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A discontinuous mixed covolume method for elliptic problems☆

Qing Yang *, Ziwen Jiang

School of Mathematical Science, Shandong Normal University, Jinan, 250014, PR China

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ABSTRACT

We develop a discontinuous mixed covolume method for elliptic problems on triangular meshes. An optimal error estimate for the approximation of velocity is obtained in a mesh-dependent norm. First-order L^2-error estimates are derived for the approximations of both velocity and pressure.

1. Introduction

Consider the variable coefficient Poisson equation in a polynomial domain $\Omega \subset \mathbf{R}^2$,

$$\begin{cases} -\nabla \cdot (D\nabla p) = f, & \mathbf{x} = (x, y) \in \Omega, \\ D\nabla p \cdot \mathbf{n} = 0, & \mathbf{x} \in \partial\Omega, \end{cases} \tag{1}$$

where $D = (a_{ij})_{2\times 2}$ is a symmetric, bounded matrix function which satisfies the following condition: there exist two positive constants $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \xi^T \xi \leqslant \xi^T D \xi \leqslant \alpha_2 \xi^T \xi, \quad \forall \xi = (\xi_1, \xi_2) \in \mathbf{R}^2. \tag{2}$$

The function f satisfies the compatibility condition $\int_\Omega f \mathrm{d}\mathbf{x} = 0$. Furthermore, we assume that the matrix

$$\tau = D^{-1} = (\tau_{ij})_{2\times 2}$$

is locally Lipschitz.

Let $\mathbf{u} = -D\nabla p$, and rewrite the above equation as the system of first-order partial differential equations

$$\begin{cases} \tau \mathbf{u} + \nabla p = 0, & \mathbf{x} \in \Omega, \\ \operatorname{div} \mathbf{u} = f, & \mathbf{x} \in \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0, & \mathbf{x} \in \partial\Omega. \end{cases} \tag{3}$$

This system can be interpreted as modeling an incompressible single-phase flow in a reservoir, ignoring gravitational effects. The first equation is Darcy's law and the second represents conservation of mass, with f standing for a source or sink term.

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* Corresponding author.
E-mail address: sd_yangq@163.com (Q. Yang).

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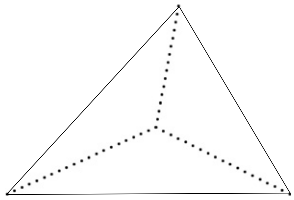

Fig. 1. Element $T \in T_h$ for a triangular mesh.

For the case when D is a diagonal matrix, Chou and Kwak [1] introduced a kind of mixed covolume method. They established the first-order error estimates for both velocity and pressure. Yang [2] extended the mixed covolume scheme given in [1] to a parabolic problem with nondiagonal diffusion tensor.

The study of discontinuous Galerkin methods has been a very active research area since its introduction in [3] in 1973. The discontinuous Galerkin method does not require continuity of the approximation functions across the interelement boundary but instead enforces the connection between elements by adding a penalty term. Because of the use of discontinuous functions, discontinuous Galerkin methods have the advantages of a high order of accuracy, high parallelizability, localizability, and easy handling of complicated geometries. Because of these advantages, discontinuous Galerkin methods have been used to solve hyperbolic and elliptic equations by many researchers. For example, see [4–12]. Arnold et al. [13] provided a framework for the analysis of a large class of discontinuous Galerkin methods for second-order elliptic problems. Most literature concerning discontinuous Galerkin methods for finite element approximations can be found in the references given in [13]. In [14,15], Ye developed a new discontinuous finite volume method for elliptic and Stokes problems, respectively. Based on the advantages of using discontinuous functions as trial functions as an approximation in discontinuous Galerkin methods, it is natural to consider using discontinuous trial functions in the mixed covolume method. In this study, we developed a new discontinuous mixed covolume method for elliptic problem. For the sake of simplicity and easy presentation of the main ideas of our method, we restrict ourselves to the model problem

$$\begin{cases} -\Delta p = f, & \mathbf{x} = (x, y) \in \Omega, \\ p = 0, & \mathbf{x} \in \partial\Omega, \end{cases} \tag{4}$$

where Ω is assumed to be a convex polygonal domain and f is a given function in $L^2(\Omega)$. In this case, the corresponding first-order system is

$$\begin{cases} \mathbf{u} + \nabla p = 0, & \mathbf{x} \in \Omega, \\ \operatorname{div} \mathbf{u} = f, & \mathbf{x} \in \Omega, \\ p = 0, & \mathbf{x} \in \partial\Omega. \end{cases} \tag{5}$$

In classical mixed covolume methods, the fluid velocity is approximated by piecewise linear functions in the lowest-order Raviart–Thomas space. In our methods, velocity is approximated by fully discontinuous piecewise linear functions and pressure is approximated by piecewise constant functions, respectively, on a triangular mesh. An optimal error estimate for the approximation of velocity is obtained in a mesh-dependent norm. First-order L^2-error estimates are derived for the approximations of both velocity and pressure. Our method and analysis can also be applied to solve more complex problems.

Throughout this article we use C to denote a generic constant independent of the discretization parameters.

2. Preliminaries and notations

We will use the standard definitions for the Sobolev spaces $H^s(K)$ and their associated inner products $(\cdot, \cdot)_{s,K}$, norms $\|\cdot\|_{s,K}$ and seminorms $|\cdot|_{s,K}$. The space $H^0(K)$ coincides with $L^2(K)$, in which case the norm and the inner product are denoted by $\|\cdot\|_K$ and $(\cdot, \cdot)_K$, respectively. If $K = \Omega$, we drop K.

Let $\mathcal{R}_h = \{K\}$ be a triangulation of the domain Ω; as usual, we assume the triangles K to be shape-regular. For a given triangulation $\mathcal{R}_h$, we construct a dual mesh $\mathcal{T}_h$ based upon the primal partition $\mathcal{R}_h$. Each triangle in $\mathcal{R}_h$ can be divided into three subtriangles by connecting the barycenter of the triangle to their corner nodes, as shown in Fig. 1. Then we define the dual partition $\mathcal{T}_h$ to be the union of the triangles shown in Fig. 1. Let $P_k(T)$ consist of all the polynomials functions of degree less than or equal to k defined on T. We define the finite-dimensional trial function space for velocity on $\mathcal{R}_h$ by

$$V_h := \{\mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_K \in P_1(K)^2,\ \forall K \in \mathcal{R}_h\}.$$

Define the finite-dimensional test function space W_h for velocity associated with the dual partition $\mathcal{T}_h$ as

$$W_h := \{\mathbf{w} \in L^2(\Omega)^2 : \mathbf{w}|_T \in P_0(T)^2,\ \forall T \in T_h\}.$$

Let Q_h be the finite-dimensional space for pressure:

$$Q_h := \{q \in L^2(\Omega) : q|_K \in P_0(K),\ \forall K \in \mathcal{R}_h\}.$$

Let Γ denote the union of the boundary of the triangles K of $\mathcal{R}_h$ and $\Gamma^\circ := \Gamma \setminus \partial\Omega$. The traces of functions in V_h and Q_h are double valued on Γ°. Let e be an interior edge shared by two triangles K_1 and K_2 in $\mathcal{R}_h$. Define the unit normal vectors

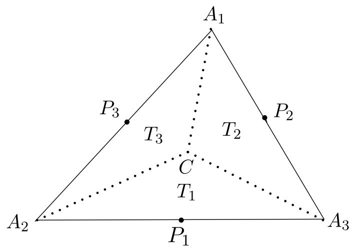

Fig. 2. Element $T \in T_h$ for a triangular mesh.

$\mathbf{n}_1$ and $\mathbf{n}_2$ on e pointing exterior to K_1 and K_2, respectively. Next, we introduce some traces operators that we will use in our numerical formulation. We define the average $\{\cdot\}$ and jump $[\cdot]$ on e for scalar q and vector $\mathbf{v}$, respectively.

$$\{q\} = \frac{1}{2}(q|_{\partial K_1} + q|_{\partial K_2}), \qquad [q] = q|_{\partial K_1}\mathbf{n}_1 + q|_{\partial K_2}\mathbf{n}_2,$$

$$\{\mathbf{v}\} = \frac{1}{2}(\mathbf{v}|_{\partial K_1} + \mathbf{v}|_{\partial K_2}), \qquad [\mathbf{v}] = \mathbf{v}|_{\partial K_1}\cdot\mathbf{n}_1 + \mathbf{v}|_{\partial K_2}\cdot\mathbf{n}_2.$$

If e is an edge on the boundary of Ω, we set

$$\{q\} = q, \qquad [\mathbf{w}] = \mathbf{w}\cdot\mathbf{n},$$

where $\mathbf{n}$ is the outward unit normal. We do not require either of the quantities $[q]$ or $\{\mathbf{v}\}$ on boundary edges, and we leave them undefined.

Multiplying the first and second equations in system (5) by $\mathbf{w} \in W_h$ and $q \in Q_h$, respectively, and using the integration by parts formula in the first equation, we have

$$\sum_{T\in\mathcal{T}_h}\int_T \mathbf{u}\cdot\mathbf{w}\mathrm{d}x + \sum_{T\in\mathcal{T}_h}\int_{\partial T} p\mathbf{w}\cdot\mathbf{n}\mathrm{d}s = 0 \tag{6}$$

and

$$\sum_{K\in\mathcal{R}_h}\int_K \nabla\cdot\mathbf{u}q\mathrm{d}x = (f, q), \tag{7}$$

where $\mathbf{n}$ is the unit outward normal vector on ∂T. Let $T_j \in \mathcal{T}_h$ ($j = 1, 2, 3$) be the triangles in $K \in \mathcal{R}_h$ (Fig. 2). Then we have

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T} p\mathbf{w}\cdot\mathbf{n}\mathrm{d}s = \sum_{K\in\mathcal{R}_h}\sum_{j=1}^{3}\int_{A_{j+1}CA_j} p\mathbf{w}\cdot\mathbf{n}\mathrm{d}s + \sum_{K\in\mathcal{R}_h}\int_{\partial K} p\mathbf{w}\cdot\mathbf{n}\mathrm{d}s, \tag{8}$$

where $A_4 = A_1$. A straightforward computation gives

$$\sum_{K\in\mathcal{R}_h}\int_{\partial K} q\mathbf{w}\cdot\mathbf{n}\mathrm{d}s = \sum_{e\in\Gamma^\circ}\int_e [q]\cdot\{\mathbf{v}\}\mathrm{d}s + \sum_{e\in\Gamma}\int_e \{q\}[\mathbf{v}]\mathrm{d}s. \tag{9}$$

Let $\int_\Gamma q\mathrm{d}s = \sum_{e\in\Gamma}\int_e q\mathrm{d}s$. Using (9) and the fact that $[p] = 0$ for $p \in H^1(\Omega)$ on Γ°, (8) becomes

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T} p\mathbf{w}\cdot\mathbf{n}\mathrm{d}s = \sum_{K\in\mathcal{R}_h}\sum_{j=1}^{3}\int_{A_{j+1}CA_j} p\mathbf{w}\cdot\mathbf{n}\mathrm{d}s + \sum_{e\in\Gamma}\int_e \{p\}[\mathbf{v}]\mathrm{d}s.$$

Let

$$a_0(\mathbf{v}, \mathbf{w}) := \sum_{T\in\mathcal{T}_h}\int_T \mathbf{v}\cdot\mathbf{w}\mathrm{d}x,$$

$$b(\mathbf{w}, q) := \sum_{K\in\mathcal{R}_h}\sum_{j=1}^{3}\int_{A_{j+1}CA_j} q\mathbf{w}\cdot n\mathrm{d}s + \sum_{e\in\Gamma}\int_e \{q\}[\mathbf{w}]\mathrm{d}s$$

and

$$c_0(\mathbf{v}, q) := \sum_{K\in\mathcal{R}_h}\int_K \nabla\cdot\mathbf{v}q\mathrm{d}x.$$

Using the above bilinear forms, it is clear that system (6)–(7) can be rewritten as the following:

$$a_0(\mathbf{u}, \mathbf{w}) + b(\mathbf{w}, p) = 0, \quad \forall\mathbf{w} \in W_h, \tag{10}$$

$$c_0(\mathbf{u}, q) = (f, q), \quad \forall q \in Q_h. \tag{11}$$

Let $V(h)=V_h+H^2(\Omega)^2$. Define a mapping $\gamma:V(h)\to W_h$ as

$$\gamma\mathbf{v}|_T=\frac{1}{h_e}\int_e \mathbf{v}|_T\mathrm{d}s,\quad T\in\mathcal{T}_h,$$

where h_e is the length of the edge e. For $\mathbf{v}=(v_1,v_2)$, γv_i $(i=1,2)$ is defined as

$$\gamma v_i|_T=\frac{1}{h_e}\int_e v_i|_T\mathrm{d}s,\quad T\in\mathcal{T}_h.$$

Using the operator γ, we define the following bilinear forms:

$$A_0(\mathbf{v},\mathbf{w}):=a_0(\mathbf{v},\gamma\mathbf{w}),\quad \forall\mathbf{v},\mathbf{w}\in V(h),$$
$$B(\mathbf{v},q):=b(\gamma\mathbf{v},q),\quad \forall\mathbf{v}\in V(h),\ \forall q\in L^2(\Omega),$$
$$C_0(\mathbf{v},q):=c_0(\mathbf{v},q),\quad \forall\mathbf{v}\in V(h),\ \forall q\in L^2(\Omega).$$

Then the system (10)–(11) are equivalent to

$$A_0(\mathbf{u},\mathbf{v})+B(\mathbf{v},p)=0,\quad \forall\mathbf{v}\in V_h, \tag{12}$$
$$C_0(\mathbf{u},q)=(f,q),\quad \forall q\in Q_h. \tag{13}$$

3. Discontinuous mixed covolume formulation

In order to define our numerical scheme, we introduce the bilinear forms as follows:

$$A(\mathbf{v},\mathbf{w}):=A_0(\mathbf{v},\mathbf{w})+\alpha\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\mathbf{v}][\mathbf{w}]\mathrm{d}s$$

and

$$C(\mathbf{v},q):=C_0(\mathbf{v},q)-\int_\Gamma\{q\}[\gamma\mathbf{v}]\mathrm{d}s,$$

where $\alpha>0$ is a parameter to be determined later. For the exact solution $(\mathbf{u},p)$ of system (5), we have

$$A_0(\mathbf{u},\mathbf{v})=A(\mathbf{u},\mathbf{v}),\quad \forall\mathbf{v}\in V_h,$$
$$C_0(\mathbf{u},q)=C(\mathbf{u},q),\quad \forall q\in Q_h.$$

Therefore, it follows from (12)–(13) that

$$A(\mathbf{u},\mathbf{v})+B(\mathbf{v},p)=0,\quad \forall\mathbf{v}\in V_h, \tag{14}$$
$$C(\mathbf{u},q)=(f,q),\quad \forall q\in Q_h. \tag{15}$$

The discontinuous mixed covolume scheme for (5) reads as follows. Seek $(\mathbf{u}_h,p_h)\in V_h\times Q_h$ such that

$$A(\mathbf{u}_h,\mathbf{v})+B(\mathbf{v},p_h)=0,\quad \forall\mathbf{v}\in V_h, \tag{16}$$
$$C(\mathbf{u}_h,q)=(f,q),\quad \forall q\in Q_h. \tag{17}$$

Let

$$B_*(\mathbf{v},q)=\sum_{K\in\mathcal{R}_h}\sum_{j=1}^{3}\int_{A_{j+1}CA_j}q\gamma\mathbf{v}\cdot n\mathrm{d}s.$$

Thus

$$B(\mathbf{v},q)=B_*(\mathbf{v},q)+\sum_{e\in\Gamma}\int_e\{q\}[\gamma\mathbf{v}]\mathrm{d}s.$$

To consider the positive definite property of bilinear forms $A(\cdot,\cdot)$ and the boundedness of bilinear forms $A(\cdot,\cdot)$, $B(\cdot,\cdot)$ and $C(\cdot,\cdot)$, we define the following norms for $\mathbf{v}\in V(h)$:

$$|||\mathbf{v}|||_{\mathrm{div}}^2=\|\mathbf{v}\|^2+\|\nabla_h\cdot\mathbf{v}\|^2+\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\mathbf{v}]^2\mathrm{d}s,$$
$$|||\mathbf{v}|||_1^2=\|\mathbf{v}\|^2+|\mathbf{v}|_{1,h}^2+\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\mathbf{v}]^2\mathrm{d}s,$$
$$|||\mathbf{v}|||=|||\mathbf{v}|||_1^2+\sum_{K\in\mathcal{R}_h}h_K^2|\mathbf{v}|_{2,K}^2,$$

where $\nabla_h\cdot\mathbf{v}$ is the function whose restriction to each element $K\in\mathcal{R}_h$ is equal to $\nabla\cdot\mathbf{v}$ and $|\mathbf{v}|_{1,h}^2=\sum_{K\in\mathcal{R}_h}|\mathbf{v}|_{1,K}^2$.

Let K be an element with e as an edge. It is well known that there exists a constant C such that, for any function $g \in H^2(K)$,

$$\|g\|_e^2 \le C(h_K^{-1}\|g\|_K^2 + h_K|g|_{1,K}^2), \tag{18}$$

$$\left\|\frac{\partial g}{\partial n}\right\|_e^2 \le C(h_K^{-1}|g|_{1,K}^2 + h_K|g|_{2,K}^2), \tag{19}$$

where C depends only on the minimum angle of K.

Lemma 3.1. *For $\mathbf{v}, \mathbf{w} \in V(h)$, we have*

$$A(\mathbf{v}, \mathbf{w}) \le C|||\mathbf{v}|||_{\mathtt{div}}|||\mathbf{w}|||_{\mathtt{div}}. \tag{20}$$

Proof. The definition of $A(\mathbf{v}, \mathbf{w})$ and the Cauchy–Schwarz inequality imply that

$$\begin{aligned}|A(\mathbf{v}, \mathbf{w})| &\le \|\mathbf{v}\|\,\|\gamma\mathbf{w}\| + \alpha\left(\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\mathbf{v}]^2\mathrm{d}s\right)^{\frac{1}{2}}\left(\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\mathbf{w}]^2\mathrm{d}s\right)^{\frac{1}{2}}\\ &\le C|||\mathbf{v}|||_{\mathtt{div}}|||\mathbf{w}|||_{\mathtt{div}}. \quad \square\end{aligned}$$

As in [15], we can prove the following lemma.

Lemma 3.2. *For any $(\mathbf{v}, q) \in V(h) \times L^2(\Omega)$, we have*

$$B_*(\mathbf{v}, q) = -(\nabla_h\cdot\mathbf{v}, q) + \sum_{K\in\mathcal{R}_h}\int_{\partial K}(\mathbf{v}-\gamma\mathbf{v})\cdot nq\mathrm{d}s + \sum_{K\in\mathcal{R}_h}(\nabla q, \gamma\mathbf{v}-\mathbf{v})_K. \tag{21}$$

Furthermore, if $q \in Q_h$, then

$$B_*(\mathbf{v}, q) = -(\nabla_h\cdot\mathbf{v}, q) \tag{22}$$

and

$$B(\mathbf{v}, q) = -C(\mathbf{v}, q). \tag{23}$$

Lemma 3.3. *For $(\mathbf{v}, q) \in V(h) \times L^2(\Omega)$, we have*

$$B(\mathbf{v}, q) \le C|||\mathbf{v}|||\left(\|q\| + \left(\sum_{K\in\mathcal{R}_h}h_K^2|q|_{1,K}^2\right)^{\frac{1}{2}}\right). \tag{24}$$

If $(\mathbf{v}, q) \in V_h \times Q_h$, then

$$B(\mathbf{v}, q) \le C|||\mathbf{v}|||\,\|q\|.$$

Proof. By Lemma 3.2, the inequality (18), and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|B_*(\mathbf{v}, q)| &\le |(\nabla_h\cdot\mathbf{v}, q)| + \left|\sum_{K\in\mathcal{R}_h}\int_{\partial K}(\mathbf{v}-\gamma\mathbf{v})\cdot nq\mathrm{d}s\right| + \left|\sum_{K\in\mathcal{R}_h}(\nabla q, \gamma\mathbf{v}-\mathbf{v})_K\right|\\ &\le C\left(|\mathbf{v}|_{1,h}\|q\| + \sum_{K\in\mathcal{R}_h}(h_K^{-1}\|\mathbf{v}-\gamma\mathbf{v}\|_K^2 + h_K|\mathbf{v}-\gamma\mathbf{v}|_{1,K}^2)^{\frac{1}{2}}(h_K^{-1}\|q\|_K^2 + h_K|q|_{1,K}^2)^{\frac{1}{2}} + \sum_{K\in\mathcal{R}_h}h|q|_{1,K}|\mathbf{v}|_{1,K}\right)\\ &\le C\left(|\mathbf{v}|_{1,h}\|q\| + |\mathbf{v}|_{1,h}\left(\|q\| + \left(\sum_{K\in\mathcal{R}_h}h_K^2|q|_{1,K}^2\right)^{\frac{1}{2}}\right) + |\mathbf{v}|_{1,h}\left(\sum_{K\in\mathcal{R}_h}h_K^2|q|_{1,K}^2\right)^{\frac{1}{2}}\right)\\ &\le C|||\mathbf{v}|||\left(\|q\| + \left(\sum_{K\in\mathcal{R}_h}h_K^2|q|_{1,K}^2\right)^{\frac{1}{2}}\right).\end{aligned}$$

The Cauchy–Schwarz inequality implies that

$$[\gamma\mathbf{v}]_e^2 = \left(\frac{1}{h_e}\int_e[\mathbf{v}]\mathrm{d}s\right)^2 \le \left(\frac{1}{h_e}\right)^2\int_e[\mathbf{v}]^2\mathrm{d}s\int_e\mathrm{d}s = \frac{1}{h_e}\int_e[\mathbf{v}]^2\mathrm{d}s. \tag{25}$$

Then

$$\sum_{e\in\Gamma}\int_e \{q\}[\gamma\mathbf{v}]\mathrm{d}s \le \left(\sum_{e\in\Gamma} h_e\int_e \{q\}^2\mathrm{d}s\right)^{\frac{1}{2}}\left(\sum_{e\in\Gamma} h_e^{-1}\int_e [\gamma\mathbf{v}]^2\mathrm{d}s\right)^{\frac{1}{2}}$$
$$\le \left(\sum_{e\in\Gamma} h_e\int_e \{q\}^2\mathrm{d}s\right)^{\frac{1}{2}}\left(\sum_{e\in\Gamma} h_e^{-1}\int_e [\mathbf{v}]^2\mathrm{d}s\right)^{\frac{1}{2}}$$
$$\le C|||\mathbf{v}|||\left(\|q\|^2+\sum_{K\in\mathcal{R}_h} h_K^2|q|_{1,K}^2\right)^{\frac{1}{2}}.$$

Noticing the definition of $B(\cdot,\cdot)$, we obtain (24) immediately. If $(\mathbf{v},q)\in V_h\times Q_h$, we have $|q|_{1,K}=0$. This completes the proof. □

Let

$$Z_h=\{\mathbf{v}|\mathbf{v}\in V_h, C(\mathbf{v},q)=0, \forall q\in Q_h\}.$$

We will prove the coercivity of the bilinear form $A(\cdot,\cdot)$ in Z_h in the following lemma.

Lemma 3.4. *For any $\mathbf{v}\in Z_h$, there is a constant C independent of h such that, for α large enough,*

$$A(\mathbf{v},\mathbf{v})\ge C|||\mathbf{v}|||^2_{\mathtt{div}}. \tag{26}$$

Proof. Recalling that

$$A(\mathbf{v},\mathbf{v})=A_0(\mathbf{v},\mathbf{v})+\alpha\sum_{e\in\Gamma}\frac{1}{h_e}\int_e[\mathbf{v}]^2\mathrm{d}s,$$

we first prove that there is a constant C_1 independent of h such that $A_0(\mathbf{v},\mathbf{v})\ge C_1\|\mathbf{v}\|^2$. For $\mathbf{v}=(v_1,v_2)\in V_h$, we have $\gamma v_i|_{T_j}=v_i|_{T_j}(P_j)$ (Fig. 2). Thus

$$A_0(\mathbf{v},\mathbf{v})=\sum_{T\in T_h}\int_T \mathbf{v}\cdot\boldsymbol{\gamma v}\mathrm{d}x=\sum_{K\in\mathcal{R}_h} I_K,$$

where

$$I_K=\sum_{j=1}^{3}\left(v_1(P_j)\int_{T_j}v_1\mathrm{d}x+v_2(P_j)\int_{T_j}v_2\mathrm{d}x\right)$$
$$\int_{T_1}v_1\mathrm{d}x=\frac{1}{3}(v_1(A_2)+v_1(A_3)+v_1(C))\frac{S_K}{3}=\frac{1}{3}\left[2v_1(P_1)+\frac{1}{3}(v_1(P_1)+v_1(P_2)+v_1(P_3))\right]\frac{S_K}{3}$$
$$=\frac{S_K}{3}\left[\frac{7}{9}v_1(P_1)+\frac{1}{9}v_1(P_2)+\frac{1}{9}v_1(P_3)\right],$$

where S_K denotes the area of triangle K. Similarly, we have

$$\int_{T_1}v_2\mathrm{d}x=\frac{S_K}{3}\left[\frac{7}{9}v_2(P_1)+\frac{1}{9}v_2(P_2)+\frac{1}{9}v_2(P_3)\right].$$

The integration of v_1 and v_2 on T_2 and T_3 can be computed similarly. As in [2], we have

$$I_K=\frac{2}{3}\int_K\mathbf{v}\cdot\mathbf{v}\mathrm{d}x+\frac{S_K}{3}\mathbf{v}(C)\cdot\mathbf{v}(C)\ge\frac{2}{3}\|\mathbf{v}\|_K^2.$$

Then we can prove that

$$A_0(\mathbf{v},\mathbf{v})\ge C_1\|\mathbf{v}\|^2$$

with $C_1=\frac{2}{3}$.

For $\mathbf{v}\in Z_h$, choosing $q=\nabla_h\cdot\mathbf{v}$, and from the definition of $C(\cdot,\cdot)$, we have

$$0=C(\mathbf{v},\nabla_h\cdot\mathbf{v})=(\nabla_h\cdot\mathbf{v},\nabla_h\cdot\mathbf{v})-\int_\Gamma\{\nabla_h\cdot\mathbf{v}\}[\gamma\mathbf{v}]\mathrm{d}s.$$

The trace inequality (18) and the inequality (25) give that, for $\mathbf{v} \in V_h$,

$$\begin{aligned}\int_\Gamma \{\nabla_h \cdot \mathbf{v}\}[\gamma \mathbf{v}]\mathrm{d}s &\le \left(\sum_{e\in\Gamma}\int_e h_e\{\nabla\cdot\mathbf{v}\}^2\mathrm{d}s\right)^{\frac12}\left(\sum_{e\in\Gamma}\int_e h_e^{-1}[\gamma\mathbf{v}]^2\mathrm{d}s\right)^{\frac12} \\ &\le C_2\left(\sum_{K\in\mathcal{R}_h}(\|\nabla\cdot\mathbf{v}\|_K^2 + h_K^2|\nabla\cdot\mathbf{v}|_{1,K}^2)\right)^{\frac12}\left(\sum_{e\in\Gamma}h_e^{-1}\int_e[\mathbf{v}]^2\mathrm{d}s\right)^{\frac12} \\ &= C_2\|\nabla_h\cdot\mathbf{v}\|\left(\sum_{e\in\Gamma}h_e^{-1}\int_e[\mathbf{v}]^2\mathrm{d}s\right)^{\frac12}.\end{aligned}$$

By using the above inequalities, we have that

$$\begin{aligned}A(\mathbf{v},\mathbf{v}) &= A(\mathbf{v},\mathbf{v}) + C(\mathbf{v},\nabla_h\cdot\mathbf{v}) \\ &= A_0(\mathbf{v},\mathbf{v}) + \alpha\sum_{e\in\Gamma}h_e^{-1}\int_e[\mathbf{v}]^2\mathrm{d}s + (\nabla_h\cdot\mathbf{v},\nabla_h\cdot\mathbf{v}) - \int_\Gamma\{\nabla_h\cdot\mathbf{v}\}[\gamma\mathbf{v}]\mathrm{d}s \\ &\ge C_1\|\mathbf{v}\|^2 + \alpha\sum_{e\in\Gamma}h_e^{-1}\int_e[\mathbf{v}]^2\mathrm{d}s + \|\nabla_h\cdot\mathbf{v}\|^2 - C_2\|\nabla_h\cdot\mathbf{v}\|\left(\sum_{e\in\Gamma}h_e^{-1}\int_e[\mathbf{v}]^2\mathrm{d}s\right)^{\frac12} \ge C|||\mathbf{v}|||_{\mathtt{div}}^2.\end{aligned}$$

The last inequality is obtained by using the ε-Cauchy inequality and choosing α large enough. Thus we obtain the conclusion of this lemma. □

4. Error estimates

We will derive an optimal error estimates for velocity in the norm $|||\cdot|||_{\mathtt{div}}$ and for pressure in the L^2-norm. A first-order error estimate for velocity in the L^2-norm will be obtained.

Let e be an interior edge shared by two elements K_1 and K_2 in $\mathcal{R}_h$. If $\int_e \mathbf{v}|_{K_1}\mathrm{d}s = \int_e \mathbf{v}|_{K_2}\mathrm{d}s$, we say that $\mathbf{v}$ is continuous on e. We say that $\mathbf{v}$ is zero at $e \in \partial\Omega$ if $\int_e \mathbf{v}\mathrm{d}s = 0$. Define a subspace $\hat{V}_h$ of V_h by

$$\hat{V}_h = \{\mathbf{v}\in L^2(\Omega)^2 : \forall K\in\mathcal{R}_h, \mathbf{v}|_K\in P_1(K)^2, \mathtt{is\ continuous\ at}\ e\in\Gamma^\circ \mathtt{and\ is\ zero\ at}\ e\in\partial\Omega\}.$$

It has been proved in [16,17] that the following discrete inf-sup condition is satisfied; i.e., there exists a positive constant β_0 such that

$$\sup_{\mathbf{v}\in\hat{V}_h}\frac{(\nabla_h\cdot\mathbf{v},q)}{|\mathbf{v}|_{1,h}} \ge \beta_0\|q\|, \quad \forall q\in Q_h. \tag{27}$$

Lemma 4.1. *The bilinear form $C(\cdot,\cdot)$ satisfies the discrete inf-sup condition*

$$\sup_{\mathbf{v}\in V_h}\frac{C(\mathbf{v},q)}{|||\mathbf{v}|||} \ge \beta\|q\|, \quad \forall q\in Q_h, \tag{28}$$

where β is a positive constant independent of the mesh size h.

Proof. For $\mathbf{v}\in\hat{V}_h\subset V_h$, we have $C(\mathbf{v},q) = (\nabla_h\cdot\mathbf{v},q)$ and

$$|||\mathbf{v}|||_1^2 = \|\mathbf{v}\|^2 + |\mathbf{v}|_{1,h}^2 + \sum_{e\in\Gamma}[\gamma\mathbf{v}]_e^2 = \|\mathbf{v}\|^2 + |\mathbf{v}|_{1,h}^2 \le C_1|\mathbf{v}|_{1,h}^2.$$

We have used Poincaré's inequality to obtain the above inequality. The standard inverse inequality implies that there is a constant C_2 such that

$$|||\mathbf{v}||| \le C_2|||\mathbf{v}|||_1, \quad \forall\mathbf{v}\in V_h.$$

Then (27) implies that, for any $q\in Q_h$,

$$\beta_0\|q\| \le \sup_{\mathbf{v}\in\hat{V}_h}\frac{(\nabla_h\cdot\mathbf{v},q)}{|\mathbf{v}|_{1,h}} \le \frac{1}{\sqrt{C_1}}\sup_{\mathbf{v}\in\hat{V}_h}\frac{C(\mathbf{v},q)}{|||\mathbf{v}|||_1} \le \frac{1}{C_2\sqrt{C_1}}\sup_{\mathbf{v}\in V_h}\frac{C(\mathbf{v},q)}{|||\mathbf{v}|||}.$$

With $\beta = \beta_0C_2\sqrt{C_1}$, we have proved (28). □

Define an operator π_K from $H^1(K)$ to $P_1(K)$ by requiring that, for any $v \in H^1(K)$,

$$\int_{e_i} \pi_K v \mathrm{d}s = \int_{e_i} v \mathrm{d}s, \quad \text{for } i = 1, 2, 3, \tag{29}$$

where e_i, $i = 1, 2, 3$, are the three sides of the element $K \in \mathcal{R}_h$. It was proved in [14] that

$$|\pi_K v - v|_{s,K} \le h^{2-s} |v|_{2,K}, \quad \forall v \in H^2(K),\ s = 0, 1, 2. \tag{30}$$

For any $\mathbf{v} \in H_0^1(\Omega)^2$, define $\Pi_1 \mathbf{v} \in V_h$ by

$$(\Pi_1 \mathbf{v})_i|_K = \pi_K v_i, \quad \forall K \in \mathcal{R}_h,\ i = 1, 2. \tag{31}$$

Using the definition of Π_1 and integration by parts, we can show that

$$C(\mathbf{v} - \Pi_1 \mathbf{v}, q) = 0, \quad \forall q \in Q_h. \tag{32}$$

The inequalities (18) and (30) imply that

$$\begin{aligned} \sum_{e \in \Gamma} h_e^{-1} \int_e [\mathbf{u} - \Pi_1 \mathbf{u}]^2 \mathrm{d}s &\le C \left(|\mathbf{u} - \Pi_1 \mathbf{u}|_{1,h}^2 + \sum_{K \in \mathcal{R}_h} h^{-2} \|\mathbf{u} - \Pi_1 \mathbf{u}\|_K^2 \right) \\ &\le Ch^2 \|\mathbf{u}\|_2^2. \end{aligned} \tag{33}$$

The definition of the norm $|||\cdot|||_{\mathtt{div}}$, together with (33) and (30), gives that

$$\begin{aligned} |||\mathbf{u} - \Pi_1 \mathbf{u}|||_{\mathtt{div}}^2 &\le \|\mathbf{u} - \Pi_1 \mathbf{u}\|^2 + 2|\mathbf{u} - \Pi_1 \mathbf{u}|_{1,h}^2 + \sum_{e \in \Gamma} h_e^{-1} \int_e [\mathbf{u} - \Pi_1 \mathbf{u}]^2 \mathrm{d}s \\ &\le Ch^2 \|\mathbf{u}\|_2^2. \end{aligned} \tag{34}$$

Let Π_2 be the L^2 projection from $L_0^2(\Omega)$ to the finite element space Q_h.

Theorem 4.2. *Let $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ be the solution of (16)–(17) and $(\mathbf{u}, p) \in H^2(\Omega)^2 \times H^1(\Omega)$ be the solution of (5). Then there exists a constant C independent of h such that*

$$|||\mathbf{u} - \mathbf{u}_h|||_{\mathtt{div}} + \|p - p_h\| \le Ch(\|\mathbf{u}\|_2 + \|p\|_1). \tag{35}$$

Proof. Let

$$\xi_h = \mathbf{u}_h - \Pi_1 \mathbf{u}, \qquad \eta_h = p_h - \Pi_2 p$$

be the error between the finite volume solution $(\mathbf{u}_h, p_h)$ and the projection $(\Pi_1 \mathbf{u}, \Pi_2 p)$ of the exact solution. Denote by

$$\xi = \mathbf{u} - \Pi_1 \mathbf{u}, \qquad \eta = p - \Pi_2 p$$

the error between the exact solution $(\mathbf{u}, p)$ and its projection. Subtracting (16) and (17) from (14) and (15), respectively, and using Lemma 3.2 gives that

$$A(\xi_h, \mathbf{v}) - B(\mathbf{v}, \eta_h) = A(\xi, \mathbf{v}) - B(\mathbf{v}, \eta), \quad \forall \mathbf{v} \in V_h, \tag{36}$$

$$C(\xi_h, q) = C(\xi, q) = 0, \quad \forall q \in Q_h. \tag{37}$$

By letting $\mathbf{v} = \xi_h$ in (36) and $q = \eta_h$ in (37), the sum of (36) and (37) gives

$$A(\xi_h, \xi_h) = A(\xi, \xi_h) - B(\xi_h, \eta). \tag{38}$$

Noting that ξ_h satisfies (37), we know that $\xi_h \in Z_h$. Thus, it follows from the coercivity (26) and the boundedness (20) that

$$A(\xi_h, \xi_h) \ge C|||\xi_h|||_{\mathtt{div}}^2$$

and

$$A(\xi, \xi_h) \le C|||\xi|||_{\mathtt{div}} |||\xi_h|||_{\mathtt{div}}.$$

We now bound the last term in (38). Noting Lemma 3.2, we have

$$B(\xi_h, \eta) = -(\nabla_h \cdot \xi_h, \eta) + \sum_{K \in \mathcal{R}_h} \int_{\partial K} \eta(\xi_h - \gamma \xi_h) \cdot n \mathrm{d}s + \sum_{K \in \mathcal{R}_h} (\nabla \eta, \gamma \xi_h - \xi_h)_K + \sum_{e \in \Gamma} \int_e \{\eta\}[\gamma \xi_h] \mathrm{d}s.$$

From the definition of the operator Π_2, we know that

$$(\nabla_h \cdot \xi_h, \eta) = (\nabla_h \cdot \xi_h, p - \Pi_2 p) = 0.$$

Using the equality (9) and the definition of γ, we have that

$$\sum_{K\in\mathcal{R}_h}\int_{\partial K}\eta(\xi_h-\gamma\xi_h)\cdot\mathbf{n}\mathrm{d}s=\sum_{e\in\Gamma^\circ}\int_e[\eta]\cdot(\xi_h-\gamma\xi_h)\mathrm{d}s+\sum_{e\in\Gamma}\int_e\{\eta\}[\xi_h-\gamma\xi_h]\mathrm{d}s$$
$$=\sum_{e\in\Gamma}\int_e\{\eta\}[\xi_h-\gamma\xi_h]\mathrm{d}s.$$

Thus, we have that

$$B(\xi_h,\eta)=-\sum_{K\in\mathcal{R}_h}(\mathbf{u},\gamma\xi_h-\xi_h)_K+\sum_{e\in\Gamma}\int_e\{\eta\}[\xi_h]\mathrm{d}s.$$

Let $\bar{\mathbf{u}}=(\bar{u}_1,\bar{u}_2)$ be the interpolation of $\mathbf{u}$, which is a constant vector on the element K. For any $\mathbf{v}=(v_1,v_2)\in V_h$, we can easily deduce that $(\bar{\mathbf{u}},\gamma\mathbf{v}-\mathbf{v})=0$. In fact, from the definition of γ, we know that

$$\begin{aligned}(\bar{u}_1,\gamma v_1)_K&=\frac{S_K}{3}\bar{u}_1(\gamma v_1|_{T_1}+\gamma v_1|_{T_2}+\gamma v_1|_{T_3})\\&=\frac{S_K}{3}\bar{u}_1\left\{\frac{1}{2}[v_1(A_1)+v_1(A_2)]+\frac{1}{2}[v_1(A_2)+v_1(A_3)]+\frac{1}{2}[v_1(A_1)+v_1(A_3)]\right\}\\&=\frac{S_K}{3}\bar{u}_1(v_1(A_1)+v_1(A_2)+v_1(A_3))\end{aligned}$$

and

$$(\bar{u}_1,v_1)_K=\bar{u}_1\int_K v_1\mathrm{d}x\mathrm{d}y=\frac{S_K}{3}\bar{u}_1(v_1(A_1)+v_1(A_2)+v_1(A_3)).$$

That is, $(\bar{u}_1,\gamma v_1-v_1)=0$. Similarly, we can prove that $(\bar{u}_2,\gamma v_2-v_2)=0$. Then, we obtain that

$$\sum_{K\in\mathcal{R}_h}(\mathbf{u},\gamma\xi_h-\xi_h)_K=\sum_{K\in\mathcal{R}_h}(\mathbf{u}-\bar{\mathbf{u}},\gamma\xi_h-\xi_h)_K\le Ch|\mathbf{u}|_1\|\xi_h\|.$$

The trace inequality implies that

$$\begin{aligned}\sum_{e\in\Gamma}\int_e\{\eta\}[\xi_h]\mathrm{d}s&\le C\left(\|\eta\|^2+\sum_{K\in\mathcal{R}_h}h_K^2|\eta|_{1,K}^2\right)^{\frac{1}{2}}\left(\sum_{e\in\Gamma}h_e^{-1}\int_e[\xi_h]^2\mathrm{d}s\right)^{\frac{1}{2}}\\&\le Ch\|p\|_1|||\xi_h|||_{\mathtt{div}}.\end{aligned}$$

Thus, we can bound the last term in (38):

$$B(\xi_h,\eta)\le Ch(\|\mathbf{u}\|_1+\|p\|_1)|||\xi_h|||_{\mathtt{div}}.$$

Combining the above boundedness of (38), we have that

$$|||\xi_h|||_{\mathtt{div}}^2\le Ch(\|\mathbf{u}\|_2+\|p\|_1)|||\xi_h|||_{\mathtt{div}},$$

which implies the following:

$$|||\xi_h|||_{\mathtt{div}}\le Ch(\|\mathbf{u}\|_2+\|p\|_1).$$

Now, using the triangle inequality, (29), the definition of Π_2, and the inequality above, we get

$$|||\mathbf{u}-\mathbf{u}_h|||_{\mathtt{div}}\le|||\mathbf{u}-\Pi_1\mathbf{u}|||_{\mathtt{div}}+|||\mathbf{u}_h-\Pi_1\mathbf{u}|||_{\mathtt{div}}\le Ch(\|\mathbf{u}\|_2+\|p\|_1),\tag{39}$$

which completes the estimate for the velocity approximation.

The discrete inf-sup condition (27), (39), Lemmas 3.2, 3.4, and the inverse inequality give

$$\begin{aligned}\|p_h-\Pi_2p\|&\le\frac{1}{\beta}\sup_{\mathbf{v}\in V_h}\frac{C(\mathbf{v},p_h-\Pi_2p)}{|||\mathbf{v}|||}\le\frac{1}{\beta}\sup_{\mathbf{v}\in V_h}\frac{B(\mathbf{v},\Pi_2p-p_h)}{|||\mathbf{v}|||}\\&=\frac{1}{\beta}\sup_{\mathbf{v}\in V_h}\frac{B(\mathbf{v},p-p_h)+B(\mathbf{v},\Pi_2p-p)}{|||\mathbf{v}|||}\\&=\frac{1}{\beta}\sup_{\mathbf{v}\in V_h}\frac{A(\mathbf{u}-\mathbf{u}_h,\mathbf{v})+B(\mathbf{v},\Pi_2p-p)}{|||\mathbf{v}|||}\end{aligned}$$

$$\leq C\left(\|\!|\mathbf{u}-\mathbf{u}_h\|\!|_{\mathtt{div}} + \|\Pi_2 p - p\| + \left(\sum_{K\in\mathcal{R}_h} h_K^2 |p-\Pi_2 p|_{1,K}^2\right)^{\frac{1}{2}}\right)$$
$$\leq Ch(\|\mathbf{u}\|_2 + \|p\|_1).$$

Using the above inequality and the triangle inequality, we have completed the proof of (35). □

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