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The roots of a split quaternion

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ABSTRACT

In this work, we express De Moivre's formula for split quaternions and find roots of a split quaternion using this formula.

1. Introduction

The roots of a quaternion were given by Niven [7] and Brand [1]. Brand proved De Moivre's theorem and used it to find nth roots of a quaternion. Using De Moivre's formula to find roots of a quaternion is more useful way. Euler's and De Moivre's formulas are important for quaternions since every unit quaternion q can be written in the form $q = \cos\theta + \vec{\varepsilon}_0 \sin\theta$ and represents a rotation of 2θ about the axis $\vec{\varepsilon}_0$ in Euclidean 3-space. See [2–4] for information about these formulas for quaternions. In this work, we express Euler and De Moivre's formulas for split quaternions and examine roots of a split quaternion with respect to the causal character of the split quaternion.

Split quaternion algebra is an associative, non-commutative non-division ring with four basic elements $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfying the equalities $\mathbf{i}^2 = -1, \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i} * \mathbf{j} * \mathbf{k} = 1$. Rotations in Minkowski 3-space can be stated with split quaternions, such as expressing Euclidean rotations using quaternions. For detailed information about split quaternions, we refer the reader to Ref. [6,5,8,9]. Split quaternions $\widehat{\mathbb{H}}$ are identified with the semi-Euclidean space $\mathbb{E}_2^4$. Besides this, the subspace of $\widehat{\mathbb{H}}$ consisting of pure split quaternions $\widehat{\mathbb{H}}_0$ is identified with the Minkowski 3-space [5]. Thus, it is possible to do with split quaternions many of the things one ordinarily does in vector analysis by using Lorentzian inner and vector products.

2. Preliminaries

It is recalled that a split quaternion $q = (q_1, q_2, q_3, q_4)$ is spacelike, timelike or lightlike, if $I_q < 0$, $I_q > 0$ or $I_q = 0$ respectively where $I_q = q_1^2 + q_2^2 - q_3^2 - q_4^2$. The norm of the q is defined as $N_q = \sqrt{|q_1^2 + q_2^2 - q_3^2 - q_4^2|}$. If $N_q = 1$ then q is called the unit split quaternion and $q_0 = q/N_q$ is a unit split quaternion for $N_q \neq 0$. Also, spacelike and timelike quaternions have multiplicative inverses having the property $q * q^{-1} = q^{-1} * q = 1$. Lightlike quaternions have no inverse. Polar forms of the split quaternions are as follows:

(i) Every spacelike quaternion can be written in the form $q = N_q\,(\sinh\theta + \vec{\varepsilon}_0 \cosh\theta)$ where $\vec{\varepsilon}_0 =$ is a spacelike unit vector in $\mathbb{E}_1^3$.
(ii) Every timelike quaternion with spacelike vector part can be written in the form $q = N_q\,(\cosh\theta + \vec{\varepsilon}_0 \sinh\theta)$ where $\vec{\varepsilon}_0$ is a spacelike unit vector in $\mathbb{E}_1^3$.

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(iii) Every timelike quaternion with timelike vector part can be written in the form $q = N_q(\cos\theta + \vec{\varepsilon}_0 \sin\theta)$ where $\vec{\varepsilon}_0$ is a timelike unit vector in $\mathbb{E}_1^3$.

Scalar and vector parts of the split quaternion q are denoted by $Sq = q_1$ and $\vec{V}q = q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$ respectively. Vector parts of the split quaternions are identified with the Minkowski 3-space. The Minkowski space $\mathbb{E}_1^3$ is the Euclidean space $\mathbb{E}^3$ provided with the Lorentzian inner product $\langle\vec{u}, \vec{v}\rangle_L = -u_1v_1 + u_2v_2 + u_1v_2$ where $\vec{u} = (u_1, u_1, u_3)$, $\vec{v} = (v_1, v_2, v_3) \in \mathbb{E}^3$. We say that a vector $\vec{u}$ in E_1^3 is spacelike, lightlike or timelike if $\langle\vec{u}, \vec{u}\rangle_L > 0$, $\langle\vec{u}, \vec{u}\rangle_L = 0$ or $\langle\vec{u}, \vec{u}\rangle_L < 0$ respectively. The norm of the vector $\vec{u} \in \mathbb{E}_1^3$ is defined by $\|\vec{u}\| = \sqrt{|\langle\vec{u}, \vec{u}\rangle_L|}$.

The split quaternion product of two quaternions $p = (p_1, p_2, p_3, p_4)$ and $q = (q_1, q_2, q_3, q_4)$ is defined as

$$p * q = p_1q_1 + < \vec{V}p, \vec{V}q >_L + p_1\vec{V}q + q_1\vec{V}p + \vec{V}p \times_L \vec{V}q$$

where $\langle, \rangle_L$ and $\times_L$ are the Lorentzian inner product and vector product respectively. Also, we can express an nth power of a split quaternion $q = Sq + \vec{V}q$ as follows:

(i) if n is an even number,

$$q^n = \left[\sum_{r=0}^{\frac{n}{2}} \binom{n}{2r} (S)^{n-2r} V^r\right] + \left[\sum_{r=0}^{\frac{n}{2}-1} \binom{n}{2r+1} (S)^{n-2r-1} V^r\right] \vec{V}q,$$

(ii) if n is an odd number,

$$q^n = \left[\sum_{r=0}^{\frac{n-1}{2}} \binom{n}{2r} (S)^{n-2r} V^r\right] + \left[\sum_{r=0}^{\frac{n-1}{2}} \binom{n}{2r+1} (S)^{n-2r-1} V^r\right] \vec{V}q$$

where $S = Sq$ and $V = \left\langle\vec{V}q, \vec{V}q\right\rangle_L$.

The set of timelike quaternions denoted by

$$\mathbb{T}\widehat{\mathbb{H}} = \{q = (q_1, q_2, q_3, q_4) : q_2, q_3, q_4, q_1 \in R,\ I_q > 0\}$$

forms a group under the split quaternion product. Every rotation in the Minkowski 3-space can be expressed using unit timelike quaternions. The set of unit timelike quaternions is represented as $\mathbb{T}\widehat{\mathbb{H}}_1$. If $q = (q_1, q_2, q_3, q_4)$ is a unit timelike quaternion, using the transformation law $(q * \vec{V}r * q^{-1})_i = \sum_{j=1}^{3} R_{ij}(\vec{V}r)_j$ the corresponding rotation matrix can be found as

$$R_q = \begin{bmatrix} q_1^2 + q_2^2 + q_3^2 + q_4^2 & 2q_1q_4 - 2q_2q_3 & -2q_1q_3 - 2q_2q_4 \\ 2q_2q_3 + 2q_4q_1 & q_1^2 - q_2^2 - q_3^2 + q_4^2 & -2q_3q_4 - 2q_2q_1 \\ 2q_2q_4 - 2q_3q_1 & 2q_2q_1 - 2q_3q_4 & q_1^2 - q_2^2 + q_3^2 - q_4^2 \end{bmatrix}$$

where $r = (Sr, \vec{V}r)$. These matrices form the three-dimensional special orthogonal group $SO(1, 2)$. Moreover, the function $\varphi : S_2^3 \simeq \mathbb{T}\widehat{\mathbb{H}}_1 \to SO(1, 2)$ which sends $q = (q_1, q_2, q_3, q_4)$ to matrix R given in (*EMPTY*) is a homomorphism of group. The kernel of φ is $\{\pm 1\}$ so that rotation matrix corresponds to pairs $\pm q$ of the unit quaternion. In particular, $SO(1, 2)$ is isomorphic to the quotient group $\mathbb{T}\widehat{\mathbb{H}}_1/\{\pm 1\}$ from the first isomorphism theorem. In another words, for every rotation in Minkowski 3-space $\mathbb{E}_1^3$, there are two unit timelike quaternions that determine this rotation. These timelike quaternions are q and $-q$ [9].

3. De Moivre's formula for split quaternions

In this section we examine De Moivre's formula for split quaternions. For this, we consider the causal character of the split quaternion and we specify this formula with respect to timelike and spacelike quaternions separately. For the usual quaternions this formula was given by [2].

3.1. Timelike quaternions with spacelike vector part ($\mathbb{T}_{Sp}$)

Every timelike quaternion with spacelike vector part can be written in the form

$$q = N_q(\cosh\theta + \vec{\varepsilon}_0 \sinh\theta)$$

where $\cosh\theta = \frac{|q_1|}{r}$, $\sinh\theta = \frac{\sqrt{-q_2^2+q_3^2+q_4^2}}{r}$, $\vec{\varepsilon}_0 = \frac{q_2\mathbf{i}+q_3\mathbf{j}+q_4\mathbf{k}}{\sqrt{-q_2^2+q_3^2+q_4^2}}$ is a spacelike unit vector in $\mathbb{E}_1^3$ and $\vec{\varepsilon}_0 * \vec{\varepsilon}_0 = 1$. A unit timelike quaternion q with spacelike vector part represents a rotation of a three-dimensional non-lightlike Lorentzian vector by a hyperbolic angle 2θ about the axis of q [9].

Euler's formula for a unit timelike quaternion with spacelike vector part holds. Since $\vec{\varepsilon}^2 = \vec{\varepsilon} * \vec{\varepsilon} = 1$, we have

$$e^{\vec{\varepsilon}\theta} = \left(1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + \vec{\varepsilon}\left(\theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) = \cosh\theta + \vec{\varepsilon}\sinh\theta.$$

Moreover, this can be shown using another method. That is,

$$q = \cosh\theta + \vec{\varepsilon}\sinh\theta \Rightarrow \mathrm{d}q = (\sinh\theta + \vec{\varepsilon}\cosh\theta)\,\mathrm{d}\theta$$
$$\Rightarrow \mathrm{d}q = \vec{\varepsilon}\,(\cosh\theta + \vec{\varepsilon}\sinh\theta)\,\mathrm{d}\theta = \vec{\varepsilon}q\mathrm{d}\theta.$$

Thus, we get $\int \frac{\mathrm{d}q}{q} = \int \vec{\varepsilon}\mathrm{d}\theta \Rightarrow \ln q = \vec{\varepsilon}\theta \Rightarrow q = \mathrm{e}^{\vec{\varepsilon}\theta} = \cosh\theta + \vec{\varepsilon}\sinh\theta$. Now, let's prove De Moivre's formula for a timelike quaternion with spacelike vector part.

Theorem 1. *Let $q = N_q\,(\cosh\theta + \vec{\varepsilon}\sinh\theta)$ be a unit timelike quaternion with spacelike vector part. Then,*

$$q^n = (N_q)^n\,(\cosh n\theta + \vec{\varepsilon}\sinh n\theta)$$

for $n \in \mathbb{Z}$.

Proof. We use induction on positive integers n. Assume that $q^n = (N_q)^n\,(\cosh n\theta + \vec{\varepsilon}\sinh n\theta)$ holds. Then,

$$\begin{aligned} q^{n+1} &= (N_q)^n\,(\cosh n\theta + \vec{\varepsilon}\sinh n\theta)\,N_q\,(\cosh\theta + \vec{\varepsilon}\sinh\theta) \\ &= (N_q)^{n+1}\,(\cosh n\theta + \vec{\varepsilon}\sinh n\theta)\,(\cosh\theta + \vec{\varepsilon}\sinh\theta) \\ &= (N_q)^{n+1}\,(\cosh n\theta\cosh\theta + \sinh n\theta\sinh\theta + (\cosh n\theta\sinh\theta + \sinh n\theta\cosh\theta)\,\vec{\varepsilon}) \\ &= (N_q)^{n+1}\,(\cosh(n+1)\,\theta + \vec{\varepsilon}\sinh(n+1)\,\theta)\,. \end{aligned}$$

Hence, the formula is true. Moreover, since

$$q^{-1} = (N_q)^{-1}\,(\cosh\theta - \vec{\varepsilon}\sinh\theta) \quad \textit{and}$$
$$q^{-n} = (N_q)^{-n}\,(\cosh n\theta - \vec{\varepsilon}\sinh n\theta) = (N_q)^{-n}\,(\cosh(-n\theta) + \vec{\varepsilon}\sinh(-n\theta))\,,$$

the formula holds for all integers. ■

3.2. Timelike quaternions with timelike vector part ($\mathbb{T}_t$)

Every timelike quaternion with timelike vector part can be written in the form

$$q = N_q\,(\cos\theta + \vec{\varepsilon}_0\sin\theta)$$

where $\cos\theta = \frac{q_1}{r}$, $\sin\theta = \frac{\sqrt{q_2^2 - q_3^2 - q_4^2}}{r}$, $\vec{\varepsilon}_0 = \frac{q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}}{\sqrt{q_2^2 - q_3^2 - q_4^2}}$ is a timelike unit vector in $\mathbb{E}_1^3$ and $\vec{\varepsilon}_0 * \vec{\varepsilon}_0 = -1$. Also, a unit timelike quaternion q with timelike vector part represents a rotation of a three-dimensional non-lightlike Lorentzian vector by an angle 2θ about the axis of q.

Euler's formula for a timelike quaternion with timelike vector part also holds. Since $\vec{\varepsilon}^2 = \vec{\varepsilon} * \vec{\varepsilon} = -1$, we have

$$\mathrm{e}^{\vec{\varepsilon}\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + \vec{\varepsilon}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) = \cos\theta + \vec{\varepsilon}\sin\theta.$$

Theorem 2. *Let $q = N_q\,(\cos\theta + \vec{\varepsilon}\sin\theta)$ be a unit timelike quaternion with timelike vector part. Then,*

$$q^n = (N_q)^n\,(\cos n\theta + \vec{\varepsilon}\sin n\theta)$$

for $n \in \mathbb{N}$.

The proof of this theorem can be done using induction, similarly to the proof of the Theorem 1.

3.3. Timelike quaternion with lightlike vector part

Every unit timelike quaternion with null vector part can be written in the form $q = 1 + \vec{\varepsilon}$ where $\vec{\varepsilon}$ is a null vector. If $q = 1 + \vec{\varepsilon}$ is a unit timelike quaternion with null vector part, then $q^n = 1 + n\vec{\varepsilon}$ and only root of the equation $w^n = q$ is $1 + \frac{\vec{\varepsilon}}{n}$.

3.4. Spacelike quaternions

Every spacelike quaternion can be written in the form

$$q = N_q\,(\sinh\theta + \vec{\varepsilon}_0\cosh\theta)$$

where $\sinh\theta = \frac{q_1}{r}$, $\cosh\theta = \frac{\sqrt{-q_2^2 + q_3^2 + q_4^2}}{r}$ and $\vec{\varepsilon}_0 = \frac{q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}}{\sqrt{-q_2^2 + q_3^2 + q_4^2}}$ is a spacelike unit vector in $\mathbb{E}_1^3$.

The product of two spacelike quaternions is timelike. That is, for a spacelike quaternion $q = N_q\,(\sinh\theta + \vec{\varepsilon}_0\cosh\theta)$, $q^2 = N_q^2\,(\cosh\theta + \vec{\varepsilon}_0\sinh\theta)$

Because of that we can give De Moivre's formula for spacelike quaternions as follows.

Theorem 3. *Let* $q = N_q\left(\sinh\theta + \vec{\varepsilon}\cosh\theta\right)$ *be a unit spacelike quaternion. Then,*

$$q^n = \begin{cases} \left(N_q\right)^n\left(\sinh n\theta + \vec{\varepsilon}\cosh n\theta\right), & n \text{ is odd} \\ \left(N_q\right)^n\left(\cosh n\theta + \vec{\varepsilon}\sinh n\theta\right), & n \text{ is even}. \end{cases}$$

4. The roots of a split quaternion

In this section we want to find roots of a split quaternion using De Moivre's formula given above.

Theorem 4. *Let* $q = N_q\left(\cosh\theta + \vec{\varepsilon}\sinh\theta\right)$ *be a timelike quaternion with spacelike vector part. Then the equation* $w^n = q$ *has only one root:*

$$w = \sqrt[n]{N_q}\left(\cosh\frac{\theta}{n} + \vec{\varepsilon}\sinh\frac{\theta}{n}\right)$$

in the set of timelike quaternions $\mathbb{T}\widehat{\mathbb{H}}$.

Proof. If $w^n = q$, q will have the same unit vector as w. So, assume that $w = N\left(\cosh x + \vec{\varepsilon}\sinh x\right)$ is a root of the equation $w^n = q$. From Theorem 1, we have

$$w^n = N^n\left(\cosh nx + \vec{\varepsilon}\sinh nx\right).$$

Thus, $N^n = N_q$ and $x = \frac{\theta}{n}$. Therefore, $w = \sqrt[n]{N_q}\left(\cosh\frac{\theta}{n} + \vec{\varepsilon}\sinh\frac{\theta}{n}\right)$ is a root of the equation $w^n = q$. If we suppose that there are two roots satisfying the equality, we obtain that these roots must be equal to each other. ■

Let's find the roots of the equation $w^3 = q$ where $q = \sqrt{2} + \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - \mathbf{k}$. Since $I_q = 1$, q is a unit timelike quaternion and its vector part is a spacelike vector. Then, we can write as $q = \cosh\theta + \varepsilon\sinh\theta$. Indeed, $q = \sqrt{2} + \left(\sqrt{2}, \sqrt{2}, -1\right)1 \Rightarrow \cosh\theta = \sqrt{2}$ and $\sinh\theta = 1$.

From the last equalities, we find $\theta = \ln\left(\sqrt{2} + 1\right)$. Thus,

$$q = \cosh\ln\left(\sqrt{2} + 1\right) + \vec{\varepsilon}\sinh\ln\left(\sqrt{2} + 1\right)$$

where $\vec{\varepsilon} = \left(\sqrt{2}, \sqrt{2}, -1\right)$. And using above lemma, we find the root as

$$w = \left(\cosh\frac{\ln\left(\sqrt{2} + 1\right)}{3} + \vec{\varepsilon}\sinh\frac{\ln\left(\sqrt{2} + 1\right)}{3}\right).$$

Theorem 5. *Let* $q = N_q\left(\cos\theta + \vec{\varepsilon}\sin\theta\right)$ *be a timelike quaternion with timelike vector part. Then the equation* $w^n = q$ *has* n *roots in the timelike quaternions* $\mathbb{T}\widehat{\mathbb{H}}$*, and they are*

$$w_m = \sqrt[n]{N_q}\left(\cos\frac{\theta + 2m\pi}{n} + \vec{\varepsilon}\sin\frac{\theta + 2mn}{n}\right)$$

where $m = 0, 1, 2, \ldots, n - 1$.

Proof. We assume that $w = N\left(\cos\varphi + \vec{\varepsilon}\sin\varphi\right)$ is a root of the equation, since the vector parts of w and q are the same. So, we find

$$N_q = N^n, \quad \cos n\varphi = \cos\theta \quad \text{and} \quad \sin n\varphi = \sin\theta$$

using Theorem 2. Thus, the nth roots of q are

$$w_m = \sqrt[n]{N_q}\left(\cos\frac{\theta + 2m\pi}{n} + \vec{\varepsilon}\sin\frac{\theta + 2mn}{n}\right)$$

for $m = 0, 1, 2, \ldots, n - 1$. ■

Let's find the roots of the equation $w^3 = -1 + \sqrt{3}\mathbf{i}$. $q = -1 + \sqrt{3}\mathbf{i}$ is a timelike quaternion with timelike part such that $N_q = 2$. Then, q can be written as $q = 2\left(\cos\left(\frac{2\pi}{3} + 2m\pi\right) + \vec{\varepsilon}\sin\left(\frac{2\pi}{3} + 2m\pi\right)\right)$ for $m \in N^+$. From Theorem 5, the roots of the equation $w^3 = q$ are

$$w_m = \sqrt[3]{2}\left(\cos\frac{\frac{2\pi}{3} + 2m\pi}{3} + \vec{\varepsilon}\sin\frac{\frac{2\pi}{3} + 2mn}{3}\right)$$

for $m = 0, 1, 2$. So, $w_0 = \sqrt[3]{2}\left(\cos\frac{2\pi}{9} + \vec{\varepsilon}\sin\frac{2\pi}{9}\right)$, $w_1 = \sqrt[3]{2}\left(\cos\frac{8\pi}{9} + \vec{\varepsilon}\sin\frac{8\pi}{9}\right)$, $w_3 = \sqrt[3]{2}\left(\cos\frac{14\pi}{9} + \vec{\varepsilon}\sin\frac{14\pi}{9}\right)$ are the roots of the given equation.

Now, we examine the solution of the equations in the form $w^n = q$ such that $q \in R$. We consider $q \in \mathbb{R}$ regarded as a timelike quaternion with timelike vector part.

Theorem 6. *Let $q \in \mathbb{R}$ be a unit timelike quaternion with $\vec{\varepsilon} = \vec{0}$. Then there are infinitely many roots of the equation $w^n = q$ in the set of timelike quaternions $\widehat{\mathbb{TH}}$.*

Proof. We must choose w such its nth power of it must be a real number with zero vector part.

Since we consider $q \in R$ regarded as a timelike quaternion with timelike vector part, we can write w as $w = N(\cos\theta + \vec{\varepsilon}\sin\theta)$, so $w^n = N^n(\cos n\theta + \vec{\varepsilon}\sin n\theta) = q$ and $N = \sqrt[n]{N_q}$, $\theta = \frac{2m\pi}{n}$ for $m = 0, 1, 2$. Thus, we find roots as

$$w_m = \sqrt[n]{N_q}\left(\cos\frac{2m\pi}{n} + \vec{\varepsilon}\sin\frac{2m\pi}{n}\right) \quad \text{for } m = 0, 1, 2, \ldots, n-1.$$

Here, if there are n roots of the equation, actually there are an infinite number of roots, because every quaternion in the form $w_m = \sqrt[n]{N_q}\left(\cos\frac{2m\pi}{n} + \vec{\varepsilon}\sin\frac{2m\pi}{n}\right)$ for $m = 0, 1, 2, \ldots$ with a timelike unit vector $\vec{\varepsilon}$ is a root of the equation and we choose infinite $\vec{\varepsilon}$. That is, for every timelike vector $\vec{\varepsilon}$, we find another timelike quaternion which satisfies the equation. ∎

Remark 7. All of the timelike quaternions satisfying the equation $w^n = q$ represent a rotation through the same angle about different axes in the Minkowski 3-space.

For example, the roots of the equation $w^4 = -4$ are

$$w_m = \sqrt[4]{4}\left(\cos\frac{\pi + 2k\pi}{4} + \vec{\varepsilon}\sin\frac{\pi + 2k\pi}{4}\right)$$

for $k = 0, 1, 2, 3$. But, for every timelike vector $\vec{\varepsilon}$, the equality $(w_m)^n = -4$ holds. That is, we can choose an infinitely timelike quaternion satisfying the equation provided that its norm is 4 and the vector part is any timelike vector. $w \in \{(1, 3, 2, 2), (1, 1, 0, 0), \left(1, 2, 1, \sqrt{2}\right), \ldots\}$.

Here, each of the quaternions $(1, 3, 2, 2), (1, 1, 0, 0), \left(1, 2, 1, \sqrt{2}\right), \ldots$ represents a rotation through 90 degrees about the different axis $\vec{\varepsilon} \in \{(3, 2, 2), (1, 0, 0), \left(2, 1, \sqrt{2}\right), \ldots\}$.

Remark 8. If n is an even number, then some of the roots of the equation $w^n = q \in \mathbb{R}^+$ can be spacelike quaternions. For example, a spacelike quaternion $w \in \{(0, 0, 1, 0), (0, 0, 0, 1), (0, 2, 2, 1), \ldots\}$ is a root of $w^4 = 1$. But, observe that we solve the equation $w^n = q$ in the set of timelike quaternions in Theorem 6.

Now let's state what the roots of a spacelike quaternion are in the set of spacelike quaternions. For this case, it is important that n is odd or even, according to Theorem 3.

Theorem 9. *Let $q = N_q(\sinh\theta + \vec{\varepsilon}\cosh\theta)$ be a spacelike quaternion. Then the solution of the equation $w^n = q$ in the spacelike quaternions*

(i) *doesn't exist if n is an even number,*
(ii) *has only one spacelike quaternion root $w = \sqrt[n]{N_q}\left(\sinh\frac{\theta}{n} + \vec{\varepsilon}\cosh\frac{\theta}{n}\right)$ if n is an odd number.*

Proof. If n is an even number, the nth power of a spacelike quaternion will be a timelike quaternion and in this case there is no solution.

So, let $w = N(\sinh\varphi + \vec{\varepsilon}\cosh\varphi)$ be a root of the equation $w^n = q$ such that n is an odd number. Then

$$w^n = N^n(\sinh n\varphi + \vec{\varepsilon}\cosh n\varphi) = N_q(\sinh\theta + \vec{\varepsilon}\cosh\theta)$$

and we have $\varphi = \frac{\theta}{n}$ and $N = \sqrt[n]{N_q}$. ∎

Let's find the roots of the equation $x^3 = 1 + \sqrt{2}\mathbf{k}$. Since $I_q = -1$, q is a unit spacelike quaternion. Then, it can be written in the form $q = \sinh\theta + \vec{\varepsilon}\cosh\theta$ where $\vec{\varepsilon} = (0, 0, 1)$, $\cosh\theta = \sqrt{2}$ and $\sinh\theta = 1$. So,

$$x = \sinh\frac{\ln\left(\sqrt{2}+1\right)}{3} + \vec{\varepsilon}\cosh\frac{\ln\left(\sqrt{2}+1\right)}{3}$$

is the only root of the equation $x^3 = 1 + \sqrt{2}k$.

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References

[1] L. Brand, The roots of a quaternion, Amer. Math. Monthly 49 (8) (1942) 519–520.
[2] E. Cho, De Moivre's formula for quaternions, Appl. Math. Lett. 11 (6) (1998) 33–35.
[3] E. Cho, Euler's formula and De Moivre's formula for quaternions, Missouri J. Math. Sci. 11 (2) (1999) 80–83.
[4] A.J. Hanson, Visualizing Quaternion, Morgan-Kaufmann/Elsevier, 2005.
[5] J. Inoguchi, Timelike surfaces of constant mean curvature in Minkowski 3-space, Tokyo J. Math. 21 (1) (1998) 141–152.
[6] J Inoguchi, M Toda, Timelike minimal surfaces via loop groups, Acta Appl. Math. 83 (3) (2004) 313–355.
[7] I. Niven, The roots of a quaternion, Amer. Math. Monthly 449 (6) (1942) 386–388.
[8] M. Özdemir, A.A. Ergin, Some geometric applications of split quaternions, in: Proc. 16th Int. Conf. Jangjeon Math. Soc. 16, 2005, pp. 108–115.
[9] M. Özdemir, A.A. Ergin, Rotations with timelike quaternions in Minkowski 3-space, J. Geom. Phys. 56 (2006) 322–336.