), which subsequently implies (17). The condition in (ii) implies that the discriminant of \( L(m) \) is nonpositive and since \( |B| > 0 \) we conclude that (17) is satisfied. Hence \( \psi \in \mathcal{P}[U, z] \), \( p < z \), and \( |p(z)| < 1 \).

The last result of this section provides an example of a \( \psi \in \mathcal{P}[\Omega, q] \) for
which $\Omega$ is not a domain in $C$ and for which $\psi(q(0), 0, 0; z) \in \Omega$ [see Remark 3].

**Theorem 6.** If $p$ is analytic in $U$ with $p(0) = 0$, then

$$|zp'(z)| + |z^2p''(z)/p(z)| < 1 \quad (18)$$

implies that $|p(z)| < 1$.

**Proof.** If we let $\psi(r, s, t; z) = |s| + |t/r|$, then (18) can be rewritten as

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega,$$

for $z \in U$. Even though $\psi$ is not continuous at $r = 0$ and $\psi(0, 0, 0; z) \notin [0, 1)$, there are nontrivial functions $p$ satisfying (18); for example, $p(z) = p_1z + p_2z^2$, with $|p_1|$ and $|p_2|$ sufficiently small. We obtain our conclusion from Theorem 1 if we show that $\psi \in \Psi[[0, 1), z]$. The proof is analogous to that of Theorem 4 and we obtain

$$\psi(e^{i\theta}, me^{i\theta}, t; z) = |me^{i\theta}| + |te^{-i\theta}| \geq m + \text{Re}[te^{-i\theta}] \geq 1,$$

for $m \geq 1$, $\text{Re}[te^{-i\theta}] \geq m(m-1)$ and $z \in U$. Hence $\psi(e^{i\theta}, me^{i\theta}, t; z) \notin [0, 1)$, $\psi \in \Psi[[0, 1), z]$, and by Theorem 1 we have $|p(z)| < 1$.

### 4. Dominants with Positive Real Part

In this section we consider differential inequalities and subordinations obtained by selecting functions in the class $\Psi[q(U), q]$, where $q(z) = (1+z)/(1-z)$. If $|z| = 1$, then $q(\zeta) = r_2 i$ (real), $\zeta q'(\zeta) = -(1+r_2^2)/2$ and $\text{Re}[1 + \zeta q''(\zeta)/q'(\zeta)] = 0$. In this case the set $\Omega = q(U) = \{z | \text{Re} z > 0\}$ will be a domain. By Definition 1 the class $\Psi[q(U), q]$ consists of those functions $\psi: C^3 \times U \rightarrow C$ that satisfy

$$\text{Re} \psi(r_2 i, s_1, t_1 + t_2 i; z) \leq 0,$$

when $r_2$ is real, $s_1 \leq -(1 + r_2^2)/2$, $s_1 + t_1 \leq 0$, and $z \in U$.

**Theorem 7.** If $p$ is analytic in $U$ with $p(0) = 1$, and if

$$\text{Re}[2p(z) - zp''(z)/p'(z) - 1] > 0,$$

then $\text{Re} p(z) > 0$.

**Proof.** If we let $q(z) = (1+z)/(1-z)$ and $\psi(r, s, t; z) = 2r - t/s - 1$ then (20) can be expressed as $\psi(p(z), zp'(z), z^2p''(z); z) \in q(U)$, for $z \in U$. Note
that \( \psi \) is not continuous at \( s = 0 \) and \( \psi(q(0), 0, 0; z) \notin \Omega = q(U) \) [see Remark 3]. Nevertheless, \( p(z) = 1 + p_1 z + p_2 z^2 \), with \( |p_1| \) and \( |p_2| \) sufficiently small is an example of a function satisfying (20). The proof follows since \( \Re \psi(r_2 i, s_1, t_1 + t_2 i; z) = -(t_1/s_1 + 1) \leq 0 \), when \( r_2 \) is real, \( s_1 < 0 \) and \( s_1 + t_1 \leq 0 \). Hence by (19) and Theorem 1, \( p(z) < q(z) = (1 + z)/(1 - z) \) and \( \Re p(z) > 0 \).

Theorem 8. Let \( B(z) \) and \( C(z) \) be functions defined on \( U \), with

\[
|\Im C(z)| \leq \Re B(z). \tag{21}
\]

If \( p \) is analytic in \( U \) with \( p(0) = 1 \), and if

\[
\Re [B(z) \cdot z p'(z) + C(z) \cdot p(z)] > 0, \tag{22}
\]

then \( \Re p(z) > 0 \).

Proof. If we let \( \psi(r, s, z) = B(z) \cdot s + C(z) \cdot r \) and \( q(z) = (1 + z)/(1 - z) \), then \( \psi: C^2 \times U \to C \) and (22) becomes \( \psi(p(z), z p'(z); z) \in q(U) \), for \( z \in U \). If \( r_2 \) is real and \( s_1 \leq -(1 + r_2^2)/2 \) then from (21) we obtain

\[
\Re \psi(r_2 i, s_1, z) = s_1 \Re B(z) - r_2 \Im C(z)
\]

\[
\leq -(1 + r_2^2) \Re B(z)/2 + r_2 |\Im C(z)|
\]

\[
\leq -(1 - r_2^2) \Re B(z)/2 \leq 0.
\]

Hence (19) is satisfied and by Theorem 1 we obtain \( p(z) < q(z) \) and \( \Re p(z) > 0 \).

If \( C(z) = 1 \) in Theorem 8, then the theorem yields (3). If \( B(z) = 1 \) in Theorem 8, then for \( |\Im C(z)| \leq 1 \) we obtain

\[
\Re [z p'(z) + C(z) \cdot p(z)] > 0 \Rightarrow \Re p(z) > 0. \tag{23}
\]

A special case of this result leads to the following corollary.

Corollary 8.1. Let \( g \) be analytic in \( U \) with \( g(0) = 1 \) and \( |\Im z g'(z)/g(z)| \leq 1 \). If \( f(z) = z + \cdots \) is analytic in \( U \) then

\[
\Re [g(z) f'(z)] > 0 \Rightarrow \Re [g(z) f(z)/z] > 0,
\]

or equivalently

\[
g(z) f'(z) < (1 + z)/(1 - z) \Rightarrow g(z) f(z)/z < (1 + z)/(1 - z).
\]

Proof. If we set \( C(z) = 1 - z g'(z)/g(z) \), then \( |\Im C(z)| \leq 1 \). If we set \( p(z) = g(z) f(z)/z \), then \( p \) is analytic in \( U \), \( p(0) = 1 \) and \( p \) satisfies
$zp'(z) + C(z) p(z) = g(z) f'(z)$. Since $\text{Re } g(z) f'(z) > 0$, by (23) we get our result.

As an example of this corollary, let $f(z) = z + \cdots$ be analytic in $U$ and let $g(z) = \lambda^z$, with $|\lambda| \leq 1$. In this case $|\text{Im } zg'(z)/g(z)| = |\text{Im } \lambda z| < 1$, and we obtain

$$\text{Re}[e^{i\lambda} f'(z)] > 0 \Rightarrow \text{Re}[e^{i\lambda} f(z)/z] > 0.$$ 

5. INTEGRAL INEQUALITIES

In this section we apply some of the differential inequalities of the previous two sections to obtain integral inequalities.

**Theorem 9.** Let $\gamma \neq 0$ be a complex number and let $\phi$ and $\varphi$ be analytic in $U$, with $\phi(z) \cdot \varphi(z) \neq 0$, $\phi(0) = \varphi(0)$, and

$$|\text{Im } (\gamma \phi(z) + z \varphi'(z))/\gamma \varphi(z)| \leq \text{Re } \phi(z)/\gamma \varphi(z). \quad (24)$$

Let $f$ be analytic in $U$ with $\text{Re } f(z) > 0$, for $z \in U$. If $F = I(f)$ is defined by

$$F(z) = \gamma z^{-\gamma} \phi(z)^{-1} \int_0^z f(t) t^{-1} \varphi(t) \, dt, \quad (25)$$

then $F$ is analytic in $U$, $F(0) = f(0)$ and $\text{Re } F(z) > 0$ for $z \in U$.

**Proof.** If we let $z = 0$ in (24) we obtain $\text{Re } \gamma > 0$. The restrictions on $\gamma$ and the conditions on $\phi$, $\varphi$ and $f$ imply that $F$ is analytic in $U$ and $F(0) = 1$. If we let $B(z) = \phi(z)/\gamma \varphi(z)$ and $C(z) = (\gamma \phi(z) + z \varphi'(z))/\gamma \varphi(z)$, then condition (24) implies condition (21) of Theorem 8. Since $\text{Re } f(z) > 0$, by differentiating (25) we obtain

$$\text{Re } [B(z) \cdot z F'(z) + C(z) F(z)] = \text{Re } f(z) > 0.$$ 

Hence condition (22) of Theorem 8 is satisfied with $p = F$, and we conclude that $\text{Re } F(z) > 0$.

**Example 1.** If we let $\phi = \varphi = 1$ then (24) reduces to $\text{Re } \gamma > 0$. Hence by Theorem 9 we obtain: if $\gamma \neq 0$, $\text{Re } \gamma > 0$, and $f$ is analytic in $U$ then

$$\text{Re } f(z) > 0 \Rightarrow \text{Re } \left[ \gamma z^{-\gamma} \int_0^z f(t) t^{-1} \, dt \right] > 0.$$ 

This result was previously obtained by D. Hallenbeck and S. Ruscheweyh [2, p. 192] using a different method of proof.
EXAMPLE 2. If we let $\varphi = \phi$ and $\gamma > 0$ then (24) reduces to
\begin{equation}
|\text{Im} \, z\phi'(z)/\phi(z)| \leq 1.
\end{equation}
In this case, if $f$ and $\phi$ are analytic in $U$ with $\phi(z) \neq 0$, and if $\gamma > 0$ then
\[ \text{Re} \, f(z) > 0 \Rightarrow \text{Re} \left[ z^{-\gamma} \phi(z)^{-1} \int_0^z f(t) \, t^{\gamma-1} \phi(t) \, dt \right] > 0. \]
The function $\phi(z) = e^{z^2}$ satisfies (26) for $|\lambda| \leq 1$. In this case we obtain
\[ \text{Re} \, f(z) > 0 \Rightarrow \text{Re} \left[ z^{-\gamma} e^{-z^2} \int_0^z f(t) \, t^{\gamma-1} e^{it} \, dt \right] > 0. \]

THEOREM 10. Let $\gamma$ be a complex number, and let $\varphi$ and $\phi$ be analytic functions in $U$ that satisfy $\phi(0) = 0$, $\varphi(0)/\phi(0) = 0$ and $\varphi(z) \cdot \phi(z) \neq 0$ for $z \neq 0$. For each $z \in U$ suppose that
\begin{equation}
1 + \gamma + \frac{\phi'(z)}{\phi(z)} \geq \varphi(z)/\phi(z) \quad \text{and} \quad \text{Re} \left[ 1 + \gamma + \frac{\phi'(z)}{\phi(z)} \right] \geq 0. \tag{27}
\end{equation}
Let $f$ be analytic in $U$, $f(0) = 0$, and $|f(z)| < 1$ for $z \in U$. If $F = J(f)$ is defined by
\begin{equation}
F(z) = z^{-\gamma} \phi(z)^{-1} \int_0^z f(t) \, t^{\gamma-1} \phi(t) \, dt, \tag{28}
\end{equation}
then $F$ is analytic in $U$, $F(0) = 0$, and $|F(z)| < 1$ for $z \in U$.

Proof. If $\phi(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots$, with $n \geq 1$, and if we let $z = 0$ in (27) we obtain $|1 + \gamma + n| > 0$ and $\text{Re} \left[ 1 + \gamma + n \right] \geq 0$. These restrictions on $\gamma$ together with the conditions on $\phi$, $\varphi$ and $f$ imply that the function $F$ is analytic in $U$ and $F(0) = 0$. If we let $B(z) = \phi(z)/\varphi(z)$ and $C(z) = \gamma \phi(z)/\varphi(z) + z \phi'(z)/\varphi(z)$, then $B(z) \neq 0$ and (27) is equivalent to (i) of Theorem 5. Since $|f(z)| < 1$, if we differentiate (28) we obtain
\[ |B(z) \cdot z F'(z) + C(z) \cdot F(z)| = |f(z)| < 1. \]
Hence (16) of Theorem 5 is satisfied with $p = F$, and we obtain $|F(z)| < 1$.

The conclusion of the theorem can also be written as $f(z) < z$ implies $F(z) < z$, or as $|f(z)| < |z|$ implies $|F(z)| < |z|$. 

EXAMPLE 3. If we let $\phi(z) = \varphi(z) = z$ and $\alpha = \gamma + 1$ then condition (27) of Theorem 10 becomes
\[ |z + 1| \geq 1 \quad \text{and} \quad \text{Re} \, [z + 1] \geq 0, \]
If \( \alpha \) satisfies both of these conditions, then by Theorem 10 we have

\[
f(z) < z \Rightarrow z^{-\alpha} \int_0^z f(t) t^{\alpha-1} dt < z.
\]

**Example 4.** Let \( \phi \) be analytic in \( U \) with \( \phi(0) = 0, \phi'(0) = 1, \) and \( \phi(z) \phi'(z) \neq 0 \) for \( z \neq 0. \) If we let \( \phi(z) = z\phi'(z) \) then (27) becomes

\[
|1 + \gamma + z\phi'(z)/\phi(z)| \geq |z\phi'(z)/\phi(z)| \quad \text{and} \quad \text{Re}[1 + \gamma + z\phi'(z)/\phi(z)] \geq 0,
\]

and for \( \gamma \neq -1 \) these conditions are equivalent to

\[
\text{Re}[(1 + \gamma)^{-1}z\phi'(z)/\phi(z)] \geq -1/2 \quad \text{and} \quad \text{Re}[1 + \gamma + z\phi'(z)/\phi(z)] \geq 0.
\]

If \( 1 + \gamma > 0, \) then these conditions will hold if

\[
\text{Re}[z\phi'(z)/\phi(z)] \geq -(1 + \gamma)/2. \quad (29)
\]

Hence by Theorem 10 we obtain: if \( \gamma > -1, \phi(z) = z + \cdots \) is analytic in \( U \) and satisfies (29), and \( \phi'(z) \neq 0, \) then

\[
f(z) < z \Rightarrow z^{-\gamma} \phi(z)^{-1} \int_0^z f(t) t^\gamma \phi'(t) dt < z,
\]

or equivalently

\[
|f(z)| < |z| \Rightarrow \left| \int_0^z f(t) t^\gamma \phi'(t) dt \right| \leq |z^\gamma + \phi(z)|. \quad (30)
\]

For the special case \( \gamma = 0, \) (29) reduces to \( \text{Re}[z\phi'(z)/\phi(z)] \geq -1/2, \) and (30) simplifies to

\[
|f(z)| < |z| \Rightarrow \left| \int_0^z f(t) \phi'(t) dt \right| \leq |z\phi(z)|.
\]

**References**