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Polynomials and spatial Pick-type theorems

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Abstract

Pick's theorem about the area of a simple lattice planar polygon has many extensions and generalizations even in the planar case. The theorem has also higher-dimensional generalizations, which are not as commonly known as the 2-dimensional case. The aim of the paper is, on one hand, to give a few new higher-dimensional generalizations of Pick's theorem and, on the other hand, collect known ones. We also study some relationships between lattice points in a lattice polyhedron which lead to some new Pick-type formulae. Another purpose of this paper is to pose several problems related to the subject of higher-dimensional Pick-type theorems. We hope that the paper may popularize the idea of determining the volume of a lattice polyhedron P by reading information contained in a lattice and the tiling of the space generated by the lattice.

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1. Introduction

Let us start with some definitions. Denote by $\mathbb{Z}^N$ the fundamental lattice of points with integer coordinates in $\mathbb{R}^N$. Elements of $\mathbb{Z}^N$ are called *lattice points*. We say that a simplex

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$\varDelta \subset \mathbb{R}^N$ is a *lattice simplex* if all its vertices belong to $\mathbb{Z}^N$. A lattice simplex $\varDelta$ is called *fundamental* if $\varDelta \cap \mathbb{Z}^N$ consists only of the vertices of $\varDelta$ (fundamental lattice 1-simplexes are simply called *fundamental lattice segments*). A set $P \subset \mathbb{R}^N$ is said to be a *polyhedron* if P is the underlying point set of a simplicial cell complex. P is called a *lattice* or *integral polyhedron* if all its vertices (0-simplexes) lie in $\mathbb{Z}^N$. Any lattice polyhedron P can be represented as the union

$$P = \bigcup_{i=1}^{m} \varDelta_i, \tag{1.1}$$

where each $\varDelta_i$ is a fundamental lattice simplex and $\varDelta_j \backslash \varDelta_k \neq \emptyset$ for $j \neq k$ (no $\varDelta_j$ is contained in another simplex). Lattice polyhedra in $\mathbb{R}^2$ are called, as usual, *lattice polygons*. A lattice polyhedron P in $\mathbb{R}^N$ is called *proper* if every $\varDelta_i$ in the union (1.1) is N-dimensional, and is called *simple* if it is homeomorphic to a closed ball.

We refer the reader to the survey [8], where many interesting problems dealing with lattice polygons and lattice polyhedra are presented. In particular, the problem of determining the area of lattice polygons and the volume of lattice polyhedra has been of interest since 1938 when H. Steinhaus retrieved Pick's area formula—published in a rather obscure journal [21]—by popularizing it in the first edition of his famous *Mathematical Snapshots*, see [26,2].

Pick's idea was to relate the area of a simple lattice polygon W to the numbers i and b of points from $\mathbb{Z}^2$ lying in the interior and on the boundary of W, respectively. He found the following elegantly simple formula:

$$A(W) = \frac{b}{2} + i - 1,$$

which has a multitude of alternative proofs and generalizations. An especially broad generalizations of Pick's theorem are given in [9] and recently in [5] and [18]. The fine expository article [25] references many of the problems related to Pick's theorem.

Looking for an analogue of Pick's formula for higher dimensions, Reeve [23] first noticed—by considering fundamental lattice tetrahedra in $\mathbb{R}^3$ similar to those in Fig. 1. below—that there can be no general expression for the volume of a lattice polyhedron P in terms of only the numbers of points from $\mathbb{Z}^3$ in the interior and on the boundary of P.

Next he introduced a secondary lattice (often called the *rational lattice*)

$$\mathbb{Z}_n^3 = \{x \in \mathbb{R}^3 : \mathrm{n}x \in \mathbb{Z}^3\}, \quad n \geqslant 1$$

(thus $\mathbb{Z}_1^3 = \mathbb{Z}^3$) which became an additional source of information sufficient to determine the volume of lattice polyhedra.

For a given lattice polyhedron P in $\mathbb{R}^N$ denoted by B_n and I_n, $n \geqslant 1$, the numbers of points of the lattice $\mathbb{Z}_n^N = \{x \in \mathbb{R}^N : \mathrm{n}x \in \mathbb{Z}^N\}$ on the boundary and in the interior of P, respectively. Thus,

$$B_n = B_n(P) = |\mathbb{Z}_n^N \cap \partial P| \text{ and } I_n = I_n(P) = |\mathbb{Z}_n^N \cap \operatorname{int} P|.$$

We will also deal with the numbers

$$G_n = G_n(P) = |\mathbb{Z}_n^N \cap P| = B_n + I_n.$$

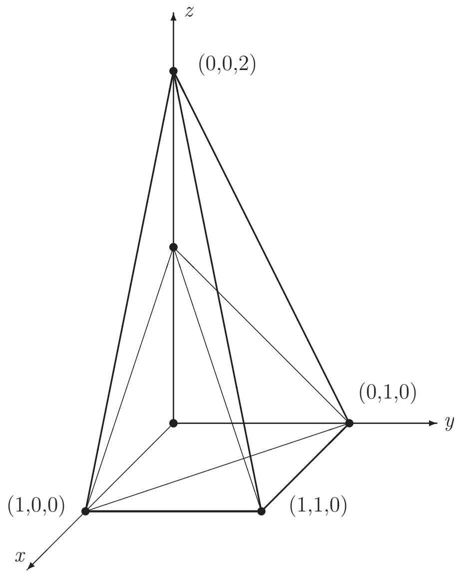

Fig. 1. Two of Reeve's fundamental lattice tetrahedra.

Reeve [22,23] showed that the volume, $V(P)$, of any proper lattice polyhedron P in $\mathbb{R}^3$ can be expressed in terms of the numbers of lattice points from $\mathbb{Z}^3$ and any lattice $\mathbb{Z}_n^3$ in the following way:

$$2n(n^2-1)V(P) = B_n - nB_1 + 2(I_n - nI_1) + (n-1)[2\chi(P) - \chi(\partial P)], \tag{1.2}$$

where $n \geqslant 2$. Here and later on $\chi(X)$ denotes the *Euler characteristic* of the set X.

Reeve's formula (1.2) was next generalized to proper lattice polyhedra in arbitrary dimensions by Macdonald [19]. Macdonald used Ehrhart's remarkable discovery that for each n the number $G_n(P)$ of lattice points from $\mathbb{Z}_n^N$ in a lattice polyhedron $P \subset \mathbb{R}^N$ is given by the following polynomial:

$$G_n(P) = V(P)n^N + a_{N-1}(P)n^{N-1} + \cdots + a_1(P)n + \chi(P) \tag{1.3}$$

in which the coefficients $a_{N-1}(P), \ldots, a_1(P)$ are some rational numbers. For an explicit description of all a_n's in the case of a lattice simplex we refer the reader to [1,3]. This polynomial is now called the *Ehrhart polynomial*. By applying this result, Macdonald first obtained the formula

$$N!V(P) = \sum_{k=0}^{N-1} (-1)^k \binom{N}{k} G_{N-k} + (-1)^N \chi(P) \tag{1.4}$$

involving lattices $\mathbb{Z}_1^N, \ldots, \mathbb{Z}_N^N$ and next found another formula expressing the volume of a proper lattice polyhedron P in $\mathbb{R}^N$ by means of $N-1$ lattices $\mathbb{Z}_1^N, \ldots, \mathbb{Z}_{N-1}^N$ as follows:

$$(N-1)N!V(P) = \sum_{k=1}^{N-1}(-1)^{k-1}\binom{N-1}{k-1}(B_{N-k}+2I_{N-k}) + (-1)^{N-1}[2\chi(P)-\chi(\partial P)]. \tag{1.5}$$

Let us notice that formula (1.5) in the case $N=2$ gives Reeve's extension of Pick's formula, as found in [23], and in the case $N=3$ coincides with Reeve's formula (1.2) when $n=2$.

The paper is organized in the following way. In Section 2 we first study relationships between numbers of points from rational lattices in an integral polyhedron and next obtain a few variants of formulae (1.4) and (1.5) in which the Euler characteristics no longer play a role. In Section 3 we present formulae that involve fewer lattice points than the above formulae. In Section 4 we show that the volume of three-dimensional lattice polyhedra is a property of the tiling of the space. In Section 5 we consider a generalization to non-proper lattice polyhedra. We end the paper with a number of remarks and open problems.

2. Rational lattice points in integral polyhedra

In this section we show that the Euler characteristic of a proper integral polyhedron P in $\mathbb{R}^N$ can be expressed by means of the numbers $G_1, \ldots, G_{N+1}$. We also study some relationships between the numbers $B_1, \ldots, B_{N+1}$ and $I_1, \ldots, I_{N+1}$, and use these relationships to obtain formulae for the volume of lattice polyhedra in which the Euler characteristics of P and its boundary ∂P no longer play a role. In this way we obtain formulae which are of a purely combinatorial character since they employ only rational lattice points.

Lemma 2.1. *Let $W_N(x) = \beta_N x^N + \beta_{N-1}x^{N-1} + \cdots + \beta_1 x + \beta_0$ be a polynomial of degree N. Then*

$$\beta_0 = \sum_{k=1}^{N+1}(-1)^{k-1}\binom{N+1}{k}W_N(k)$$

and

$$\beta_N = \frac{1}{N!}\sum_{k=1}^{N+1}(-1)^{N-k+1}\binom{N}{k-1}W_N(k).$$

Proof. $W_N(x)$ for values $x = 1, 2, \ldots, N, N+1$ gives the following system of $N+1$ independent linear equations:

$$\begin{cases} \beta_0 + \beta_1 + \cdots + \beta_{N-1} + \beta_N = W_N(1) \\ \beta_0 + 2\beta_1 + \cdots + 2^{N-1}\beta_{N-1} + 2^N\beta_N = W_N(2) \\ \beta_0 + 3\beta_1 + \cdots + 3^{N-1}\beta_{N-1} + 3^N\beta_N = W_N(3) \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \beta_0 + N\beta_1 + \cdots + N^{N-1}\beta_{N-1} + N^N\beta_N = W_N(N) \\ \beta_0 + (N+1)\beta_1 + \cdots + (N+1)^{N-1}\beta_{N-1} + (N+1)^N\beta_N = W_N(N+1) \end{cases}$$

We solve the system for β_0 and β_N. By the Cramer's rule we have

$$\beta_0 = \frac{L_0}{M} \quad \text{and} \quad \beta_N = \frac{L_N}{M},$$

where

$$L_0 = \begin{vmatrix} W_N(1) & 1 & \cdots & 1 & 1 \\ W_N(2) & 2 & \cdots & 2^{N-1} & 2^N \\ W_N(3) & 3 & \cdots & 3^{N-1} & 3^N \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ W_N(N+1) & N+1 & \cdots & (N+1)^{N-1} & (N+1)^N \end{vmatrix},$$

$$L_N = \begin{vmatrix} 1 & 1 & \cdots & 1 & W_N(1) \\ 1 & 2 & \cdots & 2^{N-1} & W_N(2) \\ 1 & 3 & \cdots & 3^{N-1} & W_N(3) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & (N+1) & \cdots & (N+1)^{N-1} & W_N(N+1) \end{vmatrix}$$

and

$$M = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & \cdots & 2^{N-1} & 2^N \\ 1 & 3 & \cdots & 3^{N-1} & 3^N \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & N+1 & \cdots & (N+1)^{N-1} & (N+1)^N \end{vmatrix}.$$

We expand now L_0 by its first column and L_N by its last column and evaluate resulting Vandermonde-type determinants. In this way we obtain

$$\beta_0 = \frac{\prod_{j=1}^{N} j! \sum_{k=1}^{N+1} (-1)^{k-1} \binom{N+1}{k} W_N(k)}{\prod_{j=1}^{N} j!} = \sum_{k=1}^{N+1} (-1)^{k-1} \binom{N+1}{k} W_N(k)$$

and

$$\beta_N = \frac{\sum_{k=1}^{N+1} (-1)^{N+k+1} W_N(k) \dfrac{\prod_{j=1}^{N} j!}{(N+1-k)!(k-1)!}}{\prod_{j=1}^{N} j!}$$
$$= \frac{1}{N!} \sum_{k=1}^{N+1} (-1)^{N-k+1} \binom{N}{k-1} W_N(k).$$

This completes the proof. □

Proposition 2.2. *If P is a proper integral polyhedron in $\mathbb{R}^N$, then the Euler characteristic, $\chi(P)$, of P is given by*

$$\chi(P) = \sum_{k=1}^{N+1} (-1)^{k-1} \binom{N+1}{k} G_k. \tag{2.1}$$

Proof. In the Ehrhart polynomial (1.3) the Euler characteristic plays the same role as β_0 in the polynomial $W_N(x)$ from Lemma 2.1. Thus, putting $W_N(k) = G_k$, we have

$$\chi(P) = \sum_{k=1}^{N+1} (-1)^{k-1} \binom{N+1}{k} G_k$$

and the proof is complete. □

Remark 2.3. For $N = 2$ our formula (2.1) coincides (up to different notation and the fact that the Euler characteristic in [23] is defined to be $-\chi(P)$) with formula (11) given in concluding remarks of [23].

Remark 2.4. Clearly the numbers G_n remain the same for any cell decomposition of P. Therefore our formula (2.1) implies (in the case of proper lattice polyhedra) the well-known fact that the Euler characteristic of P does not depend on a cell decomposition of P.

Remark 2.5. It was shown in [12] that

$$\chi(\partial P) = \sum_{k=1}^{N} (-1)^{k-1} \binom{N}{k} B_k.$$

It is a well-known fact that for simple integral polyhedra in $\mathbb{R}^N$ we have

$$\chi(\partial P) = 1 + (-1)^{N+1}.$$

Thus, if P is a simple integral polyhedron in $\mathbb{R}^N$, then

$$\sum_{k=1}^{N} (-1)^{k-1} \binom{N}{k} B_k = \begin{cases} 0 & \text{if } N \text{ is even,} \\ 2 & \text{if } N \text{ is odd.} \end{cases}$$

It is easy to observe that in the case of non-simple proper lattice polyhedra the left side of the above equality can assume any integer value. It turns out, however, that for arbitrary (possibly non-simple) proper lattice polyhedra in $\mathbb{R}^N$ there exists a strong connection between the numbers of rational boundary points that involves, in addition, the number B_{N+1}.

Theorem 2.6. *If P is a proper integral polyhedron in $\mathbb{R}^N$, then*

$$V(P) = \frac{1}{N!} \sum_{k=1}^{N+1} (-1)^{N-k+1} \binom{N}{k-1} G_k.$$

Proof. In the Ehrhart polynomial (1.3) the volume of the polyhedron plays the same role as β_N in the polynomial $W_N(x)$ from Lemma 2.1. Thus, putting $W_N(k) = G_k$, we get the required formula. $\square$

It is clear that if P is an N-dimensional proper lattice polyhedron of the form (1.1), then its boundary ∂P is an $(N-1)$-dimensional proper lattice polyhedron of the form $\partial P = \bigcup_{j=1}^{t} \delta_j$, where each δ_j is a fundamental lattice simplex. It is also clear that $B_n(P) = G_n(\partial P)$. From the two observations it follows that in the case of ∂P polynomial (1.3) is of the form

$$B_n(P) = b_{N-1}(P)n^{N-1} + b_{N-2}(P)n^{N-2} + \cdots + b_1(P)n + \chi(\partial P), \tag{2.2}$$

where $b_k(P)$, $k = 1, \ldots, N-1$, are rational numbers and b_{N-1} is the sum of $(N-1)$-dimensional volumes of δ_j counted with relation to the sublattices of $\mathbb{Z}^N$ lying in $\operatorname{aff} \delta_j$ and spanned by δ_j.

Proposition 2.7. *Let P be a proper integral polyhedron in $\mathbb{R}^N$. Then*

$$\sum_{k=1}^{N+1} (-1)^{k-1} \binom{N}{k-1} B_k = 0.$$

Proof. We can look at the polynomial (2.2) as a polynomial $W_N(x)$ of degree N with the leading coefficient $\beta_N = 0$. Putting $W_N(k) = G_k$, we have by Lemma 2.1

$$0 = \beta_N = \frac{1}{N!} \sum_{k=1}^{N+1} (-1)^{N-k+1} \binom{N}{k-1} B_k = \sum_{k=1}^{N+1} (-1)^{k-1} \binom{N}{k-1} B_k$$

and the proof is complete. $\square$

A similar close relationship exists between the numbers $I_1, \ldots, I_{N+1}$ of a proper integral polyhedron in $\mathbb{R}^N$. Obviously, $I_n(P) = G_n(P) - B_n(P)$. In view of polynomials (1.3) and (2.2) we have now

$$I_n(P) = V(P)n^N + c_{N-1}(P)n^{N-1} + \cdots + c_1(P)n + \chi(P) - \chi(\partial P), \tag{2.3}$$

where $c_j(P) = a_j(P) - b_j(P)$, $j = 1, \ldots, N-1$.

Using now the polynomial (2.3) and proceeding in a similar way as in Proposition 2.2 and Theorem 2.6 we get the following three observations:

Proposition 2.8. *Let P be a proper integral polyhedron in $\mathbb{R}^N$. Then*

$$\sum_{k=1}^{N+1} (-1)^{k-1} \binom{N+1}{k} I_k = \chi(P) - \chi(\partial P). \tag{2.4}$$

Corollary 2.9. *If P is a simple integral polyhedron in $\mathbb{R}^N$, then*

$$\sum_{k=1}^{N+1} (-1)^{k-1} \binom{N+1}{k} I_k = (-1)^N.$$

Theorem 2.10. *If P is a proper integral polyhedron in $\mathbb{R}^N$, then*

$$V(P) = \frac{1}{N!} \sum_{k=1}^{N+1} (-1)^{N-k+1} \binom{N}{k-1} I_k.$$

We end this section with eliminating the Euler characteristics from Macdonald's formula (1.5). To the new formulae in Theorems 2.6 and 2.10 we add the following:

Theorem 2.11. *If P is a proper integral polyhedron in $\mathbb{R}^N$, then its volume $V(P)$ is given by*

$$V(P) = \frac{1}{(N+1)!} \sum_{k=1}^{N} (-1)^{N-k} \binom{N-1}{k-1} (B_k + 2I_k).$$

Proof. We start with finding a formula for $2\chi(P) - \chi(\partial P)$. From Propositions 2.2 and 2.8 we have

$$\begin{aligned} 2\chi(P) - \chi(\partial P) &= [\chi(P) - \chi(\partial P)] + \chi(P) \\ &= \sum_{k=1}^{N+1} (-1)^{k-1} \binom{N+1}{k} I_k + \sum_{k=1}^{N+1} (-1)^{k-1} \binom{N+1}{k} G_k \\ &= \sum_{k=1}^{N+1} (-1)^{k-1} \binom{N+1}{k} (B_k + 2I_k). \end{aligned}$$

Substituting this into (1.5) we get

$$(N-1)N!V(P) = \sum_{k=1}^{N-1} (-1)^{N-k-1} \left[\binom{N-1}{k} - \binom{N+1}{k} \right] (B_k + 2I_k)$$
$$+ (N+1)(B_N + 2I_N) - (B_{N+1} + 2I_{N+1}).$$

On the other hand, by Theorems 2.6 and 2.10 we have

$$2N!V(P) = \sum_{k=1}^{N-1} (-1)^{N-k+1} \binom{N}{k-1} G_k + \sum_{k=1}^{N-1} (-1)^{N-k+1} \binom{N}{k-1} I_k$$
$$= \sum_{k=1}^{N-1} (-1)^{N-k+1} \binom{N}{k-1} (B_k + 2I_k) - N(B_N + 2I_N)$$
$$+ (B_{N+1} + 2I_{N+1}).$$

Now by adding (side-by-side) the last two formulae we obtain

$$(N+1)N!V(P)$$
$$= \sum_{k=1}^{N-1} (-1)^{N-k-1} \left[\binom{N-1}{k} + \binom{N}{k-1} - \binom{N+1}{k} \right] (B_k + 2I_k)$$
$$+ (B_N + 2I_N),$$

which, by the easily verified identity

$$\binom{N-1}{k} + \binom{N}{k-1} - \binom{N+1}{k} = -\binom{N-1}{k-1}$$

reduces to the desired form

$$V(P) = \frac{1}{(N+1)!} \sum_{k=1}^{N} (-1)^{N-k} \binom{N-1}{k-1} (B_k + 2I_k)$$

and ends the proof. □

3. Pick-type formulae involving fewer lattice points

In this section we give formulae for the volume of lattice polyhedra which employ either fewer lattices or fewer lattice points than in the previously mentioned Pick-type results. We start with a first result of this type due to Ehrhart. In [6], see also [7], he obtained the following formula for the volume of lattice polyhedra in even-dimensional spaces $\mathbb{R}^N$.

$$V(P)=\frac{1}{N!}\left[\sum_{j=1}^{N/2}(-1)^{(N/2)-j}\binom{N}{(N/2)-j}\left(2I_j+B_j\right)\right.$$
$$\left.+(-1)^{(N/2)}\binom{N}{N/2}\chi(P)\right].$$

(The reader is warned that in both papers of Ehrhart the formula was misprinted.) This formula, employing only lattices $\mathbb{Z}_1^N,\ldots,\mathbb{Z}_{N/2}^N$, was compared by Ehrhart to Macdonald's formula (1.5) which uses lattices $\mathbb{Z}_1^N,\ldots,\mathbb{Z}_{N-1}^N$. Recently Kołodziejczyk [17] has pointed out that the formula cannot be applied to all proper lattice polyhedra in $\mathbb{R}^N$ but is valid for lattice polyhedra that are *N-dimensional manifolds* in even-dimensional spaces $\mathbb{R}^N$. Using a result from [20] Kołodziejczyk [17] also found the following formula for the volume of lattice polyhedra that are N-dimensional manifolds in odd-dimensional spaces $\mathbb{R}^N$:

$$V(P)=\frac{1}{(N+1)!}\sum_{j=1}^{(N+1)/2}(-1)^{(N+1)/2-j}\binom{N+1}{(N+1)/2-j}j(2I_j+B_j).$$

The idea of employing fewer lattice points was proposed in [13,15]. Instead of whole lattices we use only some subsets of them consisting of so called *odd points*. In $\mathbb{R}^3$, let l_m be the subset of $\mathbb{Z}_m^3$ consisting of points with all three coordinates being *odd* multiples of $1/m$. Thus

$$l_m=\left\{(x,y,z)\in\mathbb{Z}_m^3: x=\frac{2k+1}{m}, y=\frac{2t+1}{m}, z=\frac{2s+1}{m}\quad k,t,s\in\mathbb{Z}\right\}.$$

Geometrically, l_2 consists only of the centres of the unit cubes with vertices in $\mathbb{Z}^3$ which tile the whole space. If each of these unit cubes is divided into eight subcubes with vertices from $\mathbb{Z}_2^3$ then l_4 consists of only the centres of all such subcubes.

For a polyhedron P in $\mathbb{R}^3$ denote

$$i_m=i_m(P)=l_m\cap\operatorname{int}\,P\quad\text{and}\quad b_m=b_m(P)=l_m\cap\operatorname{relint}_2\,P.$$

Thus i_m and b_m are the numbers of points from l_m lying in the interior of P and in the relative interiors of the facets of P, respectively. In order to avoid any confusion we emphasize the fact that we do not count the points of l_m lying on the edges of P.

In [15] it was shown that if P is a proper lattice polyhedron in $\mathbb{R}^3$, and m and n are any even numbers, then

$$\frac{nm^3-n^3m}{4}V(P)=nb_m-mb_n+2(ni_m-mi_n). \tag{3.1}$$

A special case of formula (3.1) which is based only on the sets l_2 and l_4, both of which are included in $\mathbb{Z}_4^3$, appeared in [13].

Reeve [24] related the volume of a proper lattice polyhedron P in $\mathbb{R}^3$ to the sums of the solid angles subtended in P at the points of $\mathbb{Z}^3$ and $\mathbb{Z}_n^3$ as follows:

$$n(n^2-1)V(P) = \sum_{x\in\mathbb{Z}_n^3} \omega(x, P) - n \sum_{x\in\mathbb{Z}^3} \omega(x, P), \tag{3.2}$$

where $\omega(x, P)$ is defined as

$$\omega(x, P) = \lim_{\varepsilon\to 0} \frac{A(x, P, \varepsilon)}{4\pi\varepsilon^2}$$

and $A(x, P, \varepsilon)$ is the area of that part of the sphere of radius ε and centre x common to both the sphere and the polyhedron P. The following, analogical to (3.2), formula involving fewer lattice points was obtained in [15]:

$$\frac{mn^3 - nm^3}{8} V(P) = m \sum_{x\in l_n} \omega(x, P) - n \sum_{x\in l_m} \omega(x, P).$$

Here m and n are any even numbers.

4. Volume as a tiling property

Let $e_1, \ldots, e_N$ be the standard basis of $\mathbb{R}^N$ and let $e_{N+i} = -e_i$ for $i = 1, \ldots, N$. For every point $x \in \mathbb{Z}_n^N \cap \partial P$ define

$$E_i(x, P) = \begin{cases} -1 \text{ if } x + \varepsilon e_i \in \operatorname{int} P & \text{for all sufficiently small } \varepsilon > 0, \\ 1 \text{ if } x + \varepsilon e_i \notin P & \text{for all sufficiently small } \varepsilon > 0, \\ 0 \text{ if } x + \varepsilon e_i \in \partial P & \text{for all sufficiently small } \varepsilon > 0. \end{cases}$$

The *boundary characteristic* of a lattice polyhedron P in $\mathbb{R}^N$ is defined as follows:

$$C_n(P) = \sum_{x\in\mathbb{Z}_n^N \cap \partial P} \sum_{i=1}^{2N} E_i(x, P).$$

The boundary characteristic was originally introduced in [4] and next applied for polygons with vertices on different planar tilings in [11]. Similar application of the boundary characteristic to proper lattice polyhedra in $\mathbb{R}^3$ resulted in the following modification of Reeve's formula

$$2n(n^2-1)V(P) = B_n - nB_1 + 2(I_n - nI_1) + \tfrac{1}{6}[nC_1(P) - C_n(P)] \tag{4.1}$$

was obtained in [14]. Having this formula we can see that the volume is indeed a tiling property in the sense that it is given only in terms of parameters defined with a close relation to the tiling. A comparison of this equation with Reeve's formula (1.2) shows that the Euler characteristic and the boundary characteristic are connected in the following way:

$$nC_1(P) - C_n(P) = 6(n-1)[2\chi(P) - \chi(\partial P)].$$

This can be interpreted as a possibility of describing the Euler characteristics of a proper lattice polyhedron by means of the standard basis of $\mathbb{R}^3$.

5. Volume of non-proper lattice polyhedra

In this section we deal with lattice polyhedra that are unions of simplexes, not necessarily N-dimensional. An $(N-1)$-dimensional fundamental lattice simplex Δ is called *two-sided* in a lattice polyhedron P if $\operatorname{relint}\Delta \cap \operatorname{cl}\operatorname{int} P = \emptyset$. Generalizing Pick's theorem to non-proper lattice polygons, Hadwiger and Wills [10] (see also [8]) proved that the area, $A(P)$, of a lattice polygon P in $\mathbb{R}^2$ is given by the following formula:

$$2A(P) = 2G_1(P) - 2\chi(P) - |\Delta^1(P)|$$

in which $|\Delta^1(P)|$ denotes the number of fundamental lattice segments on ∂P with two-sided segments contributing twice to $|\Delta^1(P)|$.

The idea of employing two-sided simplexes was considered in [16] where it was shown that the volume of non-proper lattice polyhedra in spaces $\mathbb{R}^N$, for $N \leqslant 4$, may be determined by

$$(N-1)N!V(P) = 2\sum_{k=1}^{N-1}(-1)^{k-1}\binom{N-1}{k-1}G_{N-k} + (-1)^{N-1}2\chi(P) - \sum_{\Delta^{N-1}\subset\partial P}(|\Delta^{N-1}_{N-2}| + 1)$$

and the sum extends over all $(N-1)$-dimensional fundamental lattice tetrahedra on ∂P. Here $|\Delta^{N-1}_{N-2}|$ denotes the number of lattice points from $\mathbb{Z}^{N-1}_{N-2}$ lying in the relative interior of a fundamental lattice tetrahedron Δ^{N-1} from ∂P and any two-sided tetrahedron contributes to the sum twice. (By definition $|\Delta^1_0| = 0$, and obviously $|\Delta^2_1| = 0$.) One can see that the latter formula is a generalization of Hadwiger–Wills' formula and coincides with it when $N = 2$.

6. Comments and open questions

6.1. In [15] it is shown that the numbers $g_n = i_n + b_n$ and $d_n = i_n + (1/2)b_n$ of odd points in a three-dimensional lattice polyhedron P are given by some polynomials of the same degree as the Ehrhart polynomial (1.3) but with different coefficients. It would be interesting to show that the numbers of an arbitrary dimensional lattice polyhedron also possess similar polynomiality properties.

One of the most interesting properties of the Ehrhart polynomials is the so-called *reciprocity law* which says that

$$I_n(P) = (-1)^N G_{-n}(P),$$

where $G_{-n}(P) = V(P)(-n)^N + a_{N-1}(-n)^{N-1} + \cdots + a_1(-n) + \chi(P)$ and the coefficients $a_{N-1}, \ldots, a_1$ are the same as in formula (1.3). Do we have a similar property for the numbers i_n and g_n?

6.2. Use positive answers to the questions in **6.1** to find *N*-dimensional analogues of formula (3.1). To this end show that the formulae for the volume of a lattice polyhedron P in $\mathbb{R}^N$ based on odd points and conjectured in [15] are true. Be aware that if it happens that a variant of the reciprocity law holds for odd points, it will be possible to give another formula involving fewer sets $l_m \subset \mathbb{R}^N$ of odd points than the formulae conjectured in [15].

6.3. Is it possible to give an arbitrary dimensional analogue of formula (4.1) for the volume of proper lattice polyhedra in which the Euler characteristics part would be replaced by an expression with the boundary characteristics?

6.4. Find an *N*-dimensional Hadwiger–Wills type generalization of Pick's theorem applicable to arbitrary (non-proper) lattice polyhedra.

6.5. Find algorithms for computing the numbers $I_n(P)$, $G_n(P)$ and $\chi(P)$ of a proper integral polyhedron P.

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