Global dimension of the endomorphism ring and $*^n$-modules

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Abstract

We show that, if $T$ is a selfsmall and selforthogonal module over a noetherian ring $R$ of finite global dimension with the endomorphism ring $A$, then $\text{fd}_A T \leq \text{gd}_A \leq \text{id}_R T + \text{fd}_A$. Applying the result we give answers to two questions left in [J. Wei et al., J. Algebra 168 (2) (2003) 404–418] concerning basic properties of $*^n$-modules, by showing that the flat dimension of a $*^n$-module with $n \geq 3$ over its endomorphism ring can even be arbitrarily far from the integer $n$ while the flat dimension of a $*^2$-module over its endomorphism ring is always bounded by the integer 2 and showing that $*^n$-modules are not finitely generated in general, even in case $n = 2$.

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1. Introduction and preliminaries

The tilting theory plays an important role in the representation of Artin algebra. Let $R$ be an Artin algebra and $T$ a tilting $R$-module with $A = \text{End}_R T$ (by a tilting module we mean the tilting module in sense of [9] throughout the paper). The purpose of the tilting theory is to compare $R$-mod (the category of finitely generated $R$-modules) with $A$-mod. One aspect of these is on the estimate of the global dimension of the endomorphism algebra $A$. 
A well-known result is that \( gd_R - \text{pd}_R T \leq gd A \leq gd_R + \text{pd}_R T \), see for instance [8,9].

The result was improved in [7] where it was shown that \( \text{id}_R T \leq gd A \leq \text{pd}_R T + \text{id}_R T \).

In the first part of this paper, we will study some more general case in the sense that we only assume that \( T \) is selfsmall and selforthogonal and that \( T \) is over any associated ring with identity, not only Artin algebras. A special case of our results shows that, if \( T \) is a selfsmall and selforthogonal module over a noetherian ring \( R \) of finite global dimension with the endomorphism ring \( A \), then \( \text{fd}_R T \leq gd A \leq \text{pd}_R T + \text{id}_R T \) (Corollary 2.6). Note that, for a tilting module \( T \) over an Artin algebra \( R \) with \( A = \text{End}_R T \), it always holds that \( \text{pd}_R T = \text{pd}_R T = \text{fd}_R T \), so our result extends the upper-bound-part of the corresponding one in [7].

The important tool for our investigation is the theory of \(*_n\)-modules. Recall that a \( R \)-module \( T \) is a \(*_n\)-module provided that \( T \) is selfsmall, \((n + 1)\)-quasi-projective and that \( \text{Pres}^n(T) = \text{Pres}^{n+1}(T) \) [16]. The notion of \(*_n\)-modules is a natural generalization of both \(*\)-modules (see for instance [3,4,10] etc.) and tilting modules of projective dimension \( \leq n \).

In fact, \(*\)-modules are just \(*_1\)-modules while tilting modules of projective dimension \( \leq n \) are just \(*_n\)-modules which admit finitely generated projective resolutions and \( n \)-present all injective modules [2,16].

In [16], some questions on basic properties of \(*_n\)-modules were left. The following are two questions among them.

**Question 1.** Is the flat dimension of a \(*_n\)-module over its endomorphism ring is always bounded by \( n \)?

**Question 2.** Are all \(*_n\)-modules finitely generated?

We recall that all \(*\)-modules are finitely generated [13] and that the flat dimension of a \(*\)-module over its endomorphism ring is always bounded by 1 [14]. Note that it was also proved that, for a \(*_n\)-module \( T \) with the endomorphism ring \( A \), it always holds that \( \text{Ker} \text{Tor}^A_{\geq 1}(T, -) = \text{Ker} \text{Tor}^A_{1\leq i \leq n}(T, -) \) [15].

Our investigation of the estimate of global dimensions of endomorphism rings of self-orthogonal modules turns out to be very useful to answer questions mentioned above. Indeed, it is shown, in the second part of this paper, that the flat dimension of a \(*_n\)-module with \( n \geq 3 \) over its endomorphism ring can even be arbitrarily far from the integer \( n \) (Proposition 3.3). However, the flat dimension of a \(*_2\)-module over its endomorphism ring is always bounded by the integer 2 (Proposition 3.5). In particular, we obtain that, if \( T \) is a \( w - \Sigma\)-quasi-projective \(*_2\)-module over the ring \( R \) and \( Q \) is any injective cogenerator of \( R\text{-Mod} \), then the “dual” module \( T^* = \text{Hom}_R(T, Q) \) is a cotilting module over the endomorphism ring of \( T \) (Corollary 3.6). Finally, we give an example to show that a \(*_n\)-module is not finitely generated in general, even in case \( n = 2 \) (Example 3.10). In fact, the rational \( Q \), as a \( \mathbb{Z}\)-module (\( \mathbb{Z} \) denotes the ring of all integers), is just such a module.

Throughout this paper, all rings will be associated with non-zero identity and modules will be left modules without explicit mentions. For a ring \( R \), \( R\text{-Mod} \) (\( \text{Mod-}R \)) denotes the category of all left (right) \( R\)-modules. By a subcategory, we mean a full subcategory closed under isomorphisms.
An $R$-module $T$ is selfsmall if the canonical morphism

$$\text{Hom}_R(T, T)^{(X)} \to \text{Hom}_R(T, T^{(X)})$$

is an isomorphism.

From now on, we fix $T$ a selfsmall $R$-module with the endomorphism ring $A$ and denote that $H_T = \text{Hom}_R(T, -)$. Note that $T$ is also a right $A$-module. Let $Q$ be any injective cogenerator of $R$-Mod. We fix $T^* = \text{Hom}_R(T, Q)$. Then $T^*$ is an $A$-module.

We denote that

$$T^\perp = \{ M \in R\text{-Mod} \mid \text{Ext}^i_R(T, M) = 0 \text{ for all } i \geq 1 \}$$

and

$$\text{Ker Tor}^A_{i \geq 1}(T, -) = \{ M \in A\text{-Mod} \mid \text{Tor}^A_i(T, M) = 0 \text{ for all } i \geq 1 \}.$$

For a fixed integer $n$, the subcategory $\text{Ker Tor}^A_{1 \leq i \leq n}(T, -)$ is similarly defined. Also we denote by $\text{Ad} T$ the class of modules isomorphic to direct sums of copies of the $R$-module $T$ and by $\text{Add} T$ the class of modules isomorphic to direct summands of modules in $\text{Ad} T$.

Furthermore, we denote that

$$\text{Add} T = \{ M \in R\text{-Mod} \mid \text{there exists an exact sequence } 0 \to T_m \to \cdots \to T_0 \to M \to 0 \text{ for some } m, \text{ where } T_i \in \text{Add} T \text{ for each } i \}.$$

We say that an $R$-module $T$ is selforthogonal if $\text{Ext}^i_R(T, T^*) = 0$ for all $i \geq 1$ and all $T^* \in \text{Add} T$. If $T$ has a finitely generated projective resolution, then $T$ is selforthogonal is equivalent to say that $\text{Ext}^i_R(T, T) = 0$ for all $i \geq 1$ [16].

An $R$-module $M$ is $n$-presented by $T$ if there exists an exact sequence $T_n \to \cdots \to T_2 \to T_1 \to M \to 0$ with $T_i \in \text{Add} T$ for each $i$. We denote by $\text{Pres}^n(T)$ the category of all $R$-modules $n$-presented by $T$. Of course, for every $n$, we have that $\text{Pres}^{n+1}(T) \subseteq \text{Pres}^n(T)$.

Note that $\text{Pres}^2(T)$ and $\text{Pres}^3(T)$ are just familiar subcategories $\text{Pres}(T)$ and $\text{Gen}(T)$ respectively. We denote that $\text{Copres}(T^*) = \{ N \in A\text{-Mod} \mid \text{there exists an exact sequence } 0 \to N \to K_1 \to K_2 \text{ with } K_1, K_2 \text{ products of copies of } T^* \}$ and that $\text{Cogen}(T^*) = \{ N \in A\text{-Mod} \mid N \text{ can be embedded in a product of copies of } T^* \}$.

$T$ is said to be $(n, t)$-quasi-projective (here we assume that $n \geq t \geq 1$) if, for any exact sequence $0 \to M \to T_i \to \cdots \to T_1 \to N \to 0$ with $M \in \text{Pres}^{n-t}(T)$ and $T_i \in \text{Add} T$ for each $i$, the induced sequence $0 \to H_T M \to H_T T_i \to \cdots \to H_T T_1 \to H_T N \to 0$ is exact [15]. It is easy to see that, if $T$ is $(n, t)$-quasi-projective, then $T$ is also $(m, s)$-quasi-projective, for all $m, s$ such that $m \geq n$ and $1 \leq s \leq t + m - n$. Note that notions of $(1, 1)$-quasi-projective, $(2, 1)$-quasi-projective, $(2, 2)$-quasi-projective and $(n, 1)$-quasi-projective respectively are just notions of $\Sigma$-quasi-projective [6] [11], $w$-$\Sigma$-quasi-projective [3], semi-$\Sigma$-quasi-projective [11] and $n$-quasi-projective [16], respectively.

It is well known that $(T \otimes_A \cdot, H_T)$ is a pair of adjoint functors and there are the following canonical homomorphisms for any $R$-module $M$ and any $A$-module $N$:

$$\rho_M : T \otimes_A H_T M \to M \text{ by } t \otimes f \to f(t),$$

$$\sigma_N : N \to H_T (T \otimes_A N) \text{ by } n \to [t \to t \otimes n].$$
We denote by $\text{Stat}(T)$ the class of $R$-modules $M$ such that $\rho_M$ is an isomorphism and by $\text{Costat}(T)$ the class of $A$-modules $N$ such that $\sigma_N$ is an isomorphism.

Recall from [2] that an $A$-module $K$ is 2-cotilting (1-cotilting, respectively) if

$$\text{Copres}(K) = \text{Cogen}(K), \text{respectively} = \ker \text{Ext}^i_A (\cdot, K)$$

(the definition is not the original one, but is equivalent to it by [2]). This definition is used throughout the paper. Note that in case $A$ is an Artin algebra, finitely generated 1-cotilting modules coincide with usual cotilting modules (of injective dimension $\leq 1$) in sense of [8].

Throughout the paper, $\text{gd}_R$ denotes the (left) global dimension of the ring $R$. We denote by $\text{pd}_{RT}(\text{id}_{RT}, f_{\text{d}_{TA}}$, respectively) the projective (injective, flat, respectively) dimension of the module $RT$ ($RT, TA$, respectively).

2. Global dimension of endomorphism rings

For any $M \in \hat{\text{Add}} T$, there is some $m$ such that there is an exact sequence $0 \to T_m \to \cdots \to T_0 \to M \to 0$ with $T_i \in \text{Add} T$ for each $i$, by the definition. We denote $T$-res.dim($M$) to be the minimal integer among such $m$. Then we have the following useful lemma.

**Lemma 2.1.** Let $T$ be a selforthogonal $R$-module. Then $T$-res.dim($M$) = $\text{pd}_A HT(M)$, for any $M \in \hat{\text{Add}} T$.

**Proof.** Assume that $m = T$-res.dim($M$), then there is an exact sequence of minimal length $0 \to T_m \to \cdots \to T_0 \to M \to 0$ with $T_i \in \text{Add} T$ for each $i$. By applying the functor $HT$ to the sequence, we obtain an induced exact sequence $0 \to HT T_m \to HT f_m \cdots \to HT T_0 \to HT M \to 0$, since $T$ is selforthogonal. The last sequence in fact is a projective resolution of the $A$-module $HT M$. Hence $\text{pd}_A HT M \leq m = T$-res.dim($M$). If $\text{pd}_A HT M < m$, then it is easy to check that $\text{Coker} HT f_m$ must be a projective $A$-module. Now after applying the functor $T \otimes_A$ to the sequence $0 \to HT T_m \to HT f_m HT T_m-1 \to \text{Coker} HT f_m \to 0$, we deduce that $\text{Coker} f_m \simeq \text{Coker} T \otimes_A HT f_m = T \otimes_A \text{Coker} HT f_m \in \text{Add} T$. This shows that $T$-res.dim($M$) < $m$, a contradiction. In conclusions, we have that $T$-res.dim($M$) = $\text{pd}_A HT M$, for any $M \in \hat{\text{Add}} T$.

Under some additional conditions, $T$-res.dim($M$) will be bounded by a fixed number for all $M \in \hat{\text{Add}} T$.

**Lemma 2.2.** Let $T$ be a selforthogonal $R$-module with $\text{id}_R T' \leq s < \infty$ for all $T' \in \text{Add} T$. Then $T$-res.dim($M$) $\leq s$ for any $M \in \hat{\text{Add}} T$. In particular, $\text{pd}_A HT M \leq s$.

**Proof.** For any $M \in \hat{\text{Add}} T$, there is an exact sequence $0 \to T_m \to f_m \cdots \to T_0 \to f_0 M \to 0$ with $T_i \in \text{Add} T$ for each $i$. If $m \leq s$, then we have nothing to say. So we as-
sume that $m > s$. Let $K_i = \text{Im} f_i$ for each $0 \leq i \leq m$. Note that $K_m = T_m$ and $K_0 = M$ in this case. Since $T$ is selforthogonal, we have that

$$\text{Ext}_R^j(T, K_m) = 0 \quad \text{for all } j \geq 1 \text{ and all } 0 \leq i \leq m.$$ 

Then, by applying the functor $\text{Hom}_R(\cdot, K_m)$ to the sequence, we obtain that

$$\text{Ext}_R^1(K_{m-1}, K_m) \cong \text{Ext}_R^2(K_{m-2}, K_m) \cong \cdots \cong \text{Ext}_R^m(K_0, K_m),$$

by dimension shifting. By assumptions, $\text{id}_R T' \leq s$ for all $T' \in \text{Add} T$, so we have that $\text{id} K_m \leq s < m$. It follows that $\text{Ext}_R^1(K_{m-1}, K_m) \cong \text{Ext}_R^m(K_0, K_m) = 0$. Hence the sequence $0 \to T_m \to^d T_{m-1} \to K_{m-1} \to 0$ splits. This shows that $T$-res.dim$(M)$ must be not more than $s$. □

**Lemma 2.3.** Let $T$ be a selforthogonal $R$-module with $\text{pd}_R T \leq n$. Then $\text{Pres}^n(T) \subseteq T\perp$ and $T$ is $(n+1)$-quasi-projective.

**Proof.** Take any $M \in \text{Pres}^n(T)$, then we have an exact sequence $T_n \to^j \cdots \to T_1 \to f_1 M \to 0$. Let $M_i = \text{Ker} f_i$ for each $1 \leq i \leq n$. Since $T$ is selforthogonal, we have that

$$\text{Ext}_R^j(T, T_i) = 0 \quad \text{for all } j \geq 1 \text{ and all } 1 \leq i \leq n.$$ 

Then, by applying the functor $H_T$ to the sequence, we obtain that

$$\text{Ext}_R^j(T, M) \cong \text{Ext}_R^{j+1}(T, M_i) \cong \cdots \cong \text{Ext}_R^{j+n}(T, M_n) \quad \text{for all } j \geq 1,$$

by dimension shifting. Since $\text{pd}_R T \leq n$, we deduce that

$$\text{Ext}_R^j(T, M) \cong \text{Ext}_R^{j+n}(T, M_n) = 0 \quad \text{for all } j \geq 1.$$ 

Hence $M \in T\perp$, and consequently, $\text{Pres}^n(T) \subseteq T\perp$.

Now consider any exact sequence $0 \to K \to T_N \to N \to 0$ with $K \in \text{Pres}^n(T)$ and $T_N \in \text{Add} T$. By the previous proof, we see that $K \in T\perp$. Hence, by applying the functor $H_T$ to the last sequence, we obtain that the induced sequence $0 \to H_T K \to H_T T_N \to H_T N \to 0$ is exact, i.e., $T$ is $(n+1)$-quasi-projective. □

The following result shows that a selfsmall and selforthogonal module over a ring of finite global dimension is always a $\ast^m$-module for some integer $m$.

**Proposition 2.4.** Let $T$ be a selforthogonal $R$-module with $\text{pd}_R T \leq n$ ($n \geq 1$). If $\text{gd} R = d < \infty$, then $\text{Pres}^{n+d}(T) = \text{Add} T$. In particular, $T$ is a $\ast^{n+d}$-module.

**Proof.** Obviously we have that $\overline{\text{Add} T} \subseteq \text{Pres}^{n+d}(T)$. We now show that $\text{Pres}^{n+d}(T) \subseteq \text{Add} T$ too.
For any $M \in \text{Pres}^{n+d}(T)$, we have an exact sequence $T_{n+d} \rightarrow f_{n+d} \cdots \rightarrow T_d \rightarrow f_d \cdots \rightarrow T_1 \rightarrow f_1$ $M \rightarrow 0$ with $T_i \in \text{Add} T$ for each $1 \leq i \leq n + d$. Denote that $M_i = \text{Ker} f_i$ for each $i$, then we see that $M_d \in \text{Pres}^d(T)$. We claim now $M_d \in \text{Add} T$ and then it holds that $M \in \widetilde{\text{Add}} T$.

In fact, note that $M_d \in \text{Gen}(T)$ clearly, so we have an exact sequence $0 \rightarrow N \rightarrow T((H_T M_d)_v) \rightarrow M \rightarrow 0$, $(\ast)$

where $v$ is the canonical evaluation map, which obviously remains exact after applying the functor $H_T$. By Lemma 2.3, $M_d \in \text{Pres}^d(T) \subseteq T^\perp$. Hence we deduce that $N \in T^\perp$ too. Now by applying the functor $\text{Hom}_R(\cdot, N)$ to the exact sequence $0 \rightarrow M_d \rightarrow T_d \rightarrow \cdots \rightarrow T_1 \rightarrow M \rightarrow 0$, we obtain that $\text{Ext}^1_R(M_d, N) = \text{Ext}^2_R(M_{d-1}, N) = \cdots = \text{Ext}^d_R(M_1, N) = \text{Ext}^{d+1}_R(M, N)$, by dimension shifting. Since $\text{gd } R \leq d$, we see that $\text{Ext}^1_R(M_d, N) = \text{Ext}^{d+1}(M, N) = 0$. It follows that the sequence $(\ast)$ splits and that $M_d \in \text{Add} T$, as we claimed.

Since $\text{Pres}^{n+d+1}(T) \subseteq \text{Pres}^{n+d}(T) = \widetilde{\text{Add}} T \subseteq \text{Pres}^{n+d+1}(T)$, we then get that $\text{Pres}^{n+d}(T) = \text{Pres}^{n+d+1}(T)$.

Note that $T$ is clearly $(n + d + 1)$-quasi-projective by Lemma 2.3 and that $T$ is self-small by assumptions, so we have that $T$ is a $s^{n+d}$-module [16].

Now we give the estimate of the global dimension of the endomorphism ring of a self-orthogonal module.

**Theorem 2.5.** Let $R$ be a ring of finite global dimension and $T$ be a selforthogonal $R$-module with $A = \text{End}_R T$. Assume that $\text{id } T' \leq s$ for all $T' \in \text{Add } T$. Then $\text{fd } T_A \leq \text{gd } A \leq s + \text{fd } T_A$.

**Proof.** If $\text{fd } T_A = \infty$, then $\text{gd } A = \infty$ too and we have nothing to say in this case. So we assume that $\text{fd } T_A = t < \infty$. Then it is obvious that we need only to show that $\text{gd } A \leq s + t$.

For any $A$-module $Y$, by taking the projective resolution of $Y$, we obtain an exact sequence $0 \rightarrow Y_t \rightarrow P_{t-1} \rightarrow \cdots \rightarrow P_0 \rightarrow Y \rightarrow 0$ (***)

with $P_i$ projective for each $0 \leq i \leq t - 1$. Denote by $Y_t$ the $i$th syzygy, for each $i$. We claim now $\text{pd } A Y_t \leq s$ (note that $s < \infty$ since $R$ is of finite global dimension) and so $\text{pd } A Y \leq t + s$. Then the conclusion will be followed from the arbitrariness of the choice of $Y$.

Indeed, by assumptions and Proposition 2.4, we have that $\text{Pres}^m(T) = \widetilde{\text{Add}} T$ and $T$ is a $s^m$-module for some $m$. Hence, by results in [16], we obtain that

$$\text{Ker Tor}_{i\geq1}^A(T, -) = H_T(\text{Pres}^m(T)).$$
Since \( \text{pd}_A H_T M \leq s \) for any \( M \in \text{Pres}^m(T) \) by Lemma 2.2, it follows that \( \text{pd}_A N \leq s \) for any \( N \in \text{Ker} \text{Tor}^A_{i \geq 1}(T, -) \).

Now, by applying the functor \( T \otimes_A - \) to the sequence (**), we obtain that \( \text{Tor}_j^A(T,Y_t) \simeq \cdots \simeq \text{Tor}_j^A(T,Y_1) \simeq \text{Tor}_j^A(T,Y) \) for all \( j \geq 1 \).

Since \( \text{fd}_A T \leq t < \infty \), we have that \( \text{Tor}_j^A(T,Y_t) \simeq \text{Tor}_j^A(T,Y) = 0 \) for all \( j \geq 1 \), i.e., \( Y_t \in \text{Ker} \text{Tor}_{i \geq 1}^A(T, -) \). It follows that \( \text{pd}_A Y_t \leq s \) from arguments before, as desired.

Restricting to some special cases, we have the following corollary.

**Corollary 2.6.** Let \( R \) be a ring of finite global dimension and \( T \) an \( R \)-module with \( A = \text{End}_R T \). If moreover \( R \) is noetherian or \( T \) is of finitely generated projective resolution, then \( \text{fd}_A T \leq \text{gd}_A \leq \text{id}_R T + \text{fd}_A \). In particular, if \( R \) is noetherian and \( T \) is injective, then \( \text{gd}_A = \text{fd}_A = \text{wgd}_A \), where \( \text{wgd}_A \) denotes the weak global dimension of \( A \).

**Proof.** If \( R \) is noetherian or \( T \) is of finitely generated projective resolution, then it always holds that \( \text{id}_R T' \leq \text{id}_R T \) for all \( T' \in \text{Add} T \). Hence by applying Theorem 2.5, we have that \( \text{fd}_A T \leq \text{gd}_A \leq \text{id}_R T + \text{fd}_A \).

If \( R \) is noetherian and \( T \) is injective, then clearly we have that \( \text{gd}_A = \text{fd}_A \). By the definition of the weak global dimension, it always holds that \( \text{wgd}_A = \text{gd}_A \). Since \( \text{wgd}_A \leq \text{gd}_A \) obviously, it follows that \( \text{gd}_A = \text{fd}_A = \text{wgd}_A \).

3. **Two questions on \( \ast^n \)-modules**

In this section we will give answers to questions mentioned in the first section.

Firstly we note the following result.

**Lemma 3.1.** Let \( R \) be a ring and \( T \) be a selforthogonal \( R \)-module with \( \text{pd}_R T \leq n \). If \( \text{fd}_A T = t < \infty \), where \( A = \text{End}_R T \), then \( T \) is a \( \ast^m \)-module for some \( m \).

**Proof.** By Lemma 2.3, we have that \( T \) is \((n+1)\)-quasi-projective. If \( \text{fd}_A T = t = 0 \), then \( T \) is a \( \ast^2 \)-module by [15, Theorem 4.2]. So we assume that \( t \geq 1 \). Now by [15, Proposition 3.2], a selfsmall \((m,t+1)\)-quasi-projective module \( T \) with the flat dimension of \( T \) over its endomorphism ring not more than \( t \) is a \( \ast^m \)-module. Note that \( T \) is obviously \((n+t+1,t+1)\)-quasi-projective, so \( T \) is a \( \ast^{n+t+1} \)-module.

Now we give an example of \( \ast^3 \)-modules with infinite flat dimensions over their endomorphism rings. This shows that the answer to Question 1 in the first section is negative in general.
Example 3.2. Let \( R \) be the Artin algebra defined by the following quiver over a field \( k \)

\[
\begin{array}{c}
3 \\
\uparrow \theta \\
\downarrow \lambda \\
4 \\
\uparrow \rho \\
\downarrow \gamma \\
2 \\
\end{array}
\]

with relations \( \theta \lambda = 0 = \mu \eta, \gamma \delta = \nu \rho, \rho \alpha = \lambda \mu, \) and \( \beta \nu = \eta \theta \).

Then the \( R \)-module

\[
T = \begin{pmatrix}
\frac{3}{2} & \frac{2}{7} \\
\frac{1}{6} & \frac{1}{3} \\
\frac{4}{7} & \frac{3}{2}
\end{pmatrix}
\]

is a \( \ast^3 \)-module with \( \text{fd}_A T = \infty \), where \( A = \text{End}_R T \).

**Proof.** Note that \( \text{gd}_R T = 2 \) and \( T \) is indeed a projective \( R \)-module, so \( T \) is a \( \ast^3 \)-module by Proposition 2.4.

If \( \text{fd}_A T = t < \infty \). Then \( \text{gd} A \leq 2 + t < \infty \) by Theorem 2.5. However, \( A \) is in fact the path algebra defined by the quiver

\[
\begin{array}{c}
1 \\
\uparrow \alpha \\
\downarrow \beta \\
2
\end{array}
\]

with the relation \( \alpha \beta \alpha = 0 \). It is easy to check that \( A \) is of infinite global dimension. Hence we see that \( \text{fd}_A T = \infty \). \( \square \)

More generally, we have the following result which also shows that the flat dimension of \( T_A \) for \( T \) a \( \ast^n \)-module (\( n \geq 3 \)) with \( A = \text{End}_R T \) can even be arbitrarily far from the integer \( n \).

**Proposition 3.3.** Let \( A \) be an Artin algebra of finite representation type with \( \text{gd} A = d \) (maybe infinite). Then there exists an Artin algebra \( R \) with \( \text{gd} R = 2 \), over which there is a \( \ast^3 \)-module \( T \) with \( A = \text{End}_R T \) and \( \text{fd}_A T = d \).

**Proof.** By a well-known result in the representation theory of Artin algebras, any Artin algebra of finite representation type can be obtained as the endomorphism algebra of a projective and injective module \( T \) over an Artin algebra \( R \) with \( \text{gd} R = 2 \) (see for instance
In the case, we obtain that \( A = \text{End}_RT \) and that \( T \) is in fact a \( *^3 \)-module by Proposition 2.4, since \( \text{gd} R = 2 \) and \( \text{pd}_RT \leq 1 \). Note that \( T \) is also injective, so we have that \( \text{fd}_TA = \text{gd} A = d \) by Corollary 2.6.

However, in case \( n = 2 \), we have an affirmative answer to Question 1. To see this, we need the following lemma (cf. [5, Lemma 1.4]).

**Lemma 3.4.** Let \( T \) be an \( R \)-module with \( A = \text{End}_RT \). Then
\[
\ker \text{Tor}_i^A(T, -) = \ker \text{Ext}_i^A(\cdot, T^*) \quad \text{for each } i \geq 1.
\]
In particular, \( \text{fd}_TA = \text{id}_AT^* \).

Now we can show the following.

**Proposition 3.5.** Let \( T \) be a \( *^2 \)-module with \( A = \text{End}_RT \). Then \( T^* \) is 2-cotilting. In particular, \( \text{fd}_TA \leq 2 \).

**Proof.** Since \( T \) is a \( *^2 \)-module, we have that \( \text{Pres}^2(T) \subseteq \text{Stat}(T) \) by [16]. Since \( \text{Stat}(T) \subseteq \text{Pres}^2(T) \), holds obviously, we get that \( \text{Stat}(P) = \text{Pres}^2(T) \). Then by [17] or [12], we have that \( \text{Copres} T^* = \text{Costat}(T) = H_T(\text{Pres}^2(T)) \).

By [16], it holds that
\[
H_T(\text{Pres}^2(T)) = \ker \text{Tor}_i^{\geq 1}(T, -).
\]
Hence we obtain that
\[
\text{Copres} T^* = \ker \text{Tor}_i^{\geq 1}(T, -).
\]
Now, Lemma 3.4 helps us to deduce that \( \text{Copres} T^* = \ker \text{Ext}_i^{\geq 1}(\cdot, T^*) \). It follows that \( T^* \) is 2-cotilting, by [2].

In particular, we have that \( \text{id}_AT^* \leq 2 \) [2]. So, by Lemma 3.4, we also have that \( \text{fd}_TA \leq 2 \), as we desired. \( \square \)

It is well known that, if \( T \) is a \( * \)-module, then \( T^* \) is 1-cotilting. Hence it is not surprise that \( T^* \) is 2-cotilting when \( T \) is a \( *^2 \)-module. However, if \( T \) is a \( w-\Sigma \)-quasi-projective \( *^2 \)-module, then \( T^* \) is also 1-cotilting, as the following result shows.

**Corollary 3.6.** Let \( T \) be a \( *^2 \)-module with \( A = \text{End}_RT \). If \( T \) is \( w-\Sigma \)-quasi-projective, then \( T^* \) is 1-cotilting. In particular, \( \text{fd}_TA \leq 1 \).

**Proof.** If \( T \) is \( w-\Sigma \)-quasi-projective, then we have that \( \text{Costat}(T) = H_T(\text{Pres}^2(T)) = \text{Cogen} T^* \) by [3]. Hence we obtain that
\[
\ker \text{Tor}_i^{\geq 1}(T, -) = H_T(\text{Pres}^2(T)) = \text{Cogen} T^*.
\]
It follows that $\text{Cogen}^{+} = \text{Ker} \text{Ext}_{A}^{i \geq 1}(\cdot, T^{*})$ by Lemma 3.4. Then we have that $T^{*}$ is 1-coitling by [2]. In particular, we have that $\text{fd} \, T_{A} = \text{id}_{A} T^{*} \leq 1$. □

The following result classifies the flat dimension of $T_{A}$ when $T$ is a $\ast^{2}$-module with $A = \text{End}_{R} T$.

**Proposition 3.7.** Let $T$ be a $\ast^{2}$-module with $A = \text{End}_{R} T$. Then

(1) $\text{fd} \, T_{A} \leq 1$ if and only if $T$ is w-$\Sigma$-quasi-projective.
(2) $\text{fd} \, T_{A} = 0$ if and only if $T$ is semi-$\Sigma$-quasi-projective.

**Proof.** (1) The sufficient part follows from Corollary 3.6. We now show the necessary part.

By the definition, we need to show that the functor $H_{T}$ preserves the exactness of any exact sequence $0 \to M \to TN \to N \to 0$ with $TN \in \text{Add} T$ and $M \in \text{Gen}(T)$. After applying the functor $H_{T}$, we obtain two induced exact sequences $0 \to H_{T} M \to H_{T} TN \to X \to 0$ and $0 \to X \to H_{T} N \to Y \to 0$ for some $X, Y$. Since $T$ is a $\ast^{2}$-module, we have that $H_{T} TN, H_{T} N \in \text{Ker} \text{Tor}_{i \geq 1}^{A}(T, -)$ by [16]. Hence, by dimension shifting, we obtain that

$$\text{Tor}_{1}^{A}(T, X) \simeq \text{Tor}_{2}^{A}(T, Y).$$

Since $\text{fd} \, T_{A} \leq 1$, we see that $\text{Tor}_{1}^{A}(T, X) = 0$ in fact. Hence we have the following commutative exact diagram, by applying the functor $T \otimes_{A} -$ to the exact sequence $0 \to H_{T} M \to H_{T} TN \to X \to 0$:

$$
\begin{array}{cccccc}
0 & \longrightarrow & T \otimes_{A} H_{T} M & \longrightarrow & T \otimes_{A} H_{T} TN & \longrightarrow & T \otimes_{A} X & \longrightarrow & 0 \\
0 & \longrightarrow & M & \longrightarrow & TN & \longrightarrow & N & \longrightarrow & 0.
\end{array}
$$

It follows that $\rho_{M}$ is a monomorphism from the diagram. Since $M \in \text{Gen}(T)$, $\rho_{M}$ is also an epimorphism (see for instance [3]). Hence, $\rho_{M}$ is an isomorphism. Therefore, we have that $M \in \text{Stat}(P) \subseteq \text{Pres}^{2}(T)$. Now since all three terms in the short exact sequence $0 \to M \to TN \to N \to 0$ are in $\text{Pres}^{2}(T)$, it remains exact after applying the functor $H_{T}$ by [16], as we desired.

(2) By [15]. □

The following example shows that a w-$\Sigma$-quasi-projective $\ast^{2}$-module need neither be a $\ast$-module nor be semi-$\Sigma$-quasi-projective.

**Example 3.8.** Let $R$ denote the path algebra defined by the quiver $1 \to 2 \to 3 \to 4$. Then, the partial tilting $R$-module

$$T = \frac{4}{3} \oplus_{2} \frac{4}{3} \oplus_{2} \frac{3}{2}$$
is a \(w\)-\(\Sigma\)-quasi-projective \(\ast^2\)-module, which is neither a \(\ast\)-module nor semi-\(\Sigma\)-quasi-projective.

**Proof.** \(R\) is clearly a hereditary algebra, hence \(T\) is a \(w\)-\(\Sigma\)-quasi-projective \(\ast^2\)-module by Lemma 2.3 and Proposition 2.4, since \(T\) is partial tilting.

Since the \(R\)-module \(3 \in \text{Gen}(T)\) and \(3 \notin \text{Pres}^2(T)\), we have that \(\text{Gen}(P) \neq \text{Pres}^2(T)\). Therefore, \(T\) is not a \(\ast\)-module.

Note also that we have an epimorphism
\[
\begin{array}{ccc}
  4 & \rightarrow & 4 \\
  2 & \rightarrow & 2
\end{array}
\]
in \(\text{Add} T\), which cannot split, so \(T\) is not semi-\(\Sigma\)-quasi-projective by the definition. \(\Box\)

Combining results in Section 2, we have the following result.

**Proposition 3.9.** Let \(R\) be a ring with \(\text{gd} R = 1\) and \(T\) be a selforthogonal \(R\)-module with \(A = \text{End}_RT\). Then

1. \(T^*\) is 1-tilting.
2. \(\text{gd} A \leq 2\).
3. If \(T\) is moreover semi-\(\Sigma\)-quasi-projective (specially \(T\) is projective) or injective, then \(\text{gd} A \leq 1\).
4. If \(T\) is both semi-\(\Sigma\)-quasi-projective and injective, then \(A\) is a semisimple ring.

**Proof.** By Lemma 2.3 and Proposition 2.4, we see that \(T\) is a \(w\)-\(\Sigma\)-quasi-projective \(\ast^2\)-module. Hence we have that \(T^*\) is 1-tilting by Corollary 3.6. Note that \(\text{id}_RT' \leq 1\) for all \(T' \in \text{Add} T\) since \(\text{gd} R = 1\) by assumptions, so we obtain that \(\text{gd} A \leq 2\) by Theorem 2.5.

Similarly, we get that (3) and (4) hold by applying Proposition 3.7 and Theorem 2.5. \(\Box\)

**Remark.** In the representation theory of Artin algebras, it is well known that, for a partial tilting module \(T\) over a hereditary algebra \(R\) with \(A = \text{End}_RT\), the global dimension of the endomorphism algebra \(A\) is not more than 2. This is followed from the fact that \(A\) is indeed a tilted algebra, see for instance [8]. The last proposition generalizes this result. Moreover, it also gives a cotilting module over \(A\) which is obtained directly from the \(R\)-module \(T\).

We end this paper with the following example, which shows that the answer to Question 2 in the first section is negative too in general, even in case \(n = 2\).

**Example 3.10.** The abelian group \(Q\), as a \(Z\)-module, is a \(\ast^2\)-module, which is clearly not finitely generated.

**Proof.** At first, \(Q\) is selfsmall since that, over a countable ring, any module with countable endomorphism ring is in fact selfsmall, by [1]. Secondly, \(Q\) is obviously a selforthogonal
$\mathbb{Z}$-module since it is injective and $\mathbb{Z}$ is noetherian. Note also that $gd\mathbb{Z} = 1$, so we have that $Q$ is a $*^2$-module, by Proposition 2.4. □

References