Unboundedness in a Duffing Equation with Polynomial Potentials

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In this paper, a continuous periodic function $p(t)$ is constructed such that the Duffing equation $\frac{d^2x}{dt^2} + x^{2n+1} + p(t)x' = 0$, with $p(t) \in C^0(S^1)$, $n \geq 2$ for $2n+1 > l \geq n+2$, possesses an unbounded solution, which shows that the smoothness of the coefficients of the perturbation does influence the boundedness of solutions.

1. INTRODUCTION

1.1. Background

The boundedness of Duffing equations with the constant leading term:

$$\frac{d^2x}{dt^2} + x^{2n+1} + \sum_{l=0}^{2n} p_l(t)x^{l} = 0,$$

(1)

where $p_l(t) = p_l(t+1)$, $l = 0, 1, ..., 2n, n \geq 1$, has been considered by many authors. In the early 1960’s, Littlewood [3] asked whether the solutions of the Duffing-type equations

$$\frac{d^2x}{dt^2} + V(x, t) = 0$$

(2)

are bounded for all time, i.e., $\sup_{t \in \mathbb{R}}(|x(t)| + |x'(t)|) < \infty$ holds for all solutions of Eq. (2).

For the Littlewood boundedness problem, during the past years, people have paid more attention to the special case Eq. (1), since $\frac{d^2x}{dt^2} + x^{2n+1} = 0$ is a very nice time-independent integrable system, of which all the solutions are periodic. So if $|x|$ is large enough, Eq. (1) can be treated as a perturbation of an integrable system. Then KAM theory could be applied to prove the boundedness.
The first result was due to Morris [10], who proved that all solutions of a quadratic potential

\[
\frac{d^2x}{dt^2} + 2x^3 = a(t), \quad a(t + 1) = a(t),
\]

(3)

are bounded. It is noted that \(a(t)\) is only required to be piecewise continuous.

Using the famous KAM theorem [9], Diecherhoff and Zehnder [2] generalized Morris’s results to general polynomial potentials with time periodic coefficients (see Eq. (1)). In that paper, the coefficients \(p_l(t)\) are required to be sufficiently smooth to construct a series change of variables to transform Eq. (1) into a nearly integrable systems for large energies. In fact, in [2], the smoothness on \(p_l(t)\) depends on \(l\).

An interesting problem (proposed by Dicherhoff and Zehnder) is whether or not the boundedness depends on the smoothness of the coefficients.

In [7], Laederich and Levi weaken the smoothness requirement on \(p_l(t)\) to \(c^{3+l}\). By modifying proofs in [2] and using some approximation techniques, Liu [5] proved the same results for

\[
\frac{d^2x}{dt^2} + x^{2n+1} + a(t)x = p(t) \quad \text{where} \quad a(t), p(t) \in C^0(S^1),
\]

(4)

which shows that the boundedness of solutions does not depends on the smoothness of coefficients of lower order terms.

Recently, Wang and You [11] and Yuan [12] independently proved that all solutions of Eq. (1) are bounded if \(p_l(t) \in C^2, \quad n + 1 \leq l < 2n + 1; \quad p_l(t) \in C^1, \quad 2 \leq l \leq n; \quad p_l(t) \in C^0, \quad 0 \leq l \leq 1.\)

Then the remain problem is whether or not the smoothness requirement for coefficients of higher order terms plays the same role as that for coefficients of lower order terms.

In [7], Levi and You [8] proved that the equation

\[
\frac{d^2x}{dt^2} + x^{2n+1} + a(t)x^{2l+1} = 0
\]

(5)

with a special discontinuous coefficient \(a(t) = K^{[t]} \mod 2, \quad 0 < K < 1\) and \(2n + 1 > 2l + 1 \geq n + 2\), possesses an oscillatory unbounded solution. This implied that the boundedness of solutions was linked with the smoothness of coefficients of higher order terms.
1.2. Main Theorem

In this paper, we prove that there exists a continuous periodic function $p(t)$ such that the corresponding equation

$$\frac{d^2x}{dt^2} + x^{2n+1} + p(t)x' = 0 \quad \text{where} \quad n \geq 2, \quad 2n + 1 > l \geq n + 2, \quad (6)$$

possesses a solution which escapes to infinity in finite time.

For simplicity, in this paper we only carry the proof out for a special case:

$$\frac{d^2x}{dt^2} + x^{2n+1} + p(t)x^{2n} = 0 \quad \text{where} \quad n \geq 2. \quad (7)$$

According to [4], the method in this paper works for a class of equations.

**Theorem.** There exists a $p(t) \in C^0(S^1)$ and $\lambda_0 > 2(n + 1)$ such that for the solution $(x(t), x'(t))$ with $(x(0), x'(0)) = (\lambda_0^{1/(n+2)}, 0)$ of the corresponding equation Eq. (7), there is a strictly increasing series $\{T_i\}_{i=0}^{\infty}$ such that $\lim_{i \to \infty} T_i = T^\infty < 1$ with $T_0 = 0$ and

$$\min_{T_i \leq t \leq T_{i+1}} \hat{\lambda}(x(t), x'(t)) \geq \lambda_0^p,$$

where $l > 1, \hat{\lambda}(x, x') = (\frac{1}{x^2 + x'^2})x^{2n+1} + \frac{1}{2}x'^2)^{1/2}\beta, \beta = 1/(n + 2)$.

**Corollary.** Equation (7) in Theorem possesses a solution $(x(t), x'(t))$ satisfying $(|x(t)| + |x'(t)|) \to \infty$ as $t \to T^\infty < 1$.

**Remark 1.** In 1967, Coffman and Ullrich [1] considered the equation

$$\frac{d^3x}{dt^3} + p(t)x^3 = 0. \quad (8)$$

Using a simple but very elegant method, they constructed a continuous function $p(t)$ such that at least one solution of Eq. (8) has a finite interval as its maximal interval of existence. However, the equation with constant leading term is of more interest since it is a nearly integrable Hamiltonian system for large $|x|$. Note that the coefficient of the leading term in Eq. (8) depends on $t$ and for Eq. (1) it does not, so the results of [1] have no relation to the problem proposed in [2]. Moreover, strongly depending on the special property of Eq. (8), its potential is homogeneous on $x$, the method of [1] seems difficult to use on Eq. (1).
Remark 2. $p(t)$ constructed in this paper is not Lipschitz continuous at $k + T^\alpha (k \in \mathbb{Z})$. Whether or not the smoothness requirement in [11–13] is the sharpest is unclear.

1.3. An Outline of the Proof

In fact, we construct the $p(t)$ we need and the solution $x(t)$ in the main theorem at the same time. First, we find that during the time when the curve spirals once around the origin, the action variable $\lambda$ increases at some times and decreases at other times. So we do not know whether the increment of $\lambda$ is positive or negative. But we can construct a time $t_1 < 1$ and modify $p(t)$ on $[0, t_1]$ so that the increment is positive and is $O(\lambda^{-1+\epsilon})$ if the initial point $(\lambda(0), \theta(0)) = (\lambda_0, 0)$ is far enough from the origin, where the “jump” $1/\tau < \frac{1}{2}$ is critical to modify $p(t)$ and to our estimation. Inductively, we can construct a series of times $t_1, t_2, \ldots, t_i, t_{i+1}, \ldots$ and modify $p(t)$ on $[t_i, t_{i+1}]$, $i = 1, 2, \ldots$, so that on every such interval $[t_i, t_{i+1}]$, the increment is positive and at least $O(1/\tau \cdot \lambda^{-1+\epsilon})$. Hence, we can construct a time $T_1 \leq 1/\tau < 1$ so that the curve spirals at least $O(1/\tau' \cdot \lambda^{-2\beta-1})$ times around the origin and $\lambda_1 = \lambda(T_1) > \lambda_0 + c \cdot 1/\tau' \cdot \lambda^{-2\beta-1}$ with $c > 0$ independent of induction steps, where $2\beta - \alpha > 1$ and $\tau'$ is used to ensure the time not more than 1. This complete an induction step: During the interval of time $[0, T_1]$, $\lambda$ increases from $\lambda_0$ to $\lambda_1 > \lambda_0 + c \cdot 1/\tau' \cdot \lambda^{-2\beta-1}$ with the jump $1/\tau$.

Inductively, we can construct a series of times $T_1, T_2, \ldots, T_i, T_{i+1}, \ldots$ such that during the interval of time $[T_k, T_{k+1}]$, $\lambda$ increases from $\lambda_k$ to $\lambda_{k+1} > \lambda_k + c \cdot 1/\tau \cdot \lambda^{-2\beta-1}$ with the jump $1/\tau^k$, where $T_{k+1} - T_k \leq 1/\tau^k$. The reason that the jump is less and less is that we have to assure $p(t)$ is continuous. Because the exponent $2\beta - \alpha > 1$, the less and less jump cannot stop the rapid increase of $\lambda$. If $1/\tau'$ is chosen small enough, we will find that $T_k \to T_\infty < 1$ as $k \to \infty$ and $\lambda(t) \to +\infty$ as $t \to T_\infty$.

2. PROOF

2.1. Notations and Basic Facts

Consider the equation

$$\frac{d^2 x}{dt^2} + x^{2n+1} = 0. \quad (9)$$

It is equivalent to

$$x' = y, \quad y' = -x^{2n+1},$$
whose Hamiltonian function is
\[ H_0(x, y) = \frac{1}{2} y^2 + \frac{1}{2(n+1)} x^{2(n+1)}. \]
Obviously, all the solutions of Eq. (9) are periodic. Suppose \((c(t), s(t))\) is a solution of Eq. (9) satisfying \((c(0), s(0)) = (1, 0)\). Let \(T^*\) be its minimal period.

From Eq. (9), we can find that \(c(t)\) and \(s(t)\) satisfy the following:

**Lemma 1.**

(i) \(c(t), s(t) \in C^\infty(R), c(t) = c(t + T^*), s(t) = s(t + T^*),\)
\(s(0) = 0, c(0) = 1\)

(ii) \(c'(t) = c(t), s'(t) = -c(t)^{2n+1}\)

(iii) \((n+1) s(t)^2 + c(t)^{2(n+1)} = 1\)

(iv) \(c(-t) = c(t) and s(-t) = -s(t)\)

(v) \(c(\frac{1}{2} T^* + t) = c(\frac{1}{2} T^* - t), s(\frac{1}{2} T^* + t) = -s(\frac{1}{2} T^* - t)\)

(vi) \(s(t) \leq 0, 0 \leq t \leq \frac{1}{2} T^*; s(t) \geq 0, \frac{1}{2} T^* < t \leq T^*\)

The action and angle variables are now defined by the mapping
\[ \Phi: R^* \times S^1 \rightarrow R^3 \setminus \{0\}, \] where \((x, y) = \Phi(\lambda, \theta)\) with \(\lambda > 0\) and \(\theta \mod 1\) being given by the formula
\[ \Phi: x = y^* \lambda^* \lambda(\theta T^*), y = \gamma^* \lambda^* \theta(\theta T^*) \]
where \(\alpha = 1/(n+2), \beta = 1 - \alpha, \gamma = 1/\pi T^*\).

We claim that \(\Phi\) is a symplectic diffeomorphism from \(R^* \times S^1\) onto \(R^3 \setminus \{0\}\). Indeed, using the Jacobian determinant \(A\) of \(\Phi\) one finds by (iii) that \(|A| = 1\); so \(\Phi\) is measure preserving. Moreover, since \((c, s)\) is a solution of a differential equation having \(T^*\) as its minimal period, one concludes that \(\Phi\) is one to one and onto. This proves the claim.

In the new coordinates, the Hamiltonian function becomes
\[ h_0(\lambda, \theta) = d \cdot \lambda^{2\beta} \] where \(d = \gamma^{2\beta}/2(n+1)\), which is independent of the angle variable \(\theta\), so that the system Eq. (9) become very simple in the \((\lambda, \theta)\)-plane:
\[ \frac{d\theta}{dt} = \frac{\partial h_0}{\partial \lambda} = 2\beta d \lambda^{2\beta-1}, \quad \frac{d\lambda}{dt} = -\frac{\partial h_0}{\partial \theta} = 0.\]

The full Eq. (7) has the Hamiltonian function
\[ H_1(x, y, t) = \frac{1}{2} y^2 + \frac{1}{2(n+1)} x^{2(n+1)} + \frac{1}{2n+1} p(t) x^{2n+1}, \]
and under the symplectic transformation $\Phi$, it is transformed into the form

$$h_1(\lambda, 0; t) = d + \int^{(2n+1)(n+2)p(t)/2n+1} \, dt,$$

where $q(\theta) = c(\theta T^*)^{2n+1} \in C^\infty(S^1)$. So the equation $X_{h_1}$ determined by $h_1$ is

$$\frac{d\theta}{dt} = 2\beta \, \frac{d^2 \theta - 1 + x_0^j(2n+1)(n+2)p(t)/2n+1, \quad (10)$$

$$\frac{dx_0^j}{dt} = \frac{1}{2n+1} \, \frac{(2n+1)(n+2)p(t)/2n+1, \quad (11)$$

From (ii) in Lemma 1, we have

$$q'(\theta) = (2n+1) T^* c(\theta T^*) s(\theta T^*).$$

2.2. Proof of Theorem

Now we define $p(t)$ in $[0, 1]$. We will construct a time $t_1 < 1$ and modify $p^0(t) = 1$ on $[0, t_1]$ so that the energy of one solution increases in $[0, t_1]$. We divide the construction into two steps: first, we construct a piecewise continuous function so that the energy of one solution of the corresponding Eq. (7) obtains a positive increment in $[0, t_1]$ as we expect. Then we modify this function into a continuous one in such a way that the modification does not influence the estimate we had obtained before.

We denote $\text{max}(|f(x)|)$ by $\|f(x)\|$ and let $\lambda_0 = \lambda_0(n, \|p\|, \|q^*\|) > 2(n+1)$ be a sufficiently large constant which will be determined later.

Denote the corresponding Hamiltonian system with coefficient function $p(t)$ by $X_p$. Suppose the solution $(\lambda(t), \theta(t))$ of $X_p$ with $(\lambda(0), \theta(0)) = (\lambda_0, 0)$ at $t = 0$ arrives at $(\lambda_1, 1)$ at $t = t_1 << 1$.

Define $p^1(t)$ to be a piecewise continuous function,

$$p^1(t) = \begin{cases} 1 & t \in [0, t_{1/2}] \\ 1 - \sigma & t \in (t_{1/2}, t_1) \\ 1 & t \in (t_1, 1], \end{cases}$$

where $0 < \sigma < 1$ and $t_1$ is the unique time which satisfies $\theta(t_1) = 1$ for the solution of the new system corresponding with $p^1$.

Obviously, $t_1 << 1$ if $\lambda_0$ is sufficiently large.

We denote the corresponding solution satisfying the same initial condition on $X_p$ by $(\lambda^1(t), \theta(t))$ and denote $\lambda_1 = \lambda^1(t_1)$. The jump $\sigma$ is used to control the increment of $\lambda$. In fact, from Eq. (11), one can see that $\lambda^1(t)$ is
increasing on \([0, t_{1/2}]\) and decreasing on \([t_{1/2}, t_1]\). Because \(\lambda > 1 - \sigma\), we can expect that the difference between the increment on \([0, t_{1/2}]\) and the decrement on \([t_{1/2}, t_1]\) is positive and very large. This is the key idea of our example.

In the following, all \(c_i > 0\) are independent of induction steps.

**Lemma 2.**

\[
\lambda_0 + c_1 \lambda_0^{1-x} \leq \lambda_{1/2} \leq \lambda_0 + c_2 \lambda_0^{1-x},
\]

\[
c_3 \lambda_0^{-(2\sigma-1)} \leq t_{1/2} \leq c_4 \lambda_0^{-(2\sigma-1)}.
\]

**Proof.** Because \(q'(\theta) \leq 0\) if \(t \in [0, t_{1/2}]\), \(\lambda^1\) is an increasing function on this interval. We use Eq. (10), Eq. (11) to calculate that

\[
t_{1/2} \leq \frac{1}{2} \left( 2\beta d i \lambda_0^{2\sigma-1} - x \|q\| \lambda_0^{(2n+1)(\sigma-1)} \right) \leq \frac{1}{2\beta d} \lambda_0^{-(2\sigma-1)} = c_4 \lambda_0^{-(2\sigma-1)}
\]

if \(\lambda_0\) is sufficiently large and

\[
\int_{\lambda_0}^{\lambda_{1/2}} \frac{dt}{\lambda_0^{(2n+1)(\sigma-1)}} = \int_{t_{1/2}}^{t_1} - E q'(\theta(t)) \, dt
\]

where \(E = 1/(2n + 1), \gamma^{(2n+1)(\sigma-2)} > 0\). Then we obtain

\[
(2n + 1) \left( \frac{1}{\lambda_0^{(2n+1)(\sigma-1)}} - \frac{1}{\lambda_{1/2}^{(2n+1)(\sigma-1)}} \right) \leq E \|q\| \cdot \lambda_{1/2} \leq \frac{E \|q\|}{2\beta d} \lambda_0^{-(2\sigma-1)},
\]

i.e.,

\[
\lambda_{1/2} \leq \lambda_0 \left( 1 - \frac{E \|q\|}{(2n+1) \|\lambda_0^{1-x} \|} \right)^{-1/(2n+1)(\sigma-1)}
\]

\[
\leq \lambda_0 \left( 1 + \frac{E \|q\|}{(2n+1) \|\lambda_0^{1-x} \|} \right)^{-1/(2n+1)(\sigma-1)}
\]

\[
= \lambda_0 (1 + c_2 \lambda_0^{-\sigma}).
\]

Combining with Eq. (10) and Eq. (14), we have

\[
t_{1/2} \geq \frac{1}{2} \left( 2\beta d i \lambda_0^{2\sigma-1} + x \|q\| \lambda_{1/2}^{(2n+1)(\sigma-1)} \right) \leq \frac{1}{2\beta d} \lambda_0^{-(2\sigma-1)} = c_3 \lambda_0^{-(2\sigma-1)}
\]
If $\lambda_0$ is sufficiently large. Hence from Eq. (14)
\[
\dot{\lambda}_{1/2} \equiv \dot{\lambda}_0 + \left( \frac{1}{8bd} \lambda_0^{-1(2\beta - 1)} \right) E \frac{1}{8} \| q' \| \lambda_0^{(2n+1)\alpha} = \lambda_0 + c_1 \lambda_0^1 \alpha \text{.}
\]
It can be seen that the increment is $O(\lambda_0^{1-a})$, therefore, after spiraling one
time around the origin, $\lambda^1$ increases notably.

**Lemma 3.**

\[
|t_{1/2} - \frac{1}{2}(2\beta d \lambda_0^{2\beta - 1})^{-1} - | \leq c_5 \lambda_0^{-1(2\beta - 1) - a} \text{,}
\]
\[
|t_{1/2} - (t_1 - t_{1/2}) - \frac{1}{2}(2\beta d \lambda_0^{2\beta - 1})^{-1} - | \leq c_6 \lambda_0^{-1(2\beta - 1) - a} \text{.}
\]
It follows that
\[
|(t_1 - t_{1/2}) - t_{1/2}| \leq c_7 \lambda_0^{-1(2\beta - 1) - a} \text{.}
\]

**Proof.** From Eq. (10),
\[
\theta_{1/2} - \theta_0 = \frac{1}{2} \int_{t_{1/2}}^{t_{1/2}} (2\beta d \lambda_0^{2\beta - 1} + \alpha \lambda_0^{(2n+1)/(2n+2)} q(\theta) \dot{\lambda}_0^{(2n+1)\alpha - 1} \text{.) } dt \text{.} \tag{15}
\]

In view of Lemma 2,
\[
|2\beta d \lambda_0^{2\beta - 1} - 2\beta d \lambda_0^{2\beta - 1}(t) + \alpha \lambda_0^{(2n+1)/(2n+2)} q(\theta) \lambda_0^{(2n+1)\alpha - 1}(t) - 2(\alpha \lambda_0^{(2n+1)/(2n+2)} q(\theta) \dot{\lambda}_0^{(2n+1)\alpha - 1})| \leq 2(\alpha \lambda_0^{(2n+1)/(2n+2)} q(\theta) \dot{\lambda}_0^{(2n+1)\alpha - 1}) \int_{t_{1/2}}^{t_{1/2}} (2\beta d \lambda_0^{2\beta - 1})^{-1} \text{.) } dt \text{.}
\]
if $t \in [0, t_{1/2}]$. Therefore,
\[
|t_{1/2} - \frac{1}{2}(2\beta d \lambda_0^{2\beta - 1})^{-1} - | \leq c_5 \lambda_0^{-1(2\beta - 1) - a} \text{.}
\]

with Eq. (15). Because on $[t_{1/2}, t_1]$, $\lambda$ is decreasing from Eq. (11), we con-
clude that $\lambda_1 > \lambda_0 - \lambda_0^{-1(1/2)\alpha} \text{ if } \lambda_0$ is sufficiently large. Indeed, if $\lambda_1 \leq \lambda_0 - \lambda_0^{-1(1/2)\alpha}$, then $\exists t' \leq t_1$, s.t. $\lambda_1(t') = \lambda_0 - \lambda_0^{-1(1/2)\alpha}$.

From Eq. (11), we get
\[
\int_{t_{1/2}}^{t_1} \frac{d\lambda}{\dot{\lambda}_0^{(2n+1)\alpha}} = \int_{t_{1/2}}^{t_1} - E p(t') q'(\theta(t)) \text{ dt.}
\]
Then
\[
t'_{1} - t_{1/2} \geq (\|q'\|)^{-1} \int_{t_{1/2}}^{t_1} \frac{\dot{\lambda}_0^{(2n+1)\alpha}}{\dot{\lambda}_0^{(2n+1)\alpha}} = (2E \|q'\|)^{-1} \lambda_0^{-1(2n+1)\alpha - 1} (1/2) \alpha \text{.}
\]
In view of Eq. (10), we have
\[ \phi(t_1') - \phi(t_{1/2}) \geq \frac{1}{2} \phi \int_{t_0}^{t_1'} \frac{\phi}{\beta E}\, d\phi \left(-\frac{1}{\phi} - 1\right) - \frac{1}{2} \left(\phi_{t_1'} - \phi_{t_{1/2}}\right) \]
\[ = \frac{d\phi}{\beta E}\, \phi^{\frac{1}{2}} \cdot \phi_{t_1'} > 1 \]
if \( \phi_{t_0} \) is sufficiently large, which contradicts \( \phi(t_1') - \phi(t_{1/2}) \leq \frac{1}{2} \). Hence, \( \phi_{t_0} > \phi(t_{1/2}) \) is satisfied.

With the same method in Lemma 2 and the above, we have the estimates
\[ |(t_1 - t_{1/2}) - \frac{1}{2}\phi \int_{t_0}^{t_1} \frac{\phi}{\beta E}\, d\phi| \leq c_6 \phi_{t_0}^{-(\sigma - 1)} \]

**Lemma 4.** \( \phi_{t_1} \geq \phi_{t_0} + c_8 \phi_{t_0}^{1-\sigma} \)

**Proof.** In view of Lemma 2, one finds \( \phi(t_{1/2} - t) + \phi(t_{1/2} + t) \leq c_9 \phi_{t_0}^{1-\sigma} \) if \( t \in [0, t_{1/2}] \). Without losing generality, we assume that \( t_{1/2} < 2t_{1/2} \).

From Eq. (11), it can be shown that
\[ \int_{t_{1/2}}^{t_1} \frac{d\phi}{\beta E}\, \phi = \left[ \frac{d\phi}{\beta E}\, \phi \right]_{t_{1/2}}^{t_1} - Eq'\phi(t) \, dt \]

and
\[ \int_{t_{1/2}}^{t_1} \frac{d\phi}{\beta E}\, \phi = \left[ \frac{d\phi}{\beta E}\, \phi \right]_{t_{1/2}}^{t_1} - E(1 - \sigma) \, q'(\phi(t)) \, dt \]

\[ = \left[ \frac{d\phi}{\beta E}\, \phi \right]_{t_{1/2}}^{t_1} + \left[ \frac{d\phi}{\beta E}\, \phi \right]_{2t_{1/2}}^{t_1} \]

\[ = \left[ \frac{d\phi}{\beta E}\, \phi \right]_{t_{1/2}}^{t_1} - E(1 - \sigma) \, q'(\phi(t)) \, dt \]

\[ = \left[ \frac{d\phi}{\beta E}\, \phi \right]_{t_{1/2}}^{t_1} - E(1 - \sigma) \, q'(\phi(t)) \, dt \]

\[ + \left[ \frac{d\phi}{\beta E}\, \phi \right]_{t_{1/2}}^{t_1} - E(1 - \sigma) \, q'(\phi(t)) \, dt \]
From Eq. (16) and Eq. (17), we have
\[
\frac{1}{(2n + 1) \pi - 1} \left| (1 - \sigma) (\lambda_0^{-(2n + 1) \pi_1 + 1} - \lambda_1^{-(2n + 1) \pi_1 + 1}) \right.
\[
- \left( \lambda_1^{-(2n + 1) \pi_1 + 1} - \lambda_1^{-(2n + 1) \pi_1 + 1} \right) \right|
\leq t_{1/2} E |q''| \max_{t \in [0, t_{1/2}]} [\sigma \theta(t_{1/2} + t) + \theta(t_{1/2} - (t_{1/2} - t))]
\[
+ E |t_{1/2} - t| - \|q''\|
\leq c_{10} \lambda_0^{-(2\beta - 1) - \sigma}.
\]

In the above, the second inequality comes from Lemma 2 and Lemma 3. It follows that
\[
\lambda_1^{-(2n + 1) \pi_1 + 1} \leq (1 - \sigma) \lambda_0^{-(2n + 1) \pi_1 + 1} + (2n + 1) \pi_1 - (2n + 1) \pi_1 + 1
\[
+ c_{10} \lambda_0^{-(2\beta - 1) - \sigma}.
\]

Note that \(-(2n + 1) \pi + 1 - \pi \geq - (2\beta - 1) - \pi\), we obtain \(\lambda_1 \geq \lambda_0 + c_{10} \lambda_0^{1 - \pi}\).

Now we modify the piecewise continuous function \(p^1(t)\) into a continuous function. Being short of signs, we keep the notations unchanged in the process of modification. For example, \(p^1(t)\) denotes the continuous function modified from the original piecewise continuous function \(p^1(t)\).

First we modify \(p^1(t)\) on the interval \([t_{1/2}, t_{1/2} + \lambda_0^{-\pi}]\) to be \(\sigma(t_{1/2} - t) \lambda_0^{-\pi} + \lambda_0^{-\pi} \pi\). It is easy to see that \(\{(t, p^1(t)) \mid t \in [t_{1/2}, t_{1/2} + \lambda_0^{-\pi}]\}\) is the line segment connecting \((t_{1/2}, 1)\) and \((t_{1/2} + \lambda_0^{-\pi}, 1 - \sigma)\).

In view of the Mean Value Theorem, there must exist a unique new \(t_1\) such that \(\theta(t_1) = 1\) if we let
\[
p^1(t) = \begin{cases} 
1 - \sigma & t \in [t_{1/2}, t_{1/2} + \lambda_0^{-\pi}, t_1 - \lambda_0^{-\pi}] \\
1 + (t - t_1) \lambda_0^{-\pi} & t \in [t_1 - \lambda_0^{-\pi}, t_1].
\end{cases}
\]

Now the newest coefficient is already a continuous function. It is easy to check that Lemma 2, Lemma 3, and Lemma 4 still hold with \(\epsilon_i\) after this modification in view of \(\lambda_0^{-\pi} \approx \lambda_0^{-(2\beta - 1) - \pi}\).

We will modify \(p^0\) inductively and denote the function obtained and the corresponding solution with \((\lambda_0, \theta_0)\) as initial point by \(p^i\) and \(\theta^i(t), \lambda^i(t)\) with \(\theta^i(t_i) = i, \lambda^i(t_i) = \lambda_0\).

Suppose we have obtained \(p^0, p^1, \ldots, p^i\).
\(p^{i+1}\) is constructed by modifying \(p^i\) on the interval \([t_i, t_{i+1}]\), where \(t_{i+1}\)
satisfies \( \theta^{i+1}(t_{i+1}) = i + 1 \) in the same way as above if we regard \( \lambda_i, t_i \) as \( \lambda_0, t_0 \). All the lemmas are true after the modification.

In the process of constructing \( p^i \), we keep the jump \( \sigma = 1/\tau \) \((\tau > 2)\) unchanged until \( i = j_1 \). Then we let \( \sigma = 1/\tau^2 \) and keep it unchanged until \( i = j_2 \). Inductively, we choose \( \sigma = 1/\tau^k \) when \( \theta \in [j_{k-1}, j_k] \), where \( j_0 = 0, j_1, j_2, \ldots \) are defined as above:

Let \( j_1 = [(1/\tau^i) \lambda^2_0 - 1] \), where \( [x] = \max_{n \in \mathbb{Z}} n \leq x \) and \( \tau' > 0 \) is used to control time and determined later. It follows that

\[
T_1 = t_{j_1} < j_1 \cdot 2c_4 \lambda_0^{-2(2^\beta - 1)}
\]

\[
= 2c_4 \left( \frac{1}{\tau'} \lambda_{0}^{2^\beta - 1} \right) \lambda_0^{-2(2^\beta - 1)} < \frac{c_11}{\tau'}
\]

On the interval \([0, T_1]\), since \( \lambda_{k+1} - \lambda_k \geq c_0(1/\tau) \lambda_0^{1-\alpha} \), we have

\[
\lambda_{j_k} = \lambda^h(T_1) \geq [(1/\tau') \lambda_{0}^{2^\beta - 1}] \cdot c_0(1/\tau) \lambda_0^{1-\alpha} \geq \frac{1}{2} c_0(1/\tau') \lambda_0^{2^\beta - \alpha}.
\]

Suppose we have defined \( T_0 = 0, T_1, \ldots, T_{i} \) in the above method and the following are tenable:

\[
\lambda_{j_k} - \lambda_{j_{k-1}} = \frac{1}{2} c_8 \left( \frac{1}{\tau'} \right)^{\beta} (\lambda_{j_{k-1}})^{2^\beta - \alpha}, \quad T_{k} - T_{k-1} \leq \frac{c_11}{\tau^2},
\]

since \( \sigma = \tau_k = \tau^k, \lambda_{j_k} = \lambda^h(T_k) \geq \frac{1}{2} c_8 \left( \frac{1}{\tau'} \right)^{\beta} (\lambda_{j_{k-1}})^{2^\beta - \alpha}, \quad k = 1, 2, \ldots, i. \)

Let \( j_{i+1} - j_i = [(1/\tau') \lambda_0^{2^\beta - 1}] \), \( \sigma = \tau_{i+1} = \tau' + 1, T_{i+1} = t_{j_{i+1}} \). Similarly to the above discussion, we have the following:

\[
\lambda_{j_{i+1}} = \lambda^h(T_{i+1}) \geq \frac{1}{2} c_8 \left( \frac{1}{\tau'} \right)^{\beta} (\lambda_{j_i})^{2^\beta - \alpha}, \quad T_{i+1} - T_i \leq \frac{c_11}{\tau^2 + 1}.
\]

Consequently, \( T_{i+1} \leq c_{11}(\sum_{k=1}^{i+1} 1/\tau^k) \leq c_{11}/\tau' < 1 \) if \( \tau' \) is sufficiently large.

Let \( T_i \to T_\infty \) and \( p(t) = \lim_{i \to \infty} p^i(t) \). Because \( \max_{t_{j_i}, t_{j_{i+1}}} |p(t_i) - p(t_{j_i})| \leq 2(1/\tau') \), \( \lim_{i \to \infty} \max_{t_{j_i}, t_{j_{i+1}}} |p(t) - p(t_j)| = \lim_{i \to \infty} p(t) = 0 \), we have \( p(t) \in C^0(S^1) \) at once.

**Lemma 5.** \( \lambda_{j_{i+1}} \geq 2\lambda_{0}^{l} \) where \( l > 1 \).
Proof. First, \( \lambda_j \geq \frac{1}{2} c_2 (1/\tau^r) \lambda_0^{2r - s} \geq 2 \lambda_0^r \) if \( \lambda_0 \) sufficiently large, where \( l = 2\beta - s - \alpha/2 > 1 \). Suppose \( \lambda_j \geq 2 \lambda_0^r \). It follows that

\[
\lambda_{j+1} \geq \frac{1}{2} c_8 \frac{1}{(\tau^r)^{k+1}} \lambda_0^{2r - s} \geq \frac{\lambda_0^r}{(\tau^r)^{k+1}} \geq 2 \lambda_0^r.
\]

Proof of Theorem. \( \min_{t \in [T_j, T_{j+1}]} \dot{\lambda}^r(t) \geq \frac{1}{2} \lambda_j \geq \lambda_0^r \).

Remark. Obviously, the \( p(t) \) obtained is a bit strange. In fact, it looks like a function \( 1 - (1 - t) \sin(2\pi/(1 - t)) \) on \([0, 1)\). Moreover, the \( p(t) \) we constructed is not a Lipschitz function. In fact, \{ \( (t, p(t)) \mid \ t \in [t_j + 1/2, t_j + 1/2 + \lambda_j^{-1}] \} \) is a segment whose slope \( \geq (1/t^k)/\lambda_j^{-1} \geq \lambda_0^r t^k \rightarrow \infty \) as \( k \rightarrow \infty \).

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