Computers and Mathematics with Applications 59 (2010) 3110–3116

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative

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ARTICLE INFO

Article history:
Received 2 October 2009
Received in revised form 19 February 2010
Accepted 19 February 2010

Keywords:
Fractional Euler–Lagrange equation
Fractional derivative
Caputo derivative

ABSTRACT

This paper presents the necessary and sufficient optimality conditions for problems of the fractional calculus of variations with a Lagrangian depending on the free end-points. The fractional derivatives are defined in the sense of Caputo.

1. Introduction

Fractional calculus is one of the generalizations of classical calculus and it has been used successfully in various fields of science and engineering; see, e.g., [1–12]. In recent years, there has been a growing interest in the area of fractional variational calculus and its applications [13–28]. Applications include classical and quantum mechanics, field theory, and optimal control. In the papers cited above, the problems have been formulated mostly in terms of two types of fractional derivative, namely Riemann–Liouville and Caputo. The natural boundary conditions for fractional variational problems, in terms of the Riemann–Liouville and the Caputo derivatives, are presented in [13,14]. Here we develop the theory further by proving the necessary optimality conditions for more general problems of the fractional calculus of variations with a Lagrangian that may also depend on the unspecified end-points $y(a)$, $y(b)$. More precisely, the problem under our study consists in minimizing a functional which is defined in terms of the Caputo derivative and having no constraint on $y(a)$ and/or $y(b)$. The novelty is the dependence of the integrand L on the free end-points $y(a)$, $y(b)$. This class of problems is motivated by applications in the field of economics [29].

The paper is organized as follows. Section 2 presents the necessary definitions and concepts of the fractional calculus; our results are formulated, proved, and illustrated through an example in Section 3. The main results of the paper include necessary optimality conditions with the new natural boundary conditions (Theorem 3.1) that become sufficient under appropriate convexity assumptions (Theorem 3.3).

2. Fractional calculus

In this section we review the necessary definitions and facts from fractional calculus. For more on the subject we refer the reader to [30–33].

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doi:10.1016/j.camwa.2010.02.032

Let $f \in L_1([a, b])$ and $0 < \alpha < 1$. We begin with the left and the right Riemann–Liouville fractional integrals (RLFIs) of order α of function f which are defined as follows. The left RLFI is

$$ {}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad x \in [a, b], \tag{1}$$

and the right RLFI is

$$ {}_xI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \quad x \in [a, b], \tag{2}$$

where $\Gamma(\cdot)$ represents the Gamma function. Moreover, ${}_aI_x^0 f = {}_xI_b^0 f = f$ if f is a continuous function.

Let $f \in AC([a, b])$, where $AC([a, b])$ represents the space of absolutely continuous functions on $[a, b]$. Then, using Eqs. (1) and (2), the left and the right Riemann–Liouville and Caputo derivatives are defined as follows. The left Riemann–Liouville fractional derivative (RLFD) is

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x (x-t)^{-\alpha} f(t)\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x} {}_aI_x^{1-\alpha} f(x), \quad x \in [a, b], \tag{3}$$

the right RLFD is

$$ {}_xD_b^\alpha f(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^b (t-x)^{-\alpha} f(t)\mathrm{d}t = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right) {}_xI_b^{1-\alpha} f(x), \quad x \in [a, b], \tag{4}$$

the left Caputo fractional derivative (CFD) is

$$ {}_a^CD_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} f(t)\mathrm{d}t = {}_aI_x^{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}x} f(x), \quad x \in [a, b], \tag{5}$$

and the right CFD is

$$ {}_xD_b^\alpha f(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^b (t-x)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} f(t)\mathrm{d}t = {}_xI_b^{1-\alpha} \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x), \quad x \in [a, b], \tag{6}$$

where α is the order of the derivative.

The operators (1)–(6) are obviously linear. We now present the rule of fractional integration by parts for RLFIs (see for instance [34]). Let $0 < \alpha < 1$, $p \geq 1$, $q \geq 1$, and $1/p + 1/q \leq 1 + \alpha$. If $g \in L_p([a, b])$ and $f \in L_q([a, b])$, then

$$ \int_a^b g(x) {}_aI_x^\alpha f(x)\mathrm{d}x = \int_a^b f(x) {}_xI_b^\alpha g(x)\mathrm{d}x. \tag{7}$$

In the discussion to follow, we will also need the following formulae for fractional integrations by parts:

$$ \begin{aligned} \int_a^b g(x)\, {}_a^CD_x^\alpha f(x)\mathrm{d}x &= f(x) {}_xI_b^{1-\alpha} g(x)\Big|_{x=a}^{x=b} + \int_a^b f(x) {}_xD_b^\alpha g(x)\mathrm{d}x, \\ \int_a^b g(x)\, {}_x^CD_b^\alpha f(x)\mathrm{d}x &= -f(x) {}_aI_x^{1-\alpha} g(x)\Big|_{x=a}^{x=b} + \int_a^b f(x) {}_aD_x^\alpha g(x)\mathrm{d}x. \end{aligned} \tag{8}$$

They can be derived using Eqs. (3)–(6), the identity (7) and performing integration by parts.

3. Main results

Let us consider the following problem:

$$ \begin{aligned} &\mathcal{J}(y) = \int_a^b L(x, y(x), {}_a^CD_x^\alpha y(x), {}_x^CD_b^\beta y(x), y(a), y(b))\, \mathrm{d}x \longrightarrow \text{extr} \\ &(y(a) = y_a), \qquad (y(b) = y_b). \end{aligned} \tag{9}$$

Using parentheses around the end-point conditions means that the conditions may or may not be present. We assume that:

(i) $L(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot) \in C^1([a, b] \times \mathbb{R}^5; \mathbb{R})$;

(ii) $x \to \partial_3 L(x, y(x), {}_a^CD_x^\alpha y(x), {}_x^CD_b^\beta y(x), y(a), y(b))$ has continuous right RLFI of order $1-\alpha$ and right RLFD of order α, where $\alpha \in (0, 1)$;

(iii) $x \to \partial_4 L(x, y(x), {}_a^CD_x^\alpha y(x), {}_x^CD_b^\beta y(x), y(a), y(b))$ has continuous left RLFI of order $1-\beta$ and left RLFD of order β, where $\beta \in (0, 1)$.

Remark 1. We are assuming that the admissible functions y are such that ${}_a^CD_x^\alpha y(x)$ and ${}_x^CD_b^\beta y(x)$ exist on the closed interval $[a, b]$.

In this work we denote by $\partial_i L$, $i = 1, \ldots, 6$, the partial derivative of function $L(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot)$ with respect to its ith argument.

3.1. Necessary optimality conditions

The next theorem gives the necessary optimality conditions for the problem (9).

Theorem 3.1. *Let y be a local extremizer to problem (9). Then, y satisfies the fractional Euler–Lagrange equation*

$$\begin{aligned}\partial_2 L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x), y(a), y(b)) + {}_x D_b^\alpha \partial_3 L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x), y(a), y(b)) \\ + {}_a D_x^\beta \partial_4 L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x), y(a), y(b)) = 0 \end{aligned} \tag{10}$$

for all $x \in [a, b]$. Moreover, if $y(a)$ is not specified, then

$$\begin{aligned}\int_a^b \partial_5 L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x), y(a), y(b))\, \mathrm{d}x = \Big[{}_x I_b^{1-\alpha} \partial_3 L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x), y(a), y(b)) \\ - {}_a I_x^{1-\beta} \partial_4 L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x), y(a), y(b)) \Big]\Big|_{x=a}; \end{aligned} \tag{11}$$

and if $y(b)$ is not specified, then

$$\begin{aligned}\int_a^b \partial_6 L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x), y(a), y(b))\, \mathrm{d}x = \Big[{}_a I_x^{1-\beta} \partial_4 L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x), y(a), y(b)) \\ - {}_x I_b^{1-\alpha} \partial_3 L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x), y(a), y(b)) \Big]\Big|_{x=b}. \end{aligned} \tag{12}$$

Proof. Suppose that y is an extremizer of $\mathcal{J}$. We can proceed as Lagrange did, by considering the value of $\mathcal{J}$ at a nearby function $\tilde{y} = y + \varepsilon h$, where $\varepsilon \in \mathbb{R}$ is a small parameter, and h is an arbitrary admissible variation. We do not require $h(a) = 0$ or $h(b) = 0$ in the case when $y(a)$ or $y(b)$, respectively, is free (it is possible that both are free). Let

$$\varphi(\varepsilon) = \int_a^b L(x, y(x) + \varepsilon h(x), {}_a^C D_x^\alpha (y(x) + \varepsilon h(x)), {}_x^C D_b^\beta (y(x) + \varepsilon h(x)), y(a) + \varepsilon h(a), y(b) + \varepsilon h(b))\, \mathrm{d}x.$$

Since ${}_a^C D_x^\alpha$ and ${}_x^C D_b^\beta$ are linear operators, it follows that

$$\begin{aligned} {}_a^C D_x^\alpha (y(x) + \varepsilon h(x)) &= {}_a^C D_x^\alpha y(x) + \varepsilon {}_a^C D_x^\alpha h(x) \\ {}_x^C D_b^\beta (y(x) + \varepsilon h(x)) &= {}_x^C D_b^\beta y(x) + \varepsilon {}_x^C D_b^\beta h(x). \end{aligned}$$

A necessary condition for y to be an extremizer is given by

$$\begin{aligned} &\left.\frac{\mathrm{d}\varphi}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = 0 \\ &\Leftrightarrow \int_a^b \Big[\partial_2 L(\cdots) h(x) + \partial_3 L(\cdots) {}_a^C D_x^\alpha h(x) + \partial_4 L(\cdots) {}_x^C D_b^\beta h(x) + \partial_5 L(\cdots) h(a) + \partial_6 L(\cdots) h(b) \Big] \mathrm{d}x = 0, \end{aligned} \tag{13}$$

where $(\cdots) = \left(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x), y(a), y(b)\right)$. Using formulae (8) for integration by parts, the second and the third integrals can be written as

$$\begin{aligned} \int_a^b \partial_3 L(\cdots) {}_a^C D_x^\alpha h(x) \mathrm{d}x &= \int_a^b h(x) {}_x D_b^\alpha \partial_3 L(\cdots) \mathrm{d}x + {}_x I_b^{1-\alpha} \partial_3 L(\cdots) h(x) \Big|_{x=a}^{x=b}, \\ \int_a^b \partial_4 L(\cdots) {}_x^C D_b^\beta h(x) \mathrm{d}x &= \int_a^b h(x) {}_a D_x^\beta \partial_4 L(\cdots) \mathrm{d}x - {}_a I_x^{1-\beta} \partial_4 L(\cdots) h(x) \Big|_{x=a}^{x=b}. \end{aligned} \tag{14}$$

Substituting (14) into (13), we get

$$\begin{aligned} \int_a^b \left[\partial_2 L(\cdots) + {}_x D_b^\alpha \partial_3 L(\cdots) + {}_a D_x^\beta \partial_4 L(\cdots) \right] h(x) + {}_x I_b^{1-\alpha} \partial_3 L(\cdots) h(x) \Big|_{x=a}^{x=b} - {}_a I_x^{1-\beta} \partial_4 L(\cdots) h(x) \Big|_{x=a}^{x=b} \\ + \int_a^b (\partial_5 L(\cdots) h(a) + \partial_6 L(\cdots) h(b))\, \mathrm{d}x = 0. \end{aligned} \tag{15}$$

We first consider functions $h(t)$ such that $h(a) = h(b) = 0$. Then, by the fundamental lemma of the calculus of variations, we deduce that

$$\partial_2 L(\cdots) + {}_x D_b^\alpha \partial_3 L(\cdots) + {}_a D_x^\beta \partial_4 L(\cdots) = 0 \tag{16}$$

for all $x \in [a, b]$. Therefore, in order for y to be an extremizer to the problem (9), y must be a solution of the fractional Euler–Lagrange equation. But if y is a solution of (16), the first integral in expression (15) vanishes, and then the condition (13) takes the form

$$h(b)\left\{\int_a^b \partial_6 L(\cdots)\mathrm{d}x - \left[{}_aI_x^{1-\beta}\partial_4 L(\cdots) - {}_xI_b^{1-\alpha}\partial_3 L(\cdots)\right]\Big|_{x=b}\right\}$$
$$+\,h(a)\left\{\int_a^b \partial_5 L(\cdots)\mathrm{d}x - \left[{}_xI_b^{1-\alpha}\partial_3 L(\cdots) - {}_aI_x^{1-\beta}\partial_4 L(\cdots)\right]\Big|_{x=a}\right\} = 0.$$

If $y(a) = y_a$ and $y(b) = y_b$ are given in the formulation of problem (9), then the latter equation is trivially satisfied since $h(a) = h(b) = 0$. When $y(a)$ is free, then

$$\int_a^b \partial_5 L(\cdots)\mathrm{d}x - \left[{}_xI_b^{1-\alpha}\partial_3 L(\cdots) - {}_aI_x^{1-\beta}\partial_4 L(\cdots)\right]\Big|_{x=a} = 0,$$

and when $y(b)$ is free, then

$$\int_a^b \partial_6 L(\cdots)\mathrm{d}x - \left[{}_aI_x^{1-\beta}\partial_4 L(\cdots) - {}_xI_b^{1-\alpha}\partial_3 L(\cdots)\right]\Big|_{x=b} = 0,$$

since $h(a)$ or $h(b)$ is, respectively, arbitrary. □

Remark 2. Conditions (10)–(12) are only necessary for an extremum. The question of sufficient conditions for an extremum is considered in Section 3.2.

In the case when L does not depend on $y(a)$ and $y(b)$, by Theorem 3.1 we obtain the following result.

Corollary 1 (*Theorem 1 of [14]*). *If y is a local extremizer to problem*

$$\mathcal{J}(y) = \int_a^b L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x))\,\mathrm{d}x \longrightarrow \mathrm{extr},$$

then y satisfies the fractional Euler–Lagrange equation

$$\partial_2 L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x)) + {}_xD_b^\alpha \partial_3 L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x)) + {}_aD_x^\beta \partial_4 L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x)) = 0$$

for all $x \in [a, b]$. Moreover, if $y(a)$ is not specified, then

$$\left[{}_xI_b^{1-\alpha}\partial_3 L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x)) - {}_aI_x^{1-\beta}\partial_4 L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x))\right]\Big|_{x=a} = 0,$$

and if $y(b)$ is not specified, then

$$\left[{}_aI_x^{1-\beta}\partial_4 L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x)) - {}_xI_b^{1-\alpha}\partial_3 L(x, y(x), {}_a^C D_x^\alpha y(x), {}_x^C D_b^\beta y(x))\right]\Big|_{x=b} = 0.$$

We note that the generalized Euler–Lagrange equation contains both the left and the right fractional derivatives. The generalized natural conditions also contain the left and the right fractional integrals. Although the functional has been written only in terms of the CFDs, the necessary conditions (10)–(12) contain Caputo fractional derivatives, Riemann–Liouville fractional derivatives and Riemann–Liouville fractional integrals.

Observe that if α goes to 1, then the operators ${}_a^C D_x^\alpha$ and ${}_aD_x^\alpha$ can be replaced with $\frac{\mathrm{d}}{\mathrm{d}x}$ and the operators ${}_x^C D_b^\alpha$ and ${}_xD_b^\alpha$ can be replaced with $-\frac{\mathrm{d}}{\mathrm{d}x}$ (see [32]). Thus, if the ${}_x^C D_b^\beta y$ term is not present in (9), then for $\alpha \to 1$ we obtain a corresponding result in the classical context of the calculus of variations [29] (see also [35, Corollary 1]).

Corollary 2. *If y is a local extremizer for*

$$\mathcal{J}(y) = \int_a^b L(x, y(x), y'(x), y(a), y(b))\,\mathrm{d}x \longrightarrow \mathrm{extr}$$
$$(y(a) = y_a), \qquad (y(b) = y_b),$$

then

$$\frac{\mathrm{d}}{\mathrm{d}x}\partial_3 L(x, y(x), y'(x), y(a), y(b)) = \partial_2 L(x, y(x), y'(x), y(a), y(b))$$

for all $x \in [a, b]$. Moreover, if $y(a)$ is free, then

$$\partial_3 L(a, y(a), y'(a), y(a), y(b)) = \int_a^b \partial_5 L(x, y(x), y'(x), y(a), y(b))\mathrm{d}x;$$

if $y(b)$ *is free, then*

$$\partial_3 L(b, y(b), y'(b), y(a), y(b)) = -\int_a^b \partial_6 L(x, y(x), y'(x), y(a), y(b))\mathrm{d}x.$$

3.2. Sufficient conditions

In this section we prove the sufficient conditions that ensure the existence of a minimum (maximum). Similarly to what happens in the classical calculus of variations, some conditions of convexity (concavity) are in order.

Definition 3.2. Given a function L, we say that $L(\underline{x}, y, z, t, u, v)$ is jointly convex (concave) in (y, z, t, u, v), if $\partial_i L$, $i = 2, \ldots, 6$, exist and are continuous and verify the following condition:

$$\begin{aligned} &L(x, y + y_1, z + z_1, t + t_1, u + u_1, v + v_1) - L(x, y, z, t, u, v) \\ &\quad \geq (\leq) \partial_2 L(\bullet) y_1 + \partial_3 L(\bullet) z_1 + \partial_4 L(\bullet) t_1 + \partial_5 L(\bullet) u_1 + \partial_6 L(\bullet) v_1 \end{aligned}$$

for all (x, y, z, t, u, v), $(x, y + y_1, z + z_1, t + t_1, u + u_1, v + v_1) \in [a, b] \times \mathbb{R}^5$, where $(\bullet) = (x, y, z, t, u, v)$.

Theorem 3.3. *Let* $L(\underline{x}, y, z, t, u, v)$ *be jointly convex (concave) in* (y, z, t, u, v). *If* y_0 *satisfies conditions (10)–(12), then* y_0 *is a global minimizer (maximizer) to problem (9).*

Proof. We shall give the proof for the convex case. Since L is jointly convex in (y, z, t, u, v) for any admissible function $y_0 + h$, we have

$$\begin{aligned} \mathcal{J}(y_0 + h) - \mathcal{J}(y_0) &= \int_a^b \Big[L(x, y_0(x) + h(x), {}_a^C D_x^\alpha (y_0(x) + h(x)), {}_x^C D_b^\beta (y_0(x) + h(x)), y_0(a) + h(a), y_0(b) + h(b)) \\ &\quad - L(x, y_0(x), {}_a^C D_x^\alpha y_0(x), {}_x^C D_b^\beta y_0(x), y_0(a), y_0(b)) \Big] \mathrm{d}x \\ &\geq \int_a^b \Big[\partial_2 L(\star) h(x) + \partial_3 L(\star) {}_a^C D_x^\alpha h(x) + \partial_4 L(\star) {}_x^C D_b^\beta h(x) + \partial_5 L(\star) h(a) + \partial_6 L(\star) h(b) \Big] \mathrm{d}x, \end{aligned}$$

where $(\star) = \Big(x, y_0(x), {}_a^C D_x^\alpha y_0(x), {}_x^C D_b^\beta y_0(x), y_0(a), y_0(b) \Big)$. We can now proceed analogously to the proof of Theorem 3.1. As the result we get

$$\begin{aligned} \mathcal{J}(y_0 + h) - \mathcal{J}(y_0) &\geq \int_a^b \left[\partial_2 L(\star) + {}_x D_b^\alpha \partial_3 L(\star) + {}_a D_x^\beta \partial_4 L(\star) \right] h(x) \\ &\quad + h(b) \left\{ \int_a^b \partial_6 L(\star) \mathrm{d}x - \left[{}_a I_x^{1-\beta} \partial_4 L(\star) - {}_x I_b^{1-\alpha} \partial_3 L(\star) \right]\Big|_{x=b} \right\} \\ &\quad + h(a) \left\{ \int_a^b \partial_5 L(\star) \mathrm{d}x - \left[{}_x I_b^{1-\alpha} \partial_3 L(\star) - {}_a I_x^{1-\beta} \partial_4 L(\star) \right]\Big|_{x=a} \right\} = 0. \end{aligned}$$

Since y_0 satisfies conditions (10)–(12), we obtain $\mathcal{J}(y_0 + h) - \mathcal{J}(y_0) \geq 0$. $\square$

3.3. Example

We shall provide an example in order to illustrate our main results.

Example 1. Consider the following problem:

$$\mathcal{J}(y) = \frac{1}{2} \int_0^1 \left[\left({}_0^C D_x^\alpha y(x) \right)^2 + \gamma y^2(0) + \lambda (y(1) - 1)^2 \right] \mathrm{d}x \longrightarrow \min \tag{17}$$

where $\gamma, \lambda \in \mathbb{R}^+$. For this problem, the generalized Euler–Lagrange equation and the natural boundary conditions (see Theorem 3.1) are given, respectively, as

$${}_x D_1^\alpha \left({}_0^C D_x^\alpha y(x) \right) = 0, \tag{18}$$

$$\int_0^1 \gamma y(0) \mathrm{d}x = {}_x I_1^{1-\alpha} \left({}_0^C D_x^\alpha y(x) \right)\Big|_{x=0}, \tag{19}$$

$$\int_0^1 \lambda (y(1) - 1) \mathrm{d}x = -{}_x I_1^{1-\alpha} \left({}_0^C D_x^\alpha y(x) \right)\Big|_{x=1}. \tag{20}$$

Note that it is difficult to solve the above fractional equations. For $0 < \alpha < 1$, a numerical method should be used. When α goes to 1, problem (17) tends to

$$\mathcal{J}(y) = \frac{1}{2}\int_0^1 \left[(y'(x))^2 + \gamma y^2(0) + \lambda(y(1)-1)^2\right] \mathrm{d}x \longrightarrow \min \tag{21}$$

and Eqs. (18)–(20) could be replaced with

$$y''(x) = 0, \tag{22}$$

$$\gamma y(0) = y'(0), \tag{23}$$

$$\lambda(y(1)-1) = -y'(1). \tag{24}$$

Solving Eqs. (22)–(24), we obtain that

$$\bar{y}(x) = \frac{\gamma\lambda}{\gamma\lambda+\lambda+\gamma}x + \frac{\lambda}{\gamma\lambda+\lambda+\gamma}$$

is a candidate for a minimizer. Observe that problem (17) satisfies the assumptions of Theorem 3.3. Therefore $\bar{y}$ is a global minimizer to problem (21).

Acknowledgements

The authors were partially supported by the R&D unit CEOC, via FCT and the EC fund FEDER/POCI 2010, and the research project UTAustin/MAT/0057/2008. The first author was also supported by Białystok University of Technology, via a project of the Polish Ministry of Science and Higher Education "Wsparcie miedzynarodowej mobilnosci naukowcow".

We are grateful to two anonymous reviewers for their comments.

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