On the asymptotics of Laguerre matrix polynomials for large \( x \) and \( n \)

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Abstract

This work deals with the asymptotics of normalized Laguerre matrix polynomials of a complex matrix parameter for \( \sqrt{n}/x = o(1) \) and \( x/\sqrt{n} = O(1) \) as \( n \to \infty \).

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1. Introduction

Laguerre matrix polynomials have been introduced and studied in [8,16,13,11,12]. Connections between the modified Bessel matrix function and Laguerre matrix polynomials have been established in [10, p. 44–46], see [17]. Applications to matrix integration may be found in [9]. Like in the corresponding problem for scalar functions, see [20], the problem of the development of matrix functions in series of Laguerre matrix polynomials requires some new results concerning the matrix operational calculus not available in the literature. In this work we address the asymptotics of normalized Laguerre matrix polynomials for \( \sqrt{n}/x = o(1) \) and \( x/\sqrt{n} = O(1) \) as \( n \to \infty \), which play an important role in the analysis of series expansions of Laguerre matrix polynomials [18].

Throughout this work, for a complex number \( z \), \( \Re(z) \) and \( \Im(z) \) denote its real and imaginary parts, respectively. For a matrix \( A \) in \( \mathbb{C}^{r \times r} \) its spectrum \( \sigma(A) \) denotes the set of all the eigenvalues of \( A \), \( \alpha(A) = \max\{\Re(z) ; z \in \sigma(A)\} \), \( \beta(A) = \min\{\Re(z) ; z \in \sigma(A)\} \). The spectral radius of \( A \) denoted by \( \rho(A) \) is the maximum of the set \( \{|z| ; z \in \sigma(A)\} \). The 2-norm of \( A \) will be denoted by

\[
\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2},
\]

where for a vector \( y \) in \( \mathbb{C}^r \), \( \|y\|_2 = (y^H y)^{1/2} \) is its euclidean norm, [3]. If \( f(z) \) and \( g(z) \) are holomorphic functions of the complex variable \( z \), which are defined in an open set \( \Omega \) of the complex plane, and \( A \) is a matrix in \( \mathbb{C}^{r \times r} \) with \( \sigma(A) \subset \Omega \), then from the properties of the matrix functional calculus [1, p. 558], it follows that \( f(A)g(A) = g(A)f(A) \). The reciprocal gamma function, denoted by \( \Gamma^{-1}(z) = 1/\Gamma(z) \), is an entire function of the

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complex variable $z$. Then the image of $\Gamma^{-1}(z)$ acting on $A$, denoted by $\Gamma^{-1}(A)$, is a well defined matrix. Furthermore, if $A+nI$ is invertible for every integer $n \geq 0$, where $I$ is the identity matrix in $\mathbb{C}^{r \times r}$, then $\Gamma(A)$ is invertible, its inverse coincides with $\Gamma^{-1}(A)$, and by [6, p. 253], it follows that $A(A+I) \cdots (A+(n-1)I) = \Gamma(A+nI)\Gamma^{-1}(A), n \geq 1$. If we denote by $\mu(A)$ the logarithmic norm of $A$, and by $\tilde{\mu}(A)$ the number $-\mu(-A)$, then [5,2]

$$\mu(A) = \max\{z; z \in \sigma((A + AH)/2)\}, \quad \tilde{\mu}(A) = \min\{z; z \in \sigma((A + AH)/2)\}. \quad (1.2)$$

By (vi) [5, p. 647] it follows that $\mu(A) = \min[\theta/\|e^{At}\| \leq e^{\theta t}, t \geq 0]$, and taking into account the Schur decomposition of $A$, by [3, p. 192–193] one gets

$$\|e^{At}\| \leq e^{\alpha(A)t} \sum_{k=0}^{r-1} \left(\|A\|^{1/2}t\right)^k / k!, \quad t \geq 0. \quad (1.3)$$

Hence $t^A = \exp(A \log t)$ verifies

$$\|t^A\| \leq \begin{cases} t^{\mu(A)}, & t \geq 1, \\ t^{\tilde{\mu}(A)}, & 0 < t \leq 1, \end{cases} \quad (1.4)$$

$$\|t^A\| \leq t^{\alpha(A)} \sum_{k=0}^{r-1} \left(\|A\|^{1/2} \log t\right)^k / k!, \quad t \geq 1, \quad (1.5)$$

$$\|t^A\| = \|e^{-A(-\log t)}\| \leq t^{\beta(A)} \sum_{k=0}^{r-1} \left(-\|A\|^{1/2} \log t\right)^k / k!, \quad 0 < t \leq 1. \quad (1.6)$$

Note that by (i) [5, p. 647] and Bendixon’s theorem [19, p. 395] it follows that

$$-\|A\| \leq \tilde{\mu}(A) \leq \beta(A) \leq \Re(z) \leq \alpha(A) \leq \mu(A) \leq \|A\|, \quad z \in \sigma(A), \quad (1.7)$$

and taking $x = -\log t$ in 3.381 (4) of [4, p. 364] one gets

$$\int_0^1 t^m \log^n t \, dt = \frac{(-1)^n \Gamma(n+1)}{(m+1)n+1} < \infty, \quad m > -1, \quad n \geq 0. \quad (1.8)$$

This work is organized as follows. Section 2 deals with some preliminary results. Section 3 is concerned with the asymptotics of Laguerre matrix polynomials for large $n$ and $x$.

2. Preliminaries

**Definition 2.1.** Given a fixed matrix $A$ in $\mathbb{C}^{r \times r}$, a positive scalar function $g(n)$ and a sequence of functions $f_n(z)$ defined in an open set $\Omega$ with $\sigma(A) \subset \Omega$, we say that the sequence $\{f_n(A)\}$ behaves $O(g(n), A)$, if $\|f_n(A)\| \leq M(A)g(n)$ for some positive constant $M(A)$ and $f_n(A)$ commutes with $A$ for $n \geq 1$.

If $\lambda$ is a complex number with $\Re(\lambda) > 0$ and $A$ is a matrix in $\mathbb{C}^{r \times r}$ with $A+nI$ invertible for every integer $n \geq 1$, the $n$-th Laguerre matrix polynomial $L_n^{(A,\lambda)}(x)$ and the $n$-th normalized Laguerre matrix polynomial $\mathcal{L}_n^{(A,\lambda)}(x)$ are defined by [8, p. 58], [12, p. 25]

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \lambda^k}{k!(n-k)!} (A + I)n[(A + I)k]^{-1} x^k,$$

$$\mathcal{L}_n^{(A,\lambda)}(x) = \frac{\lambda^{-1/2} \Gamma(A + (n + 1)I)}{\Gamma(n + 1)} \frac{1}{2} L_n^{(A,\lambda)}(x), \quad \mathcal{L}_n^{(A,\lambda)}(x) = \lambda^{-1/2} \Gamma(A + (n + 1)I) \frac{1}{2} L_n^{(A,\lambda)}(x), \quad (2.1)$$

where $(A + I)_n = (A + I)(A + 2I) \cdots (A + nI), n \geq 1, (A + I)_0 = I$. From (2.1) it is easy to show that

$$L_n^{(A,\lambda_2)}(x) = L_n^{(A,\lambda_1)}(\lambda_2 x/\lambda_1). \quad (2.2)$$
Taking into account the generating function of Laguerre matrix polynomials given in [8, p. 57]

\[ G(x, t, \lambda, A) = (1 - t)^{-(A+1)} \exp \left( \frac{-\lambda xt}{1 - t} \right) = \sum_{n \geq 0} L_n^{(A, \lambda)}(x)t^n, \quad t \in \mathbb{C}, |t| < 1, x \in \mathbb{C}, \quad (2.3) \]

one gets \((1 - t)G(x, t, \lambda, A + I) = G(x, t, \lambda, A)\). Hence \(\sum_{n \geq 0} L_n^{(A+1, \lambda)}(x)t^n - \sum_{n \geq 0} L_n^{(A+I, \lambda)}(x)t^{n+1} = \sum_{n \geq 0} L_n^{(A, \lambda)}(x)t^n\). Identifying coefficients of the last series and using (2.1) one gets

\[ \sqrt{A/(n + 1)} + \lambda \Pi_{n+1}^{(A, \lambda)}(x) - \Pi_n^{(A, \lambda)}(x) = \sqrt{\lambda/(n + 1)} \Pi_n^{(A-I, \lambda)}(x), \quad n \geq 0. \quad (2.4) \]

Using Taylor’s expansion with \(\sqrt{I + At}\) as \(t \to 0\) in (2.4) one gets

\[ \Pi_{n+1}^{(A, \lambda)}(x) - \Pi_n^{(A, \lambda)}(x) = \sqrt{\lambda/(n + 1)} \Pi_n^{(A-I, \lambda)}(x) + O(n^{-1}, A) \Pi_n^{(A, \lambda)}(x). \quad (2.5) \]

Using 3.383 (1) [4, p. 365], the asymptotic expression of the scalar hypergeometric function \(F_1(5)\) [15, p. 128] and the identity \(B(\mu, v) = \Gamma(\mu)\Gamma(v)/\Gamma(\mu + v)\), if \(\Re(\mu) > 0, \Re(v) > 0\), as \(\Re(\beta) \to \infty\), it follows that

\[ \int_0^1 x^{\nu - 1}(1 - x)^{\mu - 1}e^{\beta x} dx = B(\mu, v) \approx B(\mu, v) \frac{\Gamma(\mu + v)}{\Gamma(\nu)} \frac{e^\beta}{\beta^\nu}. \quad (2.6) \]

The next result provides an integral expression of normalized Laguerre matrix polynomials, see [20, p. 107].

**Lemma 2.1.** Let \(\lambda > 0\) and let \(A \in \mathbb{C}^{r \times r}\) such that \(\beta(A) > -1/2\). If \(H_n(\cdot)\) denote the scalar Hermite polynomials, then

\[ \Pi_n^{(A, \lambda)}(x) = \lambda^{(A+I)/2} \sqrt{\frac{\Gamma(A + (n + 1)I)}{\Gamma(n + 1)}} \frac{(-1)^n n!}{\sqrt{\pi(2n)!}} \Gamma^{-1}(A + I/2) 2^{\nu/2} (\sin \varphi)^{2A} H_{2n}(\sqrt{\lambda x} \cos \varphi) d\varphi. \quad (2.7) \]

**Proof.** Taking into account the Rodrigues formulae for matrix and scalar Hermite polynomials, [7, p. 29], (4.9.1) [14, p. 60], one gets \(H_{2n}(x, I) = I H_{2n}(x/\sqrt{2})\). Hence, taking \(t = \cos \varphi\) in (3.2) of lemma 1 [12, p. 24], and using (2.1) and (2.2), (2.7) follows. \(\Box\)

Taking into account that

\[ |H_n(x)| = e^{\frac{x^2}{2}} e^{-\frac{x^2}{2}} |H_n(x)| \leq e^{\frac{x^2}{2}} \max \left\{ e^{-\frac{x^2}{2}} |H_n(x)| \right\}, \quad (2.8) \]

and that by Stirling’s formula \(n! \approx \sqrt{2\pi n} \frac{n^n}{e^n}\) for \(n \to \infty\), one gets \(2^n \sqrt{(2n)!} n^{-\frac{1}{2}} = O(1)/(2n)!/n!\), using theorem 8.22.6 [20, p. 199] for \(|x| \leq a > 0\) and theorem 8.91.3 [20, p. 242] for \(|x| \geq a\), it follows that

\[ |H_{2n}(x)| \leq O(1)e^{\frac{x^2}{2}} n^{2n} \frac{(2n)!}{n!}. \quad (2.9) \]

where \(s = 0\) if \(|x| \leq \sqrt{(4 - \eta)n}, 0 < \eta < 4\), or \(s = 1/6\) if \(x \in \mathbb{R}\).

### 3. Asymptotic expression for large \(x > 0\)

The next result provides an asymptotic expression for normalized Laguerre polynomials for large \(x > 0\).

**Theorem 3.1.** Let \(A \in \mathbb{C}^{r \times r}, \lambda > 0, n \in \mathbb{N}, x > 0\) and \(s = 0\) if \(\lambda x < (4 - \eta)n\), otherwise \(s = 1/6\), with \(0 < \eta < 4\). Then the next expression holds as \(x \to \infty, n \to \infty, n > \rho(A) + 1:\)

\[ \Pi_n^{(A, \lambda)}(x) = O(1, A) \frac{e^{\frac{\lambda x}{2} n^{\frac{\mu}{2} + k_0 + s}}}{x^{\gamma_0 + k_0 + s}} \log^\gamma \left( \frac{\lambda x}{2\sqrt{n}} \right) \log^{\gamma_2} \left( \sqrt{n} \right), \quad (3.1) \]

where if \(\sqrt{n}/x = o(1)\) then

(i) \(\gamma_0 = \tilde{\mu}(A)\) and \(\gamma_1 = \gamma_2 = 0, or\)
and hence, by (3.5), it follows that

\[ \int_{\lambda x}^{1} \frac{4}{\sqrt{y(1-u)}} \leq 1 \]

in (2.7), by Definition 2.1 and (3.32) of [13], if \( n > \rho(A) + 1, \lambda > 0 \) and \( \beta(A) > -\frac{1}{2} \) one gets

\[ \Pi_n^{(A,\lambda)}(x) = O(1, A)K_n, \quad K_n = \frac{n^\frac{\alpha}{2}n!}{(2n)!} \int_{0}^{1} \frac{(1-u)^A}{\sqrt{u(1-u)}} H_{2n} \left( \sqrt{\lambda xu} \right) du. \]  

(3.2)

Hence, using (2.9) and Definition 2.1 it follows that

\[ \|K_n\| \leq n^\frac{\alpha}{2} \left[ K_{n_1} + K_{n_2} \right], \quad K_n = n^\frac{\alpha}{2} \left[ O(1, A)K_{n_1} + O(1, A)K_{n_2} \right], \]  

(3.3)

where \( s = 0 \) if \( \lambda x \leq (4-\eta)n \) or \( s = 1/6 \) if \( \lambda x > (4-\eta)n, 0 < \eta < 4 \), and

\[ K_{n_1} = \int_{1-\frac{\eta}{\lambda n}}^{1} \frac{\|\sqrt{n}(1-u)^A\|}{\sqrt{1-u}} e^{\frac{\lambda x}{\sqrt{n}}} du, \quad K_{n_2} = \int_{0}^{1} \frac{\|\sqrt{n}(1-u)^A\|}{\sqrt{1-u}} e^{\frac{\lambda x}{\sqrt{n}}} du. \]  

(3.4)

Note that \( \sqrt{n}(1-u) \geq 1 \) in the first integral and \( 0 \leq \sqrt{n}(1-u) \leq 1 \) in the second one. Taking \( v = \sqrt{n}(1-u) \) in integrals \( n_{n_2} \) of (3.4) and using (1.5) one gets

\[ K_{n_2} \leq \frac{e^{\frac{\lambda x}{\sqrt{n}}}}{\sqrt{n}} \sum_{k=0}^{r-1} \frac{(\|A\|\sqrt{r})^k}{k!} \int_{0}^{1} v^{\beta(A)-\frac{1}{2}} (-\log(v))^k \sqrt{1-v} e^{-\frac{\lambda x}{\sqrt{n}}} O(1) \left( \frac{\|A\|\sqrt{r})^k}{k!} \right) I_n, \]  

(3.5)

where

\[ I_n = \int_{0}^{1} v^{\beta(A)-\frac{1}{2}} (-\log(v))^k \sqrt{1-v} e^{-\frac{\lambda x}{\sqrt{n}}} dv. \]  

(3.6)

If \( \frac{\sqrt{n}}{x} = O(1), x \to \infty, n \to \infty \), by (3.6) and (1.7) one gets

\[ I_n \leq \int_{0}^{1} v^{\beta(A)-\frac{1}{2}} (-\log(v))^k \sqrt{1-v} e^{-\frac{\lambda x}{\sqrt{n}}} < \infty, \]  

(3.7)

and hence, by (3.5), it follows that

\[ K_{n_2} = O(1) \frac{e^{\frac{\lambda x}{\sqrt{n}}}}{\sqrt{n}}, \quad \frac{x}{\sqrt{n}} = O(1). \]  

(3.8)

If \( \frac{\sqrt{n}}{x} = o(1), x \to \infty, n \to \infty \), taking \( z = \frac{\|A\|\sqrt{r}}{2n} \) in (3.6) one gets

\[ I_n = \left( \frac{\lambda x}{2\sqrt{n}} \right)^{-\beta(A)-\frac{1}{2}} \int_{0}^{\frac{\lambda x}{2\sqrt{n}}} z^{\beta(A)-\frac{1}{2}} \left( \log \left( \frac{\lambda x}{2\sqrt{n}} \right) - \log(z) \right)^k e^{-z} dz \]

\[ = \left( \frac{\lambda x}{2\sqrt{n}} \right)^{-\beta(A)-\frac{1}{2}} \sum_{j=0}^{k} \binom{k}{j} \log \left( \frac{\lambda x}{2\sqrt{n}} \right)^{k-j} J_0, \]  

(3.9)

where

\[ J_0 = \int_{0}^{\frac{\lambda x}{2\sqrt{n}}} z^{\beta(A)-\frac{1}{2}} (-\log(z))^j e^{-z} dz \leq \int_{0}^{\lambda x/2\sqrt{n}} z^{\beta(A)-\frac{1}{2}} (-\log(z))^j dz + \int_{\lambda x/2\sqrt{n}}^{\infty} z^{\beta(A)-\frac{1}{2}} (-\log(z))^j e^{-z} dz. \]  

(3.10)

Let \( f(z) = Mz^\delta - \log(z), \delta > 0, M > 0 \). It is easy to show that \( f(z) > 0 \) if \( z = 1, \lim_{z \to \infty} f(z) = \infty \), that \( f(z) \) has only one minimum in \( z_0 = (M\delta)^{-\frac{1}{2}} \), and that given \( \delta > 0 \), if \( \delta \geq \delta e, f(z_0) \geq 0 \). Hence given \( \delta > 0 \), there exists

\[ f(z) = Mz^\delta - \log(z), \delta > 0, M > 0. \]  

(3.11)
\( M > 0 \) such that \( Mz^\delta \geq \log(z) \) and \( M' \) such that 
\[
M'z^{\delta'} \geq \log^k z, \quad \delta' = k\delta, z \geq 1, 
\]
and using (3.10), (1.7) and (3.11) it follows that 
\[
J_0 \leq O(1) + M \int_1^\infty z^{\beta(A)-\frac{1}{2}+\delta'} e^{-z} \, dz \leq O(1) + M \Gamma \left( \beta(A) + \frac{1}{2} + \delta' \right). 
\]

Hence by (3.5), (3.9) and (3.12) one gets 
\[
K_{n2} \leq O(1) \frac{e^{\lambda x}}{\sqrt{x}} \left( \frac{\sqrt{n}}{x} \right)^{\beta(A)} \log^{-1} \left( \frac{\lambda x}{2 \sqrt{n}} \right), \quad \frac{\sqrt{n}}{x} = o(1). 
\]

Using (1.5) in integral \( K_{n1} \) of (3.4) it follows that 
\[
K_{n1} \leq n \frac{\mu(A)}{x} \sum_{k=0}^{r-1} \left( \frac{\|A\| \sqrt{r}}{k!} \right)^k \int_0^{1-\frac{\sqrt{n}}{2n}} \frac{(1-u)^{a(A)-\frac{1}{2}}}{\sqrt{u}} e^{\frac{\lambda u}{x}} \log^k \left( \sqrt{n}(1-u) \right) \, du. 
\]

If \( \frac{\sqrt{n}}{x} = o(1) \), taking into account that 
\[
\log^k \left( \sqrt{n}(1-u) \right) e^{-\frac{\lambda u}{x}(1-u)} \leq e^{k \sqrt{n}(1-u)} e^{-\frac{\lambda u}{x} \sqrt{n}(1-u)} = e^{\sqrt{n}(1-u)(k-\frac{\lambda u}{x})} = o(1), 
\]
\[
0 < \delta < 1, 0 < u \leq 1 - \frac{1}{\sqrt{n}}, 
\]
using (3.14) and (2.6) and taking \( \delta, 0 < \delta < 1 \), it follows that 
\[
K_{n1} \leq e^{\frac{\lambda u}{x}} n \frac{\mu(A)}{x} \sum_{k=0}^{r-1} \left( \frac{\|A\| \sqrt{r}}{k!} \right)^k \int_0^{1-\frac{\sqrt{n}}{2n}} \frac{(1-u)^{a(A)-\frac{1}{2}}}{\sqrt{u}} e^{\frac{\lambda u}{x(1-u)(1-\delta)}} \log^k \left( \sqrt{n}(1-u) \right) e^{-\frac{\lambda u}{x}(1-u)} \, du 
\]
\[
= e^{\frac{\lambda u}{x}} n \frac{\mu(A)}{x} o(1) \int_0^{1-\frac{\sqrt{n}}{2n}} \frac{(1-u)^{a(A)-\frac{1}{2}}}{\sqrt{u}} e^{\frac{\lambda u}{x(1-\delta)x}} \, du 
\]
\[
= o(1) e^{\frac{\lambda u}{x}} \left( \frac{\sqrt{n}}{x} \right)^{a(A)} \log^{-1} \left( \sqrt{n} \right), \quad \frac{\sqrt{n}}{x} = o(1). 
\]

If \( \frac{\sqrt{n}}{x} = O(1) \), by (3.14) and (2.6) one gets 
\[
K_{n1} \leq n \frac{\mu(A)}{x} \sum_{k=0}^{r-1} \left( \frac{\|A\| \sqrt{r}}{k!} \right)^k \int_0^{1} (1-u)^{a(A)-\frac{1}{2}} e^{\frac{\lambda u}{x}} \, du 
\]
\[
= O(1) e^{\frac{\lambda u}{x}} \left( \frac{\sqrt{n}}{x} \right)^{a(A)} \log^{-1} \left( \sqrt{n} \right), \quad \frac{\sqrt{n}}{x} = O(1). 
\]

In an analogous way, starting from (3.4) and using the bound (1.4), if \( \hat{\mu}(A) > -\frac{1}{2} \) it is easy to show that 
\[
K_{n2} = \begin{cases} 
O(1) \frac{e^{\lambda x}}{\sqrt{x}} \left( \frac{\sqrt{n}}{x} \right)^{\hat{\mu}(A)}, & \frac{\sqrt{n}}{x} = O(1), \quad x \to \infty, \quad n \to \infty, \\
O(1) e^{\frac{\lambda x}{\sqrt{x}} \hat{\mu}(A)} \left( \frac{\sqrt{n}}{x} \right)^{\hat{\mu}(A)}, & \frac{\sqrt{n}}{x} = o(1), \quad x \to \infty, \quad n \to \infty, 
\end{cases} 
\]
\[
K_{n1} \leq n \frac{\mu(A)}{x} \int_0^{1} \frac{(1-u)^{\hat{\mu}(A)-\frac{1}{2}}}{\sqrt{u}} e^{\frac{\lambda u}{x}} \, du = O(1) e^{\frac{\lambda x}{\sqrt{x}} \hat{\mu}(A)} \left( \frac{\sqrt{n}}{x} \right)^{\hat{\mu}(A)}, \quad x \to \infty, \quad n \to \infty. 
\]

Using (3.2)–(3.4), (3.8), (3.13), (3.16)–(3.19) and (1.6) and taking into account that 
\[
\left( \frac{\sqrt{n}}{x} \right)^{\hat{\mu}(A)} \frac{1}{\sqrt{x}} \geq O(1)n^{-\frac{1}{4}}, \quad \left( \frac{\sqrt{n}}{x} \right)^{a(A)} \left( \frac{\sqrt{n}}{x} \right)^{-\frac{1}{2}} \geq O(1)n^{-\frac{1}{4}}, \quad \text{if} \quad \frac{x}{\sqrt{n}} = O(1), 
\]

(3.20)
\[
\left(\frac{\sqrt{n}}{x}\right)^{\mu(A)} \leq \left(\frac{\sqrt{n}}{x}\right)^{\mu(A)} , \left(\frac{\sqrt{n}}{x}\right)^{\alpha(A)} \leq \left(\frac{\sqrt{n}}{x}\right)^{\beta(A)} \log^{-1}\left(\frac{\lambda x}{2\sqrt{n}}\right) , \quad \text{if } \frac{\sqrt{n}}{x} = o(1). \tag{3.21}
\]

(3.1) follows with \(\gamma_0 > -\frac{1}{2}, k_0 = 0\). Taking into account that in the case of \(\gamma_1 = r - 1\) then \(\frac{\sqrt{n}}{x} = o(1)\) and

\[
\log^{\gamma_1}\left(\frac{\lambda x}{2\sqrt{n} - 1}\right) = \log^{\gamma_1}\left(\frac{\lambda x}{2\sqrt{n}}\right) \left(1 - \frac{\log\left(1 - \frac{1}{n}\right)}{2\log\left(\frac{\lambda x}{2\sqrt{n}}\right)}\right)^{\gamma_1}
= \log^{\gamma_1}\left(\frac{\lambda x}{2\sqrt{n}}\right) \left(1 + \frac{O(n^{-1})}{\log\left(\frac{\lambda x}{2\sqrt{n}}\right)}\right)
= O(1) \log^{\gamma_1}\left(\frac{\lambda x}{2\sqrt{n}}\right), \tag{3.22}
\]

\[
\log^{\gamma_2}\left(\sqrt{n} - 1\right) = \log^{\gamma_2}\left(\sqrt{n}\right) \left(1 + \frac{\log\left(1 - \frac{1}{n}\right)}{2\log(\sqrt{n})}\right) = O(1) \log^{\gamma_2}\left(\sqrt{n}\right), \tag{3.23}
\]

using (2.4) and (3.1) it follows that

\[
I_{n+1}^{(A, \lambda)}(x) = \sqrt{\frac{A + (n + 1)I}{\lambda}} I_{n}^{(A+I, \lambda)}(x) - \sqrt{\frac{n}{\lambda}} I_{n-1}^{(A+I, \lambda)}(x)
= \sqrt{n} e^{\frac{\lambda x}{2\sqrt{n}} n^{\frac{\gamma_0 + 1}{2}}} \log^{\gamma_1}\left(\frac{\lambda x}{2\sqrt{n}}\right) \log^{\gamma_2}\left(\sqrt{n}\right)
\left[O(1, A) - O(1, A) \left(\frac{n - 1}{n}\right)^{s} \left(\frac{n - 1}{n}\right)^{\frac{n^{\gamma_0 + 1}}{\gamma_1}} \left(\frac{\log\left(\frac{\lambda x}{2\sqrt{n}}\right)}{\log\left(\frac{\lambda x}{2\sqrt{n}}\right)}\right)^{\gamma_1} \left(\frac{\log\left(\sqrt{n} - 1\right)}{\log(\sqrt{n})}\right)^{\gamma_2}\right]
= O(1, A) e^{\frac{\lambda x}{2\sqrt{n}} n^{\frac{\gamma_0 + 1}{2}}} \log^{\gamma_1}\left(\frac{\lambda x}{2\sqrt{n}}\right) \log^{\gamma_2}\left(\sqrt{n}\right), \tag{3.24}
\]

which is valid for \(-\frac{1}{2} \geq \gamma_0 > -\frac{5}{2}\). Starting now from (3.24) and using (3.22), (3.23) and (2.4), in an analogous way one gets (3.1) with \(-\frac{3}{2} \geq \gamma_0 > -\frac{5}{2}\) and \(k_0 = 2\). Then, by induction (3.1) follows. Analogously it is easy to obtain the same asymptotic expression for \(I_{n+1}^{(A, \lambda)}(x)\). Hence, using (2.5) and (3.1) one gets

\[
I_{n+1}^{(A, \lambda)}(x) - I_{n}^{(A, \lambda)}(x) = e^{\frac{\lambda x}{2\sqrt{n}} n^{\frac{\gamma_0 + 1}{2}}} \log^{\gamma_1}\left(\frac{\lambda x}{2\sqrt{n}}\right) \log^{\gamma_2}\left(\sqrt{n}\right) \left[O(1, A) \left(\frac{n}{x}\right)^{k_1 - 1} + \frac{O(1, A)}{n} \left(\frac{n}{x}\right)^{k_0}\right], \tag{3.25}
\]

where \(k_1 = 0\) if \(\gamma_0 > \frac{1}{2}\) and \(\frac{3}{2} - k_1 \geq \gamma_0 \geq \frac{1}{2} - k_1, k_1 \in \mathbb{Z}\) for \(\gamma_0 \leq \frac{1}{2}\). Taking into account that if \(\gamma_0 > \frac{1}{2}\) then \(k_0 = k_1 = 0\), if \(-\frac{1}{2} < \gamma_0 \leq \frac{1}{2}\) then \(k_1 = 1, k_0 = 0\), and that if \(-\frac{3}{2} < k_0 < \gamma_0 \leq \frac{1}{2} - k_0, \gamma_0 \leq -\frac{1}{2}\), then \(k_1 = k_0 + 1\), the asymptotic expression of \(I_{n+1}^{(A, \lambda)}(x) - I_{n}^{(A, \lambda)}(x)\) follows.

**Remark 3.1.** Note that, taking into account (1.6), depending on the values of \(\hat{\mu}(A), \mu(A), \beta(A), \alpha(A)\) and the growth of the variables \(n\) and \(x\), the norm of the asymptotic expression (3.1) with \(\gamma_0\) taking the values \(\beta(A)\) and \(\alpha(A)\) could be greater than the norm of the same asymptotic expression with \(\gamma_0\) taking the values \(\hat{\mu}(A)\) and \(\mu(A)\) and \(\gamma_1 = \gamma_2 = 0\) or vice versa.
References