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Arrangements in Unitary and Orthogonal Geometry over Finite Fields*

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Let V be an n-dimensional vector space over $\mathbf{F}_q$. Let Φ be a Hermitian form with respect to an automorphism σ with $\sigma^2=1$. If $\sigma=1$ assume that q is odd. Let $\mathscr{A}$ be the arrangement of hyperplanes of V which are non-isotropic with respect to Φ, and let L be the intersection lattice of $\mathscr{A}$. We prove that the characteristic polynomial of L has $n-v$ roots $1, q, \ldots, q^{n-v-1}$ where v is the Witt index of Φ.

1. INTRODUCTION

Let K be a field and let V be a vector space of finite dimension n over K. An arrangement in V is a finite set $\mathscr{A}$ of hyperplanes, all containing the origin, such that $\bigcap_{H\in\mathscr{A}} H=0$. Let $L=L(\mathscr{A})$ be the set of intersections of elements of $\mathscr{A}$. Partially order L by reverse inclusion so that L has V as its minimal element and $\mathscr{A}$ as its set of atoms. The poset L is a finite geometric lattice with rank function $r(X)=\dim(V/X)$, $X\in L$. The characteristic polynomial $\chi(L,t)$ of L is defined by

$$\chi(L,t)=\sum_{X\in L}\mu(V,X)\,t^{\dim X} \tag{1.1}$$

where μ is the Möbius function of L. Suppose $K=\mathbf{F}_q$ is a finite field of q elements. If $\mathscr{A}$ consists of all hyperplanes in V and $M=L(\mathscr{A})$ then [1, p. 155]

$$\chi(M,t)=\prod_{i=0}^{n-1}(t-q^i). \tag{1.2}$$

Let σ be an automorphism of K with $\sigma^2=1$. Let Φ be a non-degenerate Hermitian form with respect to σ. Thus $\Phi(x,y)=\Phi(y,x)^\sigma$. We allow σ to be the identity in which case Φ is a symmetric bilinear form, but assume in

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this case that q is odd. Let $\mathscr{A}$ be the set of all hyperplanes in V which are non-isotropic with respect to Φ and let $L = L(\mathscr{A})$. In this paper we compute $\chi(L, t)$ and show that it has $n - v$ integer roots $1, q, \ldots, q^{n-v-1}$ where v is the Witt index of Φ.

(1.3) THEOREM. *Let a_k be the number of subspaces X of V such that* $\dim X = k$ *and* $\dim X + \dim \operatorname{rad} X = n$. *Then*

$$\chi(L, t) = \sum_{k=0}^{n} a_k (t-1)(t-q) \cdots (t-q^{k-1}).$$

(1.4) COROLLARY. *Let v be the Witt index of Φ. Then*

$$\chi(L, t) = (t-1)(t-q) \cdots (t-q^{n-v-1})\, \gamma(t)$$

where $\gamma(t) \in \mathbf{Z}[t]$ is a monic polynomial of degree v.

In the orthogonal case where $\sigma = 1$ and Φ is a symmetric bilinear form the values of a_k are given by (2.12). In the unitary case where $\sigma \neq 1$ it is convenient to change notation so that $K = \mathbf{F}_{q^2}$ and $x^\sigma = x^q$. The values of a_k are given using this notation in (2.15). To calculate $\chi(L, t)$ using these values of a_k one must remember to replace $t - q^i$ by $t - q^{2i}$ in formula (1.3). The polynomials $\gamma(t)$ do not in general have integer roots if $v \geqslant 2$.

In [6] we studied the arrangement of reflecting hyperplanes for a finite unitary reflection group $G \subset GL(n, \mathbf{C})$ and found that the corresponding characteristic polynomial has the form

$$\chi(L, t) = \prod_{i=1}^{n} (t - n_i) \tag{1.5}$$

where the n_i are positive integers which occur in the invariant theory of G. The proof of (1.5) was based on the equality $\chi(L, t) = P_\delta(G, t)$ where

$$P_\delta(G, t) = \sum_{g \in G} \delta(g)\, t^{k(g)}. \tag{1.6}$$

In this formula $\delta(g) = \det g$ and $k(g)$ is the dimension of the fixed point set of g. Since the group $G(\Phi)$ of isometries of Φ is generated by reflections in non-isotropic hyperplanes, (1.4) may be viewed as an analog of (1.5). Choose a monomorphism $\theta: K^\times \to \mathbf{C}^\times$ and let δ be the linear character of $G(\Phi)$ defined by $\delta(g) = \theta(\det g)$. We may ask whether $\chi(L, t) = P_\delta(G, t)$ for the groups $G = G(\Phi)$ of this paper. In case $K = \mathbf{F}_{q^2}$ and Φ is Hermitian with respect to the automorphism $x \to x^q$ we showed in [7] that

$$P_\delta(G, t) = \prod_{i=0}^{n-1} (t - (-q)^i). \tag{1.7}$$

Thus $\chi(L, t) \neq P_\delta(G, t)$ if $n > 1$. On the other hand we show in (3.5) that if $K = \mathbf{F}_q$, q odd, and Φ is a symmetric bilinear form with Witt index $v = 0$ or $v = 1$ then $\chi(L, t) = P_\delta(G, t)$. Kusuoka [5] has shown for all v that

$$P_\delta(G, t) = (t-1)(t-q)\cdots(t-q^{n-v-1})\,\beta(t) \tag{1.8}$$

where $\beta(t) \in \mathbf{Z}[t]$ is a monic polynomial of degree v. Thus $\chi(L, t)$ and $P_\delta(G, t)$ have $n - v$ roots in common. We give an example in Section 3 which shows that $\chi(L, t) \neq P_\delta(G, t)$ in general.

2. Proof of the Theorem

We use the usual terminology for Hermitian forms. The finiteness of K is not used in (2.1)–(2.3). Recall that if $\sigma = 1$ so Φ is symmetric bilinear then we assume char $K \neq 2$. Thus we may use the Witt decomposition [3, Sect. 4.2]. If X is a subspace of V let X^0 be its orthogonal subspace and let rad $X = X \cap X^0$. Say that X is non-isotropic if rad $X = 0$ and totally isotropic if rad $X = X$. In [7] we introduced a Witt decomposition adapted to X. This is described as follows. Let $Z = \text{rad}\, X$. There exist subspaces Y, Z', W such that

$$V = Y \oplus (Z \oplus Z') \oplus W, \qquad X = Y \oplus Z \tag{2.1}$$

where (i) Z' is totally isotropic with $\dim Z' = \dim Z$, and (ii) Y, $Z \oplus Z'$, W are non-isotropic and pairwise orthogonal. If $X = Z$ is totally isotropic this is the usual Witt decomposition. We define $\rho(X) = \dim X + \dim \text{rad}\, X$.

(2.2) Lemma. *Let $X \subseteq Y$ be subspaces of V. Then $\rho(X) \leqslant \rho(Y)$.*

Proof. We may assume that $\dim Y = 1 + \dim X$. Choose a basis $u_1, ..., u_m, v$ for Y where the notation is chosen so that $u_1, ..., u_m$ is a basis for X and $u_1, ..., u_j$ is a basis for rad X. The matrix $[\Phi_Y]$ of Φ_Y in this basis is

$$[\Phi_Y] = \begin{bmatrix} 0 & 0 & * \\ 0 & A & * \\ * & * & * \end{bmatrix}$$

where A is invertible of size $m - j$ and the entries in the last column are $\Phi(u_i, v)$ and $\Phi(v, v)$. Thus $\text{rank}[\Phi_Y] \leqslant 2 + \text{rank}\, A = 2 + \text{rank}[\Phi_X]$. Since $\dim(Y/\text{rad}\, Y) = \text{rank}[\Phi_Y]$ and $\dim(X/\text{rad}\, X) = \text{rank}[\Phi_X]$ we get $\dim \text{rad}\, X \leqslant 1 + \dim \text{rad}\, Y$. ∎

Let $\mathscr{A}$ be the set of all hyperplanes in V which are non-isotropic with respect to Φ and let L be the lattice of intersections of elements of $\mathscr{A}$.

(2.3) LEMMA. *Let $X \neq V$ be a subspace of V. Then $X \in L$ if and only if $\rho(X) \leqslant n-1$.*

Proof. Suppose $X \in L$ and $X \neq V$. Choose $Y \in \mathscr{A}$ such that $X \subseteq Y$. By Lemma 2.2 we have $\rho(X) \leqslant \rho(Y)$. Since Y is non-isotropic $\rho(Y) = n-1$. Conversely suppose $\rho(X) \leqslant n-1$. To show that $X \in L$ we do two special cases by explicit computation and then do the general case using the Witt decomposition (2.1).

Case (i). X is non-isotropic. Let $v_1, ..., v_m$ be an orthogonal basis for X^0 [3, Sect. 6, Theorem 1]; if $\sigma = 1$ we use the assumption that K has odd characteristic. Then $H_i = X \oplus \sum_{j \neq i} Kv_j$ is in $\mathscr{A}$ and $X = H_1 \cap \cdots \cap H_m$.

Case (ii). X is totally isotropic and $2 \dim X = \rho(X) = n-1$. Choose a Witt decomposition $V = (X \oplus X') \oplus Kv$ where X' is totally isotropic and v is non-isotropic and orthogonal to $X \oplus X'$. Choose bases $e_1, ..., e_m$ for X and $e'_1, ..., e'_m$ for X' such that $(e_i, e'_j) = \delta_{ij}$. Let $H_0 = X \oplus X'$ and for $i = 1, ..., m$ let $H_i = X \oplus \langle e'_1, ..., e'_i + v, ..., e'_m \rangle$. Clearly $H_0 \in \mathscr{A}$. Suppose $1 \leqslant i \leqslant m$ and $w \in \operatorname{rad} H_i$. Write $w = \sum a_j e_j + \sum_{j \neq i} a'_j e'_j + a'_i (e'_i + v)$ where $a_j, a'_j \in K$. Since $\Phi(v, e_j) = \Phi(v, e'_j) = 0$ for $j = 1, ..., m$ we have $0 = \Phi(w, e_k) = a'_k$ and $0 = \Phi(w, e'_k) = a_k$ for $k = 1, ..., m$. Thus $w = 0$ so $\operatorname{rad} H_i = 0$ and $H_i \in \mathscr{A}$. Since $X = H_0 \cap H_1 \cap \cdots \cap H_m$ we have $X \in L$.

Now consider the general case. Let $V = Y \oplus (Z \oplus Z') \oplus W$ be a Witt decomposition adapted to X. Then $\dim W = n - \rho(X) > 0$ by assumption. Choose a non-isotropic vector $w \in W$ and a subspace U orthogonal to Kw such that $W = Kw \oplus U$. Apply case (ii) to the totally isotropic subspace Z of $(Z \oplus Z') \oplus Kw$. Thus there exist non-isotropic subspaces $Z_1, ..., Z_m$ of codimension one in $(Z \oplus Z') \oplus Kw$ such that $Z = Z_1 \cap \cdots \cap Z_m$. Then $X_i = Y \oplus Z_i$ is non-isotropic because Y and Z_i are orthogonal, and $X = X_1 \cap \cdots \cap X_m$. Now the lemma follows from case (i) applied to each of the spaces X_i. ∎

We assume now that $K = \mathbf{F}_q$ is finite and prove Theorem 1.3. If $X \in L$ let $L^X = \{Y \in L \mid Y \geqslant X\}$. Let M be the lattice of all subspaces of V partially ordered by reverse inclusion. If $X \in L$ and $Y \in M$ and $Y \geqslant X$ then $Y \in L$ by (2.2). Thus $L^X = M^X$. Since M^X is isomorphic to the lattice of all subspaces of X (1.2) gives

$$\chi(L^X, t) = (t-1) \cdots (t - q^{k-1}) = \chi(M^X, t), \qquad k = \dim X$$

for all $X \in L$ with $X \neq V$. By Möbius inversion $t^n = \sum_{X \in L} \chi(L^X, t)$ and $t^n = \sum_{X \in M} \chi(M^X, t)$. Since $L^X = M^X$ whenever $X \in L$ and $X \neq V$ we get

$$\chi(L, t) = \sum_{\substack{X \in M \\ \rho(X) = n}} \chi(M^X, t).$$

Since a_k is the number of subspaces X of V with $\dim X=k$ and $\rho(X)=n$ this proves (1.3).

To prove (1.4) note that if $a_k\neq 0$ then there exists $X\in M$ with $\dim X=k$ and $\rho(X)=n$. Thus we have $\dim \operatorname{rad} X\leqslant v$ so $k\geqslant n-v$. ▮

To compute the a_k we use Witt's theorem on extension of isometries. If X is a subspace of V let Φ_X denote the restriction of Φ to X. We say the subspaces X, X' are isometric and write $X\approx X'$ if there exists an invertible linear map $h: X\to X'$ such that $\Phi(hx, hy)=\Phi(x, y)$ for all $x, y\in X$. Let $G(\Phi)$ be the group of isometries of V. Witt's theorem [3, Sect. 4.3, Theorem 1] states that every isometry $h: X\to X'$ may be extended to an element of $G(\Phi)$. Let A_k be the set of all subspaces X of V such that $\dim X=k$ and $\rho(X)=n$. Thus $a_k=|A_k|$.

(2.4) LEMMA. *The set A_k forms a single orbit in the action of $G(\Phi)$ on the set of subspaces of V.*

Proof. Suppose $X\in A_k$. In the Witt decomposition (2.1) we have $n=\dim X+\dim Z'+\dim W=\rho(X)+\dim W$. Thus $W=0$ and

$$V=Y\oplus(Z\oplus Z'). \tag{2.5}$$

Suppose $X_i\in A_k$ where $i=1, 2$. Let Y_i, Z_i, Z_i' be the corresponding subspaces in (2.5). Since $Z_1\oplus Z_1'\approx Z_2\oplus Z_2'$ it follows from Witt's theorem [3, Sect. 4.3, Corollary 1] that $Y_1\approx Y_2$. Thus $X_1\approx X_2$. The lemma follows using Witt's theorem again. ▮

Let $G(\Phi_X)$ be the group of isometries of Φ_X; the form Φ_X may be degenerate. Let $G=G(\Phi)$. Let $G_X=\{g\in G\mid gx=x \text{ for all } x\in X\}$ be the fixer of X. Let $\mathcal{O}_X$ be the orbit of X. As in [7] it follows from (2.5) that

$$|\mathcal{O}_X|\,|G_X|=|G: G(\Phi_X)| \tag{2.6}$$

$$|G(\Phi_X)|=|\operatorname{Hom}(Y, Z)|\,|G(\Phi_Y)|\,|GL(Z)|. \tag{2.7}$$

Since Y is non-isotropic the $|G(\Phi_Y)|$ are known and are given in (2.10), (2.11) and (2.14). Thus if $X\in A_k$ then (2.6) and (2.7) determine $a_k=|\mathcal{O}_X|$ provided we can calculate $|G_X|$. Choose a basis for V adapted to the decomposition (2.5) so that the matrix for Φ is

$$[\Phi]=\begin{bmatrix} B & 0 & 0\\ 0 & 0 & I\\ 0 & I & 0\end{bmatrix} \tag{2.8}$$

where $B^\sigma=B^{\mathrm{T}}$ is Hermitian and I is the identity matrix. If $g\in G_X$ then, since $Y\subseteq X$, g fixes Y. Since $Z\oplus Z'$ is the orthogonal complement of Y in

V we have $g(Z\oplus Z')\subseteq Z\oplus Z'$. Since $Z\subseteq X$ we have $g\in G_Z$. If $z\in Z$ and $z'\in Z'$ then $\Phi(z,(g-1)z')=\Phi(g^{-1}z,z')-\Phi(z,z')=0$ so $(g-1)z'\in Z$. Thus the matrix for g has the form

$$[g]=\begin{bmatrix} I & 0 & 0 \\ 0 & I & C \\ 0 & 0 & I \end{bmatrix}. \tag{2.9}$$

The condition $[g]^{\mathrm{T}}[\Phi][g]^{\sigma}=[\Phi]$ amounts to $C^{\sigma}+C^{\mathrm{T}}=0$. Thus $|G_X|$ is the number of square matrices C of size $\dim Z=n-k$ such that $C^{\sigma}+C^{\mathrm{T}}=0$. At this stage we must separate the cases $\sigma=1$ where G is orthogonal and $\sigma\neq 1$ where G is unitary.

The orthogonal groups. There exists a basis $e_1,\ldots,e_n$ for V such that $\Phi(e_i,e_j)=0$ for $i\neq j$, $\Phi(e_i,e_i)=1$ for $1\leqslant i\leqslant n-1$ and $\Phi(e_n,e_n)=\Delta$ where $\Delta=\Delta(\Phi)=\det\Phi(e_i,e_j)$ is the discriminant of Φ with respect to $e_1,\ldots,e_n$. Since $|K^{\times}:(K^{\times})^2|=2$ there are, up to isometry, two spaces (V,Φ) in each dimension. If n is odd then $v=(n-1)/2$, the two groups $G(\Phi)$ are isomorphic and

$$|G(\Phi)|=|O(n,q)|=2q^{(n-1)^2/4}\prod_{i=1}^{(n-1)/2}(q^{2i}-1), \qquad n\geqslant 1. \tag{2.10}$$

If $n=2m$ is even let $\varepsilon=\varepsilon(\Phi)=1$ if $(-1)^m\Delta$ is in $(K^{\times})^2$ and let $\varepsilon=\varepsilon(\Phi)=-1$ otherwise. Then $v=m$ if $\varepsilon=1$, $v=m-1$ if $\varepsilon=-1$ and

$$|G(\Phi)|=|O^{\varepsilon}(n,q)|=2q^{n(n-2)/4}(q^m-\varepsilon)\prod_{i=1}^{(n-2)/2}(q^{2i}-1), \qquad n\geqslant 2. \tag{2.11}$$

Here and in what follows, products indexed by the empty set are understood to be 1. For these facts see [2, pp. 144–147] and [3, Exercises 6.4 and 6.12]. The entries c_{ij} in (2.9) satisfy $c_{ii}=0$ so $|G_X|=q^{\binom{n-k}{2}}$. If n is odd then since Y is non-isotropic and $\dim Y=2k-n$ is also odd we have $|G(\Phi_Y)|=|O(2k-n,q)|$. Suppose $n=2m$ is even. Choose a basis for V as in (2.8). Since $\Delta(\Phi)=\Delta(\Phi_Y)\,\Delta(\Phi_{Z\oplus Z'})=(-1)^{2m-k}\Delta(\Phi_Y)$ and $\dim Y=2(k-m)$ we have $\varepsilon(\Phi_Y)=(-1)^{k-m}(-1)^{2m-k}\Delta(\Phi)=\varepsilon(\Phi)$. Thus $|G(\Phi_Y)|=|O^{\varepsilon}(2k-n,q)|$ where $\varepsilon=\varepsilon(\Phi)$. Furthermore $|\mathrm{Hom}(Y,Z)|=q^{(2k-n)(n-k)}$ and

$$|GL(Z)|=|GL(n-k,q)|=q^{(n-k)(n-k-1)/2}\prod_{i=1}^{n-k}(q^i-1).$$

This gives us all the information necessary to determine the number a_k of k-dimensional subspaces X of V with $\dim X+\dim\operatorname{rad}X=n$.

(2.12) PROPOSITION. *Let $K = \mathbf{F}_q$ where q is odd and suppose $\sigma = 1$. Let v be the index of the symmetric bilinear form Φ. Let $[\ ]$ denote the Gaussian binomial coefficient in base q.*

(i) *If n is even and $\varepsilon = 1$ then*

$$a_{v+\alpha} = \begin{bmatrix} v \\ \alpha \end{bmatrix} \prod_{i=\alpha}^{v-1} (q^i + 1), \qquad \alpha = 0, ..., v.$$

(ii) *If n is even and $\varepsilon = -1$ then*

$$a_{v+\alpha+1} = \begin{bmatrix} v \\ \alpha - 1 \end{bmatrix} \prod_{i=\alpha+1}^{v+1} (q^i + 1), \qquad \alpha = 1, ..., v+1.$$

(iii) *If n is odd then*

$$a_{v+\alpha} = \begin{bmatrix} v \\ \alpha - 1 \end{bmatrix} \prod_{i=\alpha}^{v} (q^i + 1), \qquad \alpha = 1, ..., v+1.$$

Proof. Suppose $n = 2m$ is even. If $\varepsilon = 1$ then $k = m, ..., n$. If $\varepsilon = -1$ then $k = m+1, ..., n$. Suppose first that $k \geqslant m+1$. It follows from the known values in (2.6) and (2.7) after performing all the obvious cancellations and setting $k = m + \alpha$ that

$$a_{m+\alpha} = \frac{(q^m - \varepsilon) \prod_{i=\alpha}^{m-1} (q^{2i} - 1)}{(q^\alpha - \varepsilon) \prod_{i=1}^{m-\alpha} (q^i - 1)}, \qquad \alpha = 1, ..., m.$$

If $\varepsilon = 1$ then $m = v$ and the result follows by multiplying numerator and denominator by $\prod_{i=1}^{\alpha} (q^i - 1)$. If $\varepsilon = -1$ then $m = v + 1$ and we multiply numerator and denominator by $\prod_{i=1}^{\alpha-1} (q^i - 1)$. If $k = m$ then $\varepsilon = 1$ and $\dim Y = 0$ so $|G(\Phi_Y)| = 1$. Here (2.6) and (2.7) give $a_v = 2 \prod_{i=1}^{v-1} (q^i + 1)$ so (i) holds for $\alpha = 0$ as well. This proves (i) and (ii). Formula (iii) is proved in the same way; here $n = 2v + 1$ and $k = v + 1, ..., n$. ∎

If $v = 0$ then $\gamma(t) = 1$ in (1.4). If $v = 1$ then $\gamma(t)$ has degree one. Thus if $v = 0, 1$ then $\chi(L, t)$ has integer roots. The roots of $\chi(L, t)$ in these cases are:

$$\begin{array}{lll} n=1 & & : 1 \\ n=2 & (\varepsilon=-1): & 1, q \\ n=2 & (\varepsilon=1) : & 1, q-2 \\ n=3 & & : 1, q, q^2-q-1 \\ n=4 & (\varepsilon=-1): & 1, q, q^2, q^3-q^2-1. \end{array} \qquad (2.13)$$

The unitary groups. Here it is convenient to change notation and let $\mathbf{F}_q$ be the fixed field of the involutory automorphism $\sigma: x \to x^q$ of $K = \mathbf{F}_{q^2}$. If n is even then $n = 2\nu$; if n is odd then $n = 2\nu + 1$. The entries of $C = (c_{ij})$ satisfy $c_{ji} + c_{ij}^q = 0$ so $|G_X| = q^{(n-k)^2}$. Furthermore [3, Exercises 6.3 and 6.13] any two spaces (V, Φ) of the same dimension are isometric and

$$|G(\Phi)| = |U(n, q^2)| = q^{n(n-1)/2} \prod_{i=1}^{n} (q^i - (-1)^i). \tag{2.14}$$

Since Y is non-isotropic and $\dim Y = 2k - n$ we have $|G(\Phi_Y)| = |U(2k-n, q^2)|$. Furthermore $|\mathrm{Hom}(Y, Z)| = q^{2(2k-n)(n-k)}$ and $|GL(Z)| = |GL(n-k, q^2)|$. Now arguing as in (2.12) we get

(2.15) PROPOSITION. *Let $K = \mathbf{F}_{q^2}$ and let $\sigma: x \to x^q$. Let ν be the index of the Hermitian form Φ. Let $[\ ]$ denote the Gaussian binomial coefficient in base q^2.*

(i) *If $n = 2\nu$ is even then*

$$a_{\nu+\alpha} = \begin{bmatrix} \nu \\ \alpha \end{bmatrix} \prod_{j=\alpha}^{\nu-1} (q^{2j+1} + 1), \qquad \alpha = 0, \ldots, \nu.$$

(ii) *If $n = 2\nu + 1$ is odd then*

$$a_{\nu+\alpha+1} = \begin{bmatrix} \nu \\ \alpha \end{bmatrix} \prod_{j=\alpha+1}^{\nu} (q^{2j+1} + 1), \qquad \alpha = 0, \ldots, \nu.$$

As in the orthogonal case if $\nu = 0, 1$ then $\chi(L, t)$ has integer roots. The roots of $\chi(L, t)$ in these cases are:

$$\begin{aligned} n = 1:&\quad 1 \\ n = 2:&\quad 1, q^2 - q - 1 \\ n = 3:&\quad 1, q^2, q^4 - q^3 - 1. \end{aligned} \tag{2.16}$$

3. FIXED POINT SETS

Let K be a field and let V be a vector space of dimension n over K. If $g \in GL(V) = GL(n, K)$ let $\mathrm{Fix}(g)$ denote the fixed point set of g and let $k(g) = \dim \mathrm{Fix}(g)$. For each finite subgroup G of $GL(V)$ define a polynomial $P(G, t)$ by

$$P(G, t) = \sum_{g \in G} t^{k(g)}. \tag{3.1}$$

We say that g is a reflection if it is semisimple and $k(g)=n-1$. Suppose $K=\mathbf{C}$ and $G\subset GL(V)$ is a finite group generated by reflections. Shephard and Todd [9] proved the formula

$$P(G, t)=\prod_{i=1}^{n}(t+m_i) \tag{3.2}$$

where the m_i are non-negative integers, called the exponents, which occur in the invariant theory of G. Suppose now that K is a finite field. If $K=\mathbf{F}_q$ and q is odd then the orthogonal groups $O^{\varepsilon}(n, q)$ are generated by reflections. If $K=\mathbf{F}_{q^2}$ then the unitary groups $U(n, q^2)$ are generated by reflections unless $n=q=2$ [4, p. 41]. This fact leads one to consider the sum on the left-hand side of (3.2) in case $G=O^{\varepsilon}(n, q)$ or $G=U(n, q^2)$. It was shown in [10] that if the index v of the corresponding form Φ is 0 or 1 then there exist positive integers $m_i(q)$ such that

$$P(G, t)=\prod_{i=1}^{n}(t+m_i(q)), \qquad v=0, 1. \tag{3.3}$$

Suppose $G=O^{\varepsilon}(n, q)$. If we compare the table of $m_i(q)$ given in [10, p. 440] with the roots of $\chi(L, t)$ given in (2.13) we see that for $G=O^{\varepsilon}(n, q)$ and $v=0, 1$ we have

$$\chi(L, t)=\sum_{g\in G}(-1)^{n-k(g)}\,t^{k(g)}=(-1)^n\,P(G, -t). \tag{3.4}$$

Our first aim in this section is to explain the coincidence (3.4). Let $\delta(g)=\det g$. Scherk's theorem [8] on the decomposition of orthogonal transformations into reflections shows that $\delta(g)=(-1)^{n-k(g)}$. We may view $\delta(g)\in\mathbf{C}$. Since q is odd δ is still a non-trivial character of G. Thus (3.4) is explained by

(3.5) THEOREM. *Let $G=O^{\varepsilon}(n, q)$ where q is odd. Suppose the index of the corresponding symmetric bilinear form is 0 or 1. Then*

$$\chi(L, t)=\sum_{g\in G}\delta(g)\,t^{k(g)}.$$

To prove (3.5) we prove some geometric lemmas which lead to a characterization (3.11) of L in case $v=0, 1$. If X is an isotropic hyperplane and $g\in G$ fixes X then $g=1$ [2, Theorem 3.17]. Thus there is no $g\in G$ with $X=\mathrm{Fix}(g)$. The following lemma asserts the converse.

(3.6) LEMMA. *If Φ is symmetric and X is a subspace of V which is not an isotropic hyperplane then there exists $g\in G$ such that $X=\mathrm{Fix}(g)$.*

Proof. Suppose first that X is totally isotropic. The Witt decomposition (2.1) is thus $V=(X\oplus X')\oplus W$. Choose a basis for V adapted to this decomposition so that the matrix for Φ is

$$[\Phi]=\begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & A \end{bmatrix} \tag{3.7}$$

where $A=A^{\mathsf{T}}$ is symmetric. In this basis the elements of G_X have the form $g=us$ where

$$[u]=\begin{bmatrix} I & P & -Q^{\mathsf{T}}A \\ 0 & I & 0 \\ 0 & Q & I \end{bmatrix}, \qquad [s]=\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & S \end{bmatrix} \tag{3.8}$$

are subject to the restrictions

$$P+P^{\mathsf{T}}+Q^{\mathsf{T}}AQ=0 \qquad \text{and} \qquad S^{\mathsf{T}}AS=A. \tag{3.9}$$

If $\dim X=0$ choose $S=-I$. Then $\mathrm{Fix}(g)=X$. Suppose $\dim X=1$. Since X is not an isotropic hyperplane we have $\dim V\geqslant 3$. If $\dim V=3$ then $\dim W=1$ so we have $A=a\in K^{\times}$. Choose $Q=t\in K^{\times}$ and let $P=-at^2/2$. Let $S=1$. Then (3.9) is satisfied and $X=\mathrm{Fix}(g)$. If $\dim V>3$ choose a non-isotropic vector $w\in W$. Let $T=X\oplus X'\oplus Kw$. By the case $\dim V=3$ there exists $h\in G(\Phi_T)$ with $\mathrm{Fix}(h)=X$. Choose $g\in G$ to agree with h on T and so that $g=-1$ on the orthogonal complement T^0 of T in V. Then $\mathrm{Fix}(g)=X$ If $\dim X\geqslant 2$ choose $Q=0$, $S=-I$, and let P be any invertible skew symmetric matrix. Then (3.9) is satisfied and the fact that P is invertible implies $\mathrm{Fix}(g)=X$. This completes the proof in case X is totally isotropic.

In general we use the Witt decomposition (2.1). If $Z=0$ let $g=1$ on $X=Y$ and let $g=-1$ on W. Suppose $\dim Z\geqslant 1$. If Z is an isotropic hyperplane in $Y^0=Z\oplus Z'\oplus W$ then $\dim Z=\dim Y^0-1=2\dim Z+\dim W-1$ so $W=0$ and $\dim Z=1$. Then $V=X\oplus Z'$ shows that X is an isotropic hyperplane in V, a contradiction. Thus Z is not an isotropic hyperplane of Y^0 so we may apply the first part of the argument to conclude that there exists $h\in G(\Phi_{Y^0})$ with $\mathrm{Fix}(h)=Z$. Choose $g\in G$ to agree with h on Y^0 and so that $g=1$ on Y. Then $\mathrm{Fix}(g)=X$. ∎

(3.10) COROLLARY. *If Φ is symmetric then $L\subseteq\{\mathrm{Fix}(g)\mid g\in G\}$.*

(3.11) LEMMA. *Suppose Φ is symmetric. Then $L=\{\mathrm{Fix}(g)\mid g\in G\}$ if and only if $v=0, 1$.*

Proof. Suppose $\nu = 0, 1$. Suppose $g \in G$ and let $X = \mathrm{Fix}(g)$. If $\dim X = n-1$ then X is non-isotropic, by the remark preceding (3.6), so $X \in L$. If $\dim X \leqslant n-2$ then $\rho(X) \leqslant n-2+\nu \leqslant n-1$ so $X \in L$ by (2.3). Thus $\{\mathrm{Fix}(g) \mid g \in G\} \subseteq L$. Equality follows from (3.10). Now suppose $\nu \geqslant 2$. Let Z be a totally isotropic subspace of V of dimension ν. Choose a Witt decomposition $V = (Z \oplus Z') \oplus W$. Let $T = Z \oplus W$. Then $\dim T = n - \nu \leqslant n-2$ so by Lemma 3.6 there exists $g \in G$ such that $T = \mathrm{Fix}(g)$. On the other hand $\rho(T) = (n-\nu) + \nu = n$ so $T \notin L$ by Lemma 2.3. ∎

(3.12) Lemma. *If $X \in L$ and $X \neq V$ then $\sum_{g \in G_X} \delta(g) = 0$.*

Proof. Lemma 2.3 shows that $\rho(X) \leqslant n-1$. The argument given in [7, Lemma 2.6] shows for any subspace Y of V that $G_Y \subseteq SL(V)$ if and only if $\rho(Y) = n$. Thus the restriction of δ to G_X is a non-trivial character of G_X. ∎

Now we prove Theorem 3.4. Recall that M is the lattice of all subspaces of V. If $Y \in M$ let $F_Y = \{g \in G \mid \mathrm{Fix}(g) = Y\}$. If $X \in M$ then without any assumption on ν we have

$$G_X = \bigcup_{\substack{Y \in M \\ Y \leqslant X}} F_Y \qquad \text{disjoint union.}$$

If F_Y is non-empty then there exists $g \in G$ with $\mathrm{Fix}(g) = Y$. Since $\nu = 0, 1$ we conclude from Lemma 3.11 that $Y \in L$. Thus for $X \in L$ we have

$$G_X = \bigcup_{\substack{Y \in L \\ Y \leqslant X}} F_Y. \tag{3.13}$$

For $X \in L$ let $\lambda(X) = \sum_{g \in F_X} \delta(g)$. Then (3.12) and (3.13) imply

$$\sum_{\substack{Y \in L \\ Y \leqslant X}} \lambda(Y) = \sum_{g \in G_X} \delta(g) \begin{aligned} &= 0 \quad \text{if } X \neq V \\ &= 1 \quad \text{if } X = V. \end{aligned} \tag{3.14}$$

Since $\mu(V, X)$ satisfies the same recurrence (3.14) we have $\mu(V, X) = \lambda(X)$. Thus

$$\mu(V, X) = \sum_{g \in F_X} \delta(g), \qquad X \in L. \tag{3.15}$$

It follows from (3.13) that $G = \bigcup_{X \in L} F_X$. This completes the proof of (3.4). ∎

We have seen in Theorem 1.4 that if Φ is symmetric then $\chi(L, t)$ has roots $1, q, \ldots, q^{n-\nu-1}$ for all ν. Kusuoka [5] has shown for $G = G(\Phi)$ that

$P(G, t)$ has roots $-1, -q, \ldots, -q^{n-\nu-1}$ for all ν. Although $\chi(L, t) = (-1)^n P(G, -t)$ for $\nu = 0, 1$ equality does not hold for $\nu \geqslant 2$. For example, if $n = 4$, $\nu = 2$ then it follows from (2.12) after some calculation and from Kusuoka's recursion formula that

$$\begin{aligned}\chi(L, t) &= (t-1)(t-q)(t^2 - (q^3 - 2q - 1)\,t \\ &\quad + q^5 - q^4 - 2q^3 - q^2 + 2q + 2) \\ P(G, -t) &= (t-1)(t-q)(t^2 - (q^3 - 2q - 1)\,t \\ &\quad + q^5 - q^4 - 2q^3 + q^2 + 2q).\end{aligned}$$

Thus letting $b_0(G)$ denote the number of $g \in G$ with $k(g) = 0$ we have an inequality

$$b_0(g) - |\mu(V, 0)| = 2q(q^2 - 1) > 0$$

which replaces the equality $b_0(g) = |\mu(V, 0)|$ in case $\nu = 0, 1$.

Suppose now that Φ is Hermitian with respect to the automorphism $x \to x^q$ of $K = \mathbf{F}_{q^2}$ and $G = G(\Phi) = U(n, q^2)$. If $\nu = 0, 1$ the polynomial $P(G, -t)$ again has integer roots. These are given in [10, p. 435] by

$$\begin{aligned} n = 1&: \quad q \\ n = 2&: \quad q,\ q^3 - q - 1 \\ n = 3&: \quad q,\ q^3,\ q^5 - q^3 - 1. \end{aligned} \tag{3.16}$$

Recall that if G is orthogonal then $\chi(L, t) = (-1)^n P(G, -t)$ if $\nu = 0, 1$. If we compare the roots of $P(G, -t)$ in (3.16) with the roots of $\chi(L, t)$ in (2.16) we see that $\chi(L, t) \neq (-1)^n P(G, -t)$ in the unitary case. On the other hand Kusuoka has shown in the unitary case that $P(G, -t)$ has $n - \nu$ roots q^i where $i = 1, 3, \ldots, 2(n - \nu) - 1$ while we have shown in (1.4) that $\chi(L, t)$ has $n - \nu$ roots q^i where $i = 0, 2, \ldots, 2(n - \nu) - 2$. We cannot explain this coincidence.

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