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ON THE EVOLUTION OF FERROMAGNETIC MEDIA

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Abstract—Starting from a microscopic model of ferromagnetism, information is deduced about the form of the macroscopic constitutive relationship between the fields H and M; this exhibits hysteresis and space interaction. Accordingly, the concepts of distributed and nondistributed hysteresis functionals are introduced. Constitutive relations of this type are coupled with Maxwell's equations for a distributed system, for both the quasistationary and the fast evolution cases. Several weak formulations in Sobolev spaces are introduced; existence results are proved by means of implicit time discretization, a priori estimates and compactness procedures for taking the limit in the hysteresis functional.

INTRODUCTION

This paper deals with the electromagnetic evolution of a ferromagnetic body taking account of hysteresis effects. In [8] the author studied this problem in the univariate case, representing the constitutive relation between the fields H and M by means of a "memory functional":

$$M(x, t) = [\bar{\mathcal{F}}(H(x, \cdot), M^0(x)](t) \qquad \forall t \in [0, T], \text{ a.e. in } \Omega. \tag{1}$$

Here Ω is a 3-dimensional domain, $T > 0$, $\bar{\mathcal{F}}$ is a Volterra (i.e. causal) functional, its arguments are a (continuous) function of time into $\mathbb{R}^3$ and a vector; $H(x, \cdot)$ denotes the function $t \mapsto H(x, t)$ and $M^0(x) = M(x, 0)$ a.e. in Ω; x is just a parameter (actually in [8], M was replaced by B, but this is equivalent as $B = \mu_0 H + 4\pi M$, μ_0 being a positive constant). Examples of hysteresis functionals were also given in [8]; the classical Preisach model was studied in [9], and then generalized to the vector case in [4].

Here we consider the multivariate case. The fact that the injection of $\{v \in L^2(\Omega)^3 \mid \nabla \times v \in L^2(\Omega)^3\}$ into $L^2(\Omega)^3$ is not compact prevents from using a compactness argument for proving an existence result as in [8].

If instead of (1) a relation of the form $M = \phi(H)$ with ϕ maximal monotone function $\mathbb{R}^3 \to \mathbb{R}^3$ (or graph, more generally) were used, then the previous difficulty could be overcome by means of a compensated compactness argument, as in [1]: however this does not extend to the case with hysteresis. Moreover (1) does not take into account space interactions and is essentially a constitutive relation for a nondistributed system (i.e. with no explicit dependence on space). The use of a nondistributed constitutive relation (i.e., with x acting just as a parameter) for a distributed system is a common practice, but in the case of hysteresis there is the mathematical drawback that the "smoothness" of H w.r.t. x does not imply any "smoothness" of M w.r.t. x, even for a "regular" functional $\bar{\mathcal{F}}$; this does not seem very sound also from the physical viewpoint.

Searching a more precise constitutive relation, in [10] the author examined the mathematical properties of a microscopic model of ferromagnetism, namely the classical theory of Weiss domains and Landau-Lifshitz equations. Here in section 1 we use this model for deducing some information on the structure of the constitutive relation between H and M. As a result in the case of quasistationary evolution we get

$$H(t) = [\bar{\mathcal{G}}(M(x, \cdot), H^0(x))](t) + \mathcal{A}M(t) := [\mathcal{G}(M, H^0)](t) \text{ in } \mathcal{D}'(\Omega)^3, \ \forall t \in [0, T]; \quad (2)$$

here $\bar{\mathcal{G}}$ is the inverse of $\bar{\mathcal{F}}$ (assumed to exist) and is a nondistributed hysteresis functional, $H^0(x) = H(x, 0)$ a.e. in Ω and $\mathcal{A}$ is a second order elliptic operator; $\mathcal{G}$ is an example of distributed hysteresis functional. These concepts are detailed in section 2. Eq. (2) is coupled with Maxwell's equations for quasistationary evolution, i.e. without displacement current term, or equivalently with an equation of the form

$$\frac{\partial}{\partial t}(H + M) + \nabla \mathrm{x} \nabla \mathrm{x} H = f, \quad (3)$$

with normalized constants and f datum. In section 3 we prove the existence of at least one weak solution for the system (2), (3) and another existence result for (3) coupled with the constitutive relation

$$M = [\mathcal{F}(H, M^0)] \qquad \text{in } \Omega \times [0, T], \quad (4)$$

where the distributed hysteresis functional $\mathcal{F}$ is the inverse of $\mathcal{G}$ (assumed to exist), so that (2) and (4) are equivalent. In many technical applications it is reasonable to assume the magnetostatic approximation, namely to replace Maxwell's equations by

$$\nabla \mathrm{x} H = f \qquad \nabla \cdot (\mu_0 H + 4\pi M) = 0; \quad (5)$$

we prove existence of a solution also for the system (2), (5).

In section 4 we deal with the fast evolution case. The microscopic model suggests a macroscopic constitutive relation of the form

$$H = \mathcal{G}(M, H^0) + \frac{\partial M}{\partial t}, \quad (6)$$

where the extra term $\partial M/\partial t$ (multiplied by a viscosity coefficient) represents viscosity. We prove an existence result for the problem of this constitutive relation coupled with Maxwell's equations, now including the displacement current term.

These existence results can be summarized as follows. We prove existence of a solution for the magnetostatic equations or for Maxwell's equations without displacement current term coupled with the quasistationary constitutive relation (2) or with the fast evolution relation (6) and also for Maxwell's equations with displacement current coupled with (6). The excluded case of Maxwell's equations with displacement current coupled with the quasistationary relation (2) is not physical, since the viscosity term in the constitutive relation is more important than the displacement current term. For all of these problems the uniqueness of the solution is an open question.

Also the invertibility of hysteresis functionals is an open question, even under a certain monotonicity assumption. Can the functional corresponding to the classical Preisach model be inverted? If so, the question of the identification of its inverse arises. Another question is the possible existence of periodic solutions. Finally, can any space interaction effect be introduced also in the constitutive stress-strain relationship for plasticity?

1. MICROSCOPIC AND MACROSCOPIC MODELS FOR FERROMAGNETISM

Consider a ferromagnetic body occupying a bounded domain Ω of the Euclidean space $\mathbb{R}^3$. According to the classical theory of Weiss, below a critical temperature on a microscopic scale the body is magnetically saturated, that is, denoting the microscopic magnetic field by m

$$| m(x, t) | = \mathcal{M} \quad \text{(positive constant) in } \Omega \times [0, T]. \tag{1.1}$$

The evolution of m is governed by Landau-Lifshitz equations

$$\frac{\partial m}{\partial t} = \lambda_1 m \mathrm{x} h^e - \lambda_2 m \mathrm{x}(m \mathrm{x} h^e) \qquad \text{in } \Omega \times [0, T] \tag{1.2}$$

$$h^e := h + \nabla \cdot (F \cdot \nabla m) - G \cdot m \qquad \text{in } \Omega \times [0, T] \tag{1.3}$$

$$\left(\text{i.e., } h_l^e = h_l + \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}\left(F_{ij} \frac{\partial m_l}{\partial x_i}\right) - \sum_{i=1}^{3} G_{li} m_i, \quad l = 1, 2, 3 \right),$$

where λ_1, λ_2 are constants, $\lambda_2 > 0$; h is the magnetic field (the field in Maxwell's equations, to be clearer), F and G are positive-definite 3^2-tensors. As usual x denotes the vector product, $\cdot$ the scalar product, ∇ the gradient, $\nabla\cdot$ the divergence, ∇x the curl. Note that (1.2) preserves the constraint (1.1) in time, as can be easily checked multiplying (1.2) by m. An equivalent way of writing (1.2) is Gilbert's equation

$$\frac{\partial m}{\partial t} = \mu_1 m \times \left(h^e - \frac{\mu_2}{\mu_1} \frac{\partial m}{\partial t} \right) \qquad \text{in } \Omega \times [0, T], \tag{1.4}$$

with μ_1, μ_2 constants. The transformation formulas between the couples of constants λ_1, λ_2 and μ_1, μ_2 are

$$\lambda_1 = \frac{-\mu_1}{1 + \mu_2^2 \mathcal{M}^2}; \quad \lambda_2 = \frac{-\mu_1 \mu_2}{1 + \mu_2^2 \mathcal{M}^2}.$$

The first term at the right-hand side of (1.2) causes a precession of m around h^e and is not dissipative; the second term tends to align m to h^e and is due to viscosity, hence it is dissipative. Therefore, as remarked by Callen in [3], (1.4) suggests the interpretation that the dissipation introduces a contribution proportional to $\partial m/\partial t$ into the effective magnetic field, μ_2/μ_1 representing a viscosity coefficient.

An initial condition has to be prescribed

$$m(x,0) = m^0(x) \qquad \text{in } \Omega \times [0, T], \tag{1.5}$$

with $| m^0 | = \mathcal{M}$. By physical reasons we have

$$\nu \cdot F \cdot \nabla m \left(:= \sum_{i,j=1}^{3} \nu_i F_{ij} \frac{\partial m}{\partial x_j} \right) = 0 \qquad \text{on } \partial\Omega \times [0, T], \tag{1.6}$$

where ν denotes the unitary external normal vector on $\partial\Omega$. (1.1), . . . , (1.3) are coupled with Maxwell's equations and Ohm's law:

$$\nabla \mathrm{x} h = 4\pi j + \frac{\epsilon}{c}\frac{\partial e}{\partial t} \qquad \text{in } \mathbb{R}^3 \times [0, T] \tag{1.7}$$

$$\nabla \mathrm{x} e = -\frac{1}{c}\frac{\partial}{\partial t}(h + 4\pi m) \qquad \text{in } \mathbb{R}^3 \times [0, T] \tag{1.8}$$

$$j = \sigma(e + f) \qquad \text{in } \mathbb{R}^3 \times [0, T]. \tag{1.9}$$

Here j denotes the electric current density, e the electric field, σ the electric conductivity (assumed constant in Ω and vanishing outside), ϵ the dielectric permeability and c the velocity of the light in the vacuum; f represents a given applied electromotive force. Following [2], (1.7), . . . , (1.9) are set in the whole space and not just in Ω as for (1.2), (1.3). This physical setting was studied in [10], where in particular an existence result for the corresponding weak formulation was proved. Here we deal with the macroscopic situation. Following a standard procedure, we introduce the space average operator

$$\phi \mapsto \langle\phi\rangle(x) := \frac{1}{\text{volume}(B(x, R) \cap \Omega)} \int_{B(x,R)\cap\Omega} \phi(x + r)dr \quad \text{for } x \in \Omega,$$

where $B(x, R)$ denotes the ball with center x and prescribed radius R; it is assumed that $B(x, R) \cap \Omega$ contains "many" particles, yet is "small" w.r.t. Ω. We set

$$H := \langle h\rangle,\ H^e := \langle h^e\rangle,\ M := \langle m\rangle,\ E := \langle e\rangle,\ J := \langle j\rangle. \tag{1.10}$$

By (1.1) and (1.3) we have

$$|M(x, t)| \leq \mathcal{M} \qquad \text{in } \Omega \times [0, T] \tag{1.11}$$

$$H^e = H + \nabla\cdot(F \cdot \nabla M) - G \cdot M \qquad \text{in } \Omega \times [0, T] \tag{1.12}$$

and by (1.7), . . . , (1.9)

$$\nabla x H = 4\pi J + \frac{\epsilon}{c}\frac{\partial E}{\partial t} \qquad \text{in } \mathbb{R}^3 \times [0, T] \tag{1.13}$$

$$\nabla x E = -\frac{1}{c}\frac{\partial}{\partial t}(H + 4\pi M) \qquad \text{in } \mathbb{R}^3 \times [0, T] \tag{1.14}$$

$$J = \sigma(E + f). \qquad \text{in } \mathbb{R}^3 \times [0, T]. \tag{1.15}$$

In many cases the displacement current $(\epsilon/c)\ (\partial E/\partial t)$ can be neglected (see [7], e.g.). Difficulties arise in the application of the averaging procedure to the nonlinear Landau-Lifshitz equation (1.2). Therefore instead we shall use an experimental relationship between H^e and M. At first we assume that the relaxation process described by (1.2) (or equivalently by (1.4)) is fast enough compared with the evolution described by Maxwell's equations, whose characteristic time scale depends on the datum f; in this case we introduce a quasi-stationary condition of the form

$$M(x, t) = [\bar{\mathcal{F}}(H^e(x, \cdot), M^0(x)](t) \qquad \text{in } \Omega \times [0, T] \tag{1.16}$$

with $\tilde{\mathcal{F}}$ is a "hysteresis functional" and $M^0(x) = M(x, 0)$ in Ω; this will be made precise later, however in particular it means that M does not depend just on the value of H^e at the same time, but also on the previous evolution of H^e; this fact is strictly related to the multiplicity of solutions of the stationary microscopic problem of minimization of the magnetic energy under the constraint (1.1) (see [10], section 2).

In the univariate (i.e., one-dimensional) case it is possible to construct an approximation of $\tilde{\mathcal{F}}$ by means of a classical measurement technique (see [5], § 36.3). In the multivariate situation rather than a direct measurement procedure it seems more convenient to use a model relating the vectorial case to the scalar one; in this direction see [4]. Usually the hysteresis relation is written for H and not for H^e, since in the nondistributed case (i.e., with no space dependence) $H^e = H$. However by (1.2) the evolution of m is determined by h^e, therefore it is natural to assume that the evolution of $M := \langle m \rangle$ be controlled by that of $H^e := \langle h^e \rangle$; instead H is the field appearing in Maxwell's equations.

Assuming that $\tilde{\mathcal{F}}$ can be inverted, (1.16) can be written in the form

$$H^e(x, t) = [\tilde{\mathcal{G}}(M(x, \cdot), H^{e0}(x))](t) \quad \text{in } \Omega \times [0, T], \tag{1.17}$$

with $H^{e0}(x) = H^e(x, 0)$, or also by (1.12)

$$\begin{aligned} H(x, t) = [\tilde{\mathcal{G}}(M(x, \cdot), H^{e0}(x))](t) - \nabla \cdot [F \cdot \nabla M(x, t)] + G.M(x, t) \\ := [\mathcal{G}(M, H^0)](x, t), \quad \text{in } \Omega \times [0, T], \end{aligned} \tag{1.18}$$

with $H^0(x) = H(x, 0)$ in Ω; $\tilde{\mathcal{G}} := \tilde{\mathcal{F}}^{-1}$ and $\mathcal{G}$ are hysteresis functionals. If also $\mathcal{G}$ is invertible, then (1.18) can be rewritten as

$$M(x, t) = [\mathcal{F}(H, M^0)](x, t) \qquad \text{in } \Omega \times [0, T], \tag{1.19}$$

with $M^0(x) = M(x, 0)$ in Ω; also $\mathcal{F} := \mathcal{G}^{-1}$ is a hysteresis functional.

In conclusion, assuming that all inverses exist, in the quasistationary case we have four equivalent constitutive relations (here written in shortened form)

$$\begin{cases} M = \tilde{\mathcal{F}}(H^e, M^0), & H^e = \tilde{\mathcal{G}}(M, H^{e0}), \\ M = \mathcal{F}(H, M^0), & H = \mathcal{G}(M, H^0), \end{cases} \tag{1.20}$$

$\tilde{\mathcal{F}}$ and its inverse $\tilde{\mathcal{G}}$ are set pointwise in space, whereas $\mathcal{F}$ and its inverse $\mathcal{G}$ have a global character; all of these are hysteresis functionals. This will be made precise in the next section. Each of these relations can be coupled with Maxwell's equations without displacement current term or also with magnetostatic equations.

If f is such that the relaxation process of m cannot be neglected, (1.4) suggests to replace H^e by

$$\tilde{H}^e := H^e - \frac{\mu_2}{\mu_1}\frac{\partial M}{\partial t} = H + \nabla \cdot (F \cdot \nabla M) - G \cdot M - \frac{\mu_2}{\mu_1}\frac{\partial M}{\partial t} \quad \text{in } \Omega \times [0, T] \tag{1.21}$$

(by (1.12)); in this case also the displacement current term $(\epsilon/c)(\partial E/\partial t)$ can be taken into account in Maxwell's equations; however also the magnetostatic approximation is often used in practice. In order to shorten formulae, henceforth all physical constants, even π in (1.14), will be replaced by 1; this will be immaterial for our mathematical developments.

2. HYSTERESIS FUNCTIONALS

We shall say that $(\mathfrak{X}, \mathcal{L}, \mathcal{H})$ (or more shortly $\mathcal{H}$) is a *memory functional* if and only if

$$\mathfrak{X} \text{ is a real Banach space;} \qquad \mathcal{L} \subset \mathfrak{X} \times \mathfrak{X}' \tag{2.1}$$

($\mathfrak{X}'$ denotes the dual space of $\mathfrak{X}$)

$$\mathrm{Dom}\,(\mathcal{H}) = \{(v, \xi) \in C^0([0, T]; \; \mathrm{Dom}(\mathcal{L})) \times \mathfrak{X}' \mid (v(0), \xi) \in \mathcal{L}\} \tag{2.2}$$

$$(\mathrm{Dom}\,(\mathcal{L}) := \{v \in \mathfrak{X} \mid \exists w \in \mathfrak{X}' : (v, w) \in \mathcal{L}\});$$

$$\begin{cases} \forall (v, \xi) \in \mathrm{Dom}(\mathcal{H}), \quad \mathcal{H}(v, \xi) \in C^0([0, T]; \mathfrak{X}'), \quad [\mathcal{H}(v, \xi)](0) = \xi \\ \text{and } \forall\, t \in [0, T] \quad (v(t), [\mathcal{H}(v, \xi)](t)) \in \mathcal{L}; \end{cases} \tag{2.3}$$

$$\begin{cases} \forall (v_1, \xi), (v_2, \xi) \in \mathrm{Dom}(\mathcal{H}), \forall \bar{t} \in [0, T], \quad \text{if } v_1 = v_2 \quad \text{in } [0, \bar{t}] \quad \text{then} \\ [\mathcal{H}(v_1, \xi](\bar{t}) = [\mathcal{H}(v_2, \xi)(\bar{t}), \qquad \text{(Causality)} \end{cases} \tag{2.4}$$

If $\mathfrak{X}$ is a Banach space of functions defined over a Euclidean domain Ω, then we shall say that $\mathcal{H}$ is a *distributed* memory functional; if $\mathfrak{X} = \mathbb{R}^N$ $(N \geq 1)$, then $\mathcal{H}$ will be said a *non-distributed* memory functional.

Besides the above basic properties, we introduce the following ones

$$\begin{cases} \forall (v, \xi) \in \mathrm{Dom}(\mathcal{H}), \forall \bar{t} \in [0, T], \quad \text{setting } \xi_{\bar{t}} := [\mathcal{H}(v, \xi)](\bar{t}) \quad \text{and} \\ v_{\bar{t}}(t) := v(t + \bar{t}), \quad \forall t \in [\bar{t}, T], \\ [\mathcal{H}(v_{\bar{t}}, \xi_{\bar{t}})](t - \bar{t}) = [\mathcal{H}(v, \xi)](t). \quad \text{(Semigroup property)} \end{cases} \tag{2.5}$$

$$\begin{cases} \forall (v, \xi) \in \mathrm{Dom}(\mathcal{H}), \forall s : [0, T] \to [0, T] \quad \text{increasing homeomorphism}, \forall t \in [0, T] \\ [\mathcal{H}(v \circ s, \xi)](t) = [\mathcal{H}(v, \xi)](s(t)). \qquad \text{(Rate independence)} \end{cases} \tag{2.6}$$

In the opinion of the writer, (2.5) and (2.6) are the distinguishing properties of hysteresis: they are not fulfilled by time convolution functionals representing viscosity effects. $(\mathfrak{X}, \mathcal{L}, \mathcal{H})$ will be called a *hysteresis functional* if and only if it fulfills (2.1), . . . , (2.6).

More precisely what we have defined could be named a *time-continuous* memory functional, or a *time-continuous* hysteresis functional if also (2.5) and (2.6) hold, since in (2.3) $\mathcal{H}(v, \xi) \in C^0([0, T]; \mathfrak{X}')$. We shall use also the following property

$\forall t', t'' \in [0, T](t' < t''), \forall v : [0, t'] \to \mathfrak{X}$ continuous and piecewise linear,

$\forall \xi \in \mathfrak{X}'$ such that $(v(0), \xi) \in \mathcal{L}, \forall z \in \mathfrak{X}$, set

$$v_z := \begin{cases} v & \text{in } [0, t'] \\ v(t') + \dfrac{t - t'}{t'' - t'} \mid z - v(t') \mid & \text{in } [t', t'']; \; \Phi_v(z) := [\mathcal{H}(v_z, \xi)](t''). \\ z & \text{in } [t'', T] \end{cases}$$

It is required that $\Phi_v : \mathfrak{X} \to \mathfrak{X}'$ be cyclically maximal monotone

$$\text{(Piecewise monotonicity).} \tag{2.7}$$

Remark that also this property does not hold if $\mathcal{H}$ has the form of a time convolution, whereas it is compatible with hysteresis phenomena. For applications we shall need also some continuity property of the functional $v \mapsto \mathcal{H}(v, \xi)$; this will be specified case by case.

Let $(\mathcal{X}, \mathcal{L}, \mathcal{H})$ be a memory functional; another memory functional $(\mathcal{X}', \mathcal{L}^{-1}, \mathcal{K})$ will be called its inverse and we shall write $\mathcal{K} = \mathcal{H}^{-1}$ if and only if

$$\forall (v, \xi) \in \mathrm{Dom}(\mathcal{H}), \text{ setting } w := \mathcal{H}(v, \xi), \; v = \mathcal{K}(w, v(0)) \tag{2.8}$$

$$\forall (w, \eta) \in \mathrm{Dom}(\mathcal{K}), \text{ setting } v := \mathcal{K}(w, \eta), \; w = \mathcal{H}(v, w(0)). \tag{2.9}$$

A necessary condition for the existence of the inverse of a memory functional is that the piecewise monotonicity property (2.7) be fulfilled. However, it is not evident that this property be also a sufficient condition.

Now we consider an example of a typical procedure of construction of nondistributed hysteresis functionals (see [8], § 4): $\mathcal{X} = \mathbb{R}$ and $\mathcal{H}$ is defined for input functions v in a suitable subset of $C^0([0, T])$, for instance for continuous and piecewise linear v's; then by some uniform continuity property $\mathcal{H}$ is extended to its whole domain. It is easy to see that $\mathcal{H}$ can be inverted on the class of continuous and piecewise monotone functions $[0, T] \to \mathbb{R}$ if (and only if) (2.7) holds; but it is not evident that the inverse can be extended to its whole domain and, if such an extension exists, that this be the inverse of $\mathcal{H}$.

Now we make the definitions of section 1 more accurate. In (1.16) $(\mathbb{R}^3, \bar{R}, \bar{\mathcal{F}})$ is a nondistributed hysteresis functional and $\bar{R} = \text{graph } \mathbb{R}^3 \to \{\xi \in \mathbb{R}^3 \,\|\, \xi \,| \leq \mathcal{M}\}$ is as sketched

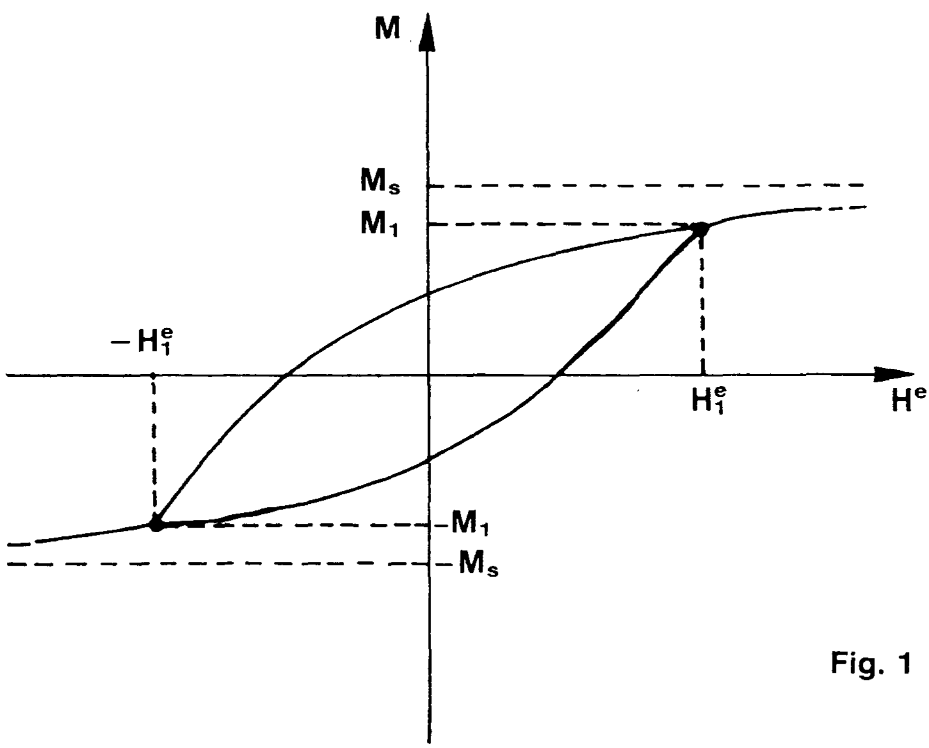

Fig. 1.

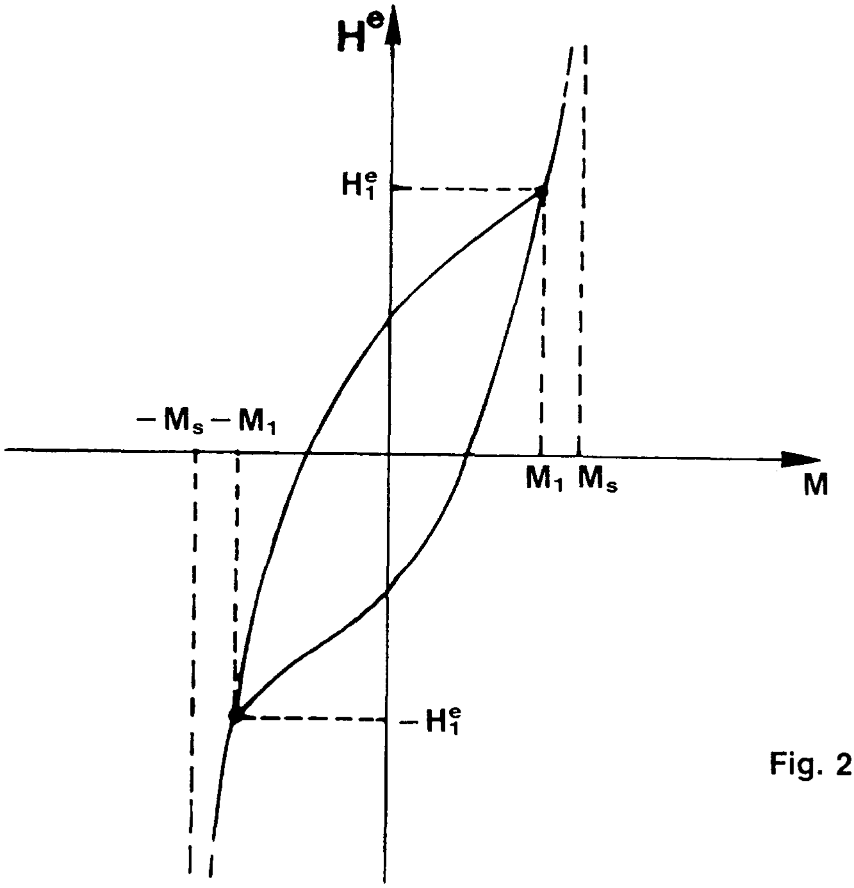

Fig. 2.

in Fig. 1. If $\bar{\mathscr{F}}$ can be inverted, (1.16) is equivalent to (1.17) with $(\mathbb{R}^3, \bar{S}, \bar{\mathscr{G}})$ nondistributed hysteresis functional, $\bar{S} := \bar{R}^{-1}$. Let the linear and continuous operator $\mathscr{A}: H^1(\Omega)^3 \to (H^1(\Omega)^3)'$ be defined by

$$ {}_{(H^1(\Omega)^3)'}\langle u, v\rangle_{H^1(\Omega)^3} := \sum_{i,j=1}^{3} \int_\Omega \sum_{l=1}^{3} \left(F_{ij} \frac{\partial u_l}{\partial x_i} \cdot \frac{\partial v_l}{\partial x_j} + G_{ij} u_i u_j\right) dx, \ \forall u, v \in H^1(\Omega)^3; \quad (2.10)$$

this is the bilinear form associated with $M \mapsto -\nabla \cdot (F \cdot \nabla M) + G \cdot M$ and with the boundary condition (1.6).

For a slow dynamics, using the expression

$$H^e = H - \mathscr{A} M \in (H^1(\Omega)^3)' \quad (2.11)$$

(see (1.12)), (1.17) can be rewritten in the form

$$H(t) = [\bar{\mathscr{G}}(M(x, \cdot), H^{e0}(x))](t) + \mathscr{A} M(t)$$
$$:= [\mathscr{G}(M(\cdot), H^{e0})](t) \quad \text{in } (H^1(\Omega)^3)', \forall t \in [0, T]. \quad (2.12)$$

Let $S = \text{graph}\{\xi \in H^1(\Omega)^3 \,\|\, \xi \,| < \mathscr{M}\} \to H^1(\Omega)^3$ be defined as follows: $\forall (M, H) \in H^1(\Omega)^3 \times (H^1(\Omega)^3)'$, $(M, H) \in S$ if an only if $H^e := H - \mathscr{A} M$ is measurable in Ω and $(M(x), H^e(x)) \in \bar{S}$ a.e. in Ω; then $(H^1(\Omega)^3, S, \mathscr{G})$ is a (distributed) hysteresis functional. If also

$\mathscr{G}$ can be inverted then (2.12) can be written in the form

$$M(t) = [\mathscr{F}(H(\cdot), M^0)](t) \quad \text{in } H^1(\Omega)^3, \quad \forall t \in [0, T], \tag{2.13}$$

with $((H^1(\Omega)^3)', R, \mathscr{F})$ hysteresis functional, $R: = S^{-1}$.

We introduce a decomposition of $\mathscr{G}$ which will be used in the next sections. Remark that the maximal loop of $\mathscr{G}: = \bar{\mathscr{F}}^{-1}$ is contained in a set of the form $Z: = \{(M, H^e) \in (\mathbb{R}^3)^2 \,\|\, M| \leq M_1, |H^e| \leq H_1^e\}$ with $M_1, H_1^e \in \mathbb{R}^+$ (see Fig. 2). We set

$$\begin{cases} \pi_1: \text{projection of } \mathbb{R}^3 \text{ onto } \{v \in \mathbb{R}^3 \mid |v| \leq H_1^e\} \\ \bar{S}_1: = \{(v, \pi_1 w) \mid (v, w) \in \bar{S}\} \\ \bar{\mathscr{G}}_1(v, \xi) = \pi_1 \bar{\mathscr{G}}(v, \xi), \quad \forall (v, \xi) \in \mathrm{Dom}\,(\bar{\mathscr{G}}); \end{cases} \tag{2.14}$$

then $(\mathbb{R}^3, \bar{S}_1, \bar{\mathscr{G}}_1)$ is a nondistributed hysteresis functional and $v \mapsto \phi(v): = \bar{\mathscr{G}}(v, \xi) - \bar{\mathscr{G}}_1(v, \xi)$ is a monotone function without hysteresis. We set also

$$\mathscr{G}_2(v): = \phi(v) + \mathscr{A} v, \qquad \forall v \in H^1(\Omega)^3. \tag{2.15}$$

Thus (2.12) can be rewritten as

$$H(t) = [\bar{\mathscr{G}}_1(M(x, \cdot), H^{e0}(x)](t) + \mathscr{G}_2(M(t)) \quad \text{in } (H^1(\Omega)^3)', \forall t \in [0, T]. \tag{2.16}$$

3. QUASISTATIONARY EVOLUTION

For the study of Maxwell's equations (1.13), (1.14) it is convenient to introduce the following functional space

$$W: = \{v \in L^2(\mathbb{R}^3)^3 \mid \nabla \mathrm{x} v \in L^2(\mathbb{R}^3)^3\},$$

a Hilbert space with norm $\| v \|_W = (\| v \|^2_{L^2(\mathbb{R}^3)^3} + \| \nabla \mathrm{x} v \|^2_{L^2(\mathbb{R}^3)^3})^{1/2}$. Identifying $L^2(\mathbb{R}^3)^3$ with its dual we get

$$W \subset L^2(\mathbb{R}^3)^3 = (L^2(\mathbb{R}^3)^3)' \subset W', \tag{3.1}$$

with continuous and dense but noncompact inclusions. We shall use also the Hilbert triplet

$$V: = H^1(\Omega)^3 \subset L^2(\Omega)^3 = (L^2(\Omega)^3)' \subset V'. \tag{3.2}$$

Any function defined just in Ω will be identified with its extension with value zero to $\mathbb{R}^3 \backslash \Omega$; thus

$$V \subset L^2(\mathbb{R}^3)^3 = (L^2(\mathbb{R}^3)^3)' \subset V', \tag{3.3}$$

$$\{v \in V \mid \nu \mathrm{x} v = 0 \text{ on } \partial\Omega\} \subset W; \tag{3.4}$$

in (3.2), . . . , (3.4) the inclusions are continuous and dense, in (3.2), (3.3) they are also compact.

As we said in section 1, if the relaxation dynamics of the elementary magnets (see (1.2), . . . , (1.4)) is much faster than the dynamics expressed by Maxwell's equations and controlled by the datum f, then we consider one of the quasi-stationary constitutive relations (1.20). In our formulation we shall use the more general concept of memory functional; however for the existence result we shall require the piecewise monotonicity property (2.7), typical of hysteresis functionals.

Let $(V, S, \mathscr{G})$ be a distributed memory functional, in the sense of (2.1), . . . , (2.4); let

$$(M^0, H^0) \in S, \qquad H^0 \in W', \qquad f \in L^2(0, T; W'). \tag{3.5}$$

We introduce a weak problem corresponding to Maxwell's equations with no displacement current term coupled with (2.14):

(P1) Find $M \in C^0([0, T]); V)$ with $M(t) \in \mathrm{Dom}(S)\ \forall t \in [0, T]$,

$$M(0) = M^0 \qquad \text{a.e. in } \Omega, \tag{3.6}$$

such that, setting

$$H(t) := [\mathscr{G}(M, H^0)](t) \qquad \text{in } V', \forall t \in [0, T], \tag{3.7}$$

then $H \in L^2(0, T; W)$ and

$$\frac{\partial}{\partial t}(H + M) + \nabla \mathrm{x} \nabla \mathrm{x} H = f \qquad \text{in } W', \text{ a.e. in }]0, T[. \tag{3.8}$$

Remark 1. By (2.3), (3.7) yields $H(0) = H^0$ in V'; by (3.8) we have also $H + M \in H^1(0, T; W')$, hence

$$(H + M)\big|_{t=0} = H^0 + M^0 \qquad \text{in } W'. \tag{3.9}$$

THEOREM 1. Assume that

$$(M^0, H^0) \in S, \quad H^0 \in W, \quad f = f_1 + f_2$$
$$\text{with} \quad f_1 \in L^2(\mathbb{R}^3 x]0, T[)^3, \quad f_2 \in W^{1,1}(0, T; W') \tag{3.10}$$

$$\left\{ \begin{array}{l} (\mathbb{R}^3, \bar{S}_1, \bar{\mathscr{G}}_1) \text{ is a piecewise monotone (non-distributed) memory} \\ \text{functional in the sense of (2.1), . . . , (2.4), (2.7)} \end{array} \right. \tag{3.11}$$

$$\forall (v, \xi) \in \mathrm{Dom}(\bar{\mathscr{G}}_1), \quad |\bar{\mathscr{G}}_1(v, \xi)| \le C_1 \ (C_1\text{: positive constant}) \tag{3.12}$$

$$\left\{ \begin{array}{l} \forall \{(v_n, \xi) \in \mathrm{Dom}(\bar{\mathscr{G}}_1)\}_{n \in \mathbb{N}}, \quad \text{if } v_n \to v \quad \text{strongly in } C^0([0, T])^3, \\ \text{then } (v, \xi) \in \mathrm{Dom}(\bar{\mathscr{G}}_1) \quad \text{and} \quad \bar{\mathscr{G}}_1(v_n, \xi) \to \bar{\mathscr{G}}_1(v, \xi) \quad \text{strongly in } C^0([0, T])^3 \end{array} \right. \tag{3.13}$$

$$\mathscr{G}_2 : \{v \in V \mid v(x) \in \mathrm{Dom}(\bar{S}_1) \text{ a.e. in } \Omega\} \to V' \quad \text{is maximal monotone} \tag{3.14}$$

$$\forall v_1, v_2 \in V, \ {}_{V'}\langle \mathscr{G}_2(v_1) - \mathscr{G}_2(v_2), v_1 - v_2 \rangle_V + C_2 \| v_1 - v_2 \|^2_{L^2(\mathbb{R}^3)^3} \ge C_3 \| v_1 - v_2 \|^2_V$$
$$(C_2, C_3\text{:} \quad \text{positive constants}) \tag{3.15}$$

$$\left\{ \begin{array}{l} H_1^0 := H^0 - \mathscr{G}_2(M^0) \quad \text{is measurable in } \Omega \quad \text{and} \\ (M^0(x), H_1^0(x)) \in \mathrm{Dom}(\bar{\mathscr{G}}_1) \text{ a.e. in } \Omega. \end{array} \right. \tag{3.16}$$

Set

$$\begin{cases} S := \{(v, w) \in V \times V' \mid v(x) \in \mathrm{Dom}(\bar{S}_1) \text{ a.e. in } \Omega, z := w - \mathscr{G}_2(v) \quad \text{is measurable in } \Omega \\ \text{and} \quad (v(x), z(x)) \in \bar{S}_1 \text{ a.e. in } \Omega\} \end{cases} \tag{3.17}$$

$$\mathrm{Dom}(\mathscr{G}) := \{(v, \xi) \in C^0([0, T]; \ \mathrm{Dom}(S)) \times V' \mid (v(0), \xi) \in S\} \tag{3.18}$$

$$\begin{cases} \forall\ (v, \xi) \in \mathrm{Dom}(\mathscr{G}), \quad \text{setting } H_1^0 := \xi - \mathscr{G}_2(v(\cdot, 0)), \\ [\mathscr{G}(v, \xi)](t) := [\mathscr{G}_1(v(x, \cdot), H_1^0(x))](t) + \mathscr{G}_2(v(\cdot, t)) \quad \text{in } V', \forall t \in [0, T]. \end{cases} \tag{3.19}$$

Then $(V, S, \mathscr{G})$ is a (distributed) memory functional in the sense of (2.1), . . . , (2.4) and problem (P1) has at least one solution such that moreover

$$H \in H^1(0, T; L^2(\mathbb{R}^3)) \cap L^\infty(0, T; W); M \in H^1(0, T; V). \tag{3.20}$$

Remark. The decomposition of $\mathscr{G}$ into the sum $\mathscr{G} = \mathscr{G}_1 + \mathscr{G}_2$ corresponds to the developments at the end of section 2. The assumptions of theorem 2 are consistent with the physical model introduced in section 1.

Proof.

(i) *Approximation.*

Let $m \in \mathbb{N}$, $k := \dfrac{T}{m}$.

(P1)$_m$ Find $M_m^n \in V$ $(n = 1, \ldots, m)$ such that, setting

$$M_m := \text{linear interpolate of } M_m(nk) := M_m^n \text{ in } [0, T] \ (M_m^0 := M^0) \tag{3.21}$$

$$H_m^n := [\mathscr{G}(M_m, H^0)](nk) \quad \text{in } V', n = 0, \ldots, m, \tag{3.22}$$

then $H_m^n \in W\ \forall n$ and

$$\frac{1}{k}(H_m^n - H_m^{n-1} + M_m^n - M_m^{n-1}) + \nabla \mathrm{x} \nabla \mathrm{x} H_m^n = f_m^n \quad \text{in } W', \quad n = 1, \ldots, m; \tag{3.23}$$

here

$$\begin{cases} f_m^n := f_{1m}^n + f_{2m}^n, \quad f_{1m}^n(x) := \dfrac{1}{k}\displaystyle\int_{(n-1)k}^{nk} f_1(x, t)dt \text{ a.e. in } \Omega, \\ f_{2m}^n = f_2(nk) \quad \text{in } V'. \end{cases}$$

(P1)$_m$ can be solved step by step. Let $n \in \{1, \ldots, m\}$ and assume that M_m^l is known for $l = 1, \ldots, n-1$; then (3.22) can be written in the form

$$H_m^n = \Phi_{M^0, \ldots, M_m^{n-1}}(M_m^n) := \Phi_m^n(M_m^n) \quad \text{in } V'$$

or also

$$M_m^n = (\Phi_m^n)^{-1}(H_m^n) \quad \text{in } V,$$

with Φ_m^n and $(\Phi_m^n)^{-1}$ cyclically maximal monotone graphs, hence subdifferentials of convex, lower semicontinuous functionals. Therefore (3.25) is equivalent to the minimization of a lower semicontinuous, strictly convex functional $W \to \mathbb{R}$; this problem has one and only one solution, which can be also numerically approximated by standard techniques, if an approximation of $\mathcal{G}$ is available.

(ii) *Estimates*

We multiply (3.23) against $H_m^n - H_m^{n-1}$ and sum for $n = 1, \ldots, l$, for a generic $l \in \{1, \ldots, m\}$. We note that by the piecewise monotonicity of $\mathcal{G}_1$ and by (3.15),

$$\frac{1}{k} \sum_{n=1}^{l} {}_{V}\langle M_m^n - M_m^{n-1}, \; H_m^n - H_m^{n-1}\rangle_{V'} \geq C_2 k \sum_{n=1}^{l} \left\| \frac{M_m^n - M_m^{n-1}}{k} \right\|_V^2; \tag{3.24}$$

by a standard procedure based on Gronwall's lemma we get

$$k \sum_{n=1}^{m} \left\| \frac{H_m^n - H_m^{n-1}}{k} \right\|_{L^2(\mathbb{R}^3)^3}^2 \leq C \quad \text{(constant independent of } m) \tag{3.25}$$

$$\max_{n=1,\ldots,m} \| H_m^n \|_W \leq C \tag{3.26}$$

$$k \sum_{n=1}^{m} \left\| \frac{M_m^n - M_m^{n-1}}{k} \right\|_V^2 \leq C. \tag{3.27}$$

We denote by H_m the function obtained by linear interpolation of $H_m(x, nk) := H_m^n(x)$ a.e. in Ω in $[0, T]$; we set $\hat{H}_m(x, t) := H_m^n(x)$ a.e. in Ω, $\hat{G}_{1m}(t) := [\mathcal{G}_1 (M_m, H_1^0)](nk)$ and $\hat{G}_{2m}(t) := \mathcal{G}_2(M_m^n)$ in V' for $(n-1)k < t \leq nk$, $n = 1, \ldots, m$; we define $\hat{M}_m$ and $\hat{f}_m$ similarly.

Thus (3.22), (3.23) and (3.25), . . . , (3.27) can be rewritten in the form

$$\hat{H}_m = \hat{G}_{1m} + \hat{G}_{2m} \qquad \text{in } V', \text{ a.e. in }]0, T[, \tag{3.28}$$

$$\frac{\partial}{\partial t}(H_m + M_m) + \nabla \mathrm{x} \nabla \mathrm{x} \hat{H}_m = \hat{f}_m \qquad \text{in } W', \text{ a.e. in }]0, T[. \tag{3.29}$$

$$\| H_m \|_{H^1(0,T;L^2(\mathbb{R}^3)^3) \cap L^\infty(0,T;W)} \leq C \quad \text{(constant independent of } m) \tag{3.30}$$

$$\| M_m \|_{H^1(0,T;V)} \leq C; \tag{3.31}$$

moreover by (3.12)

$$\| \hat{G}_{1m} \|_{L^\infty(\Omega \times]0,T[)} \leq C \tag{3.32}$$

and then by comparison in (3.28)

$$\| \hat{G}_{2m} \|_{L^\infty(0,T;L^2(\Omega)^3)} \leq C. \tag{3.33}$$

(iii) *Limit*

By the previous estimates there exist H, M, G_1, G_2 such that, possibly taking subsequences,

$$H_m \to H \quad \text{weakly in } H^1(0, T; L^2(\mathbb{R}^3)^3), \quad \text{weakly star in } L^\infty(0, T; W) \tag{3.34}$$

$$M_m \to M \quad \text{weakly in } H^1(0, T; V) \tag{3.35}$$

$$\hat{G}_{1m} \to G_1 \quad \text{weakly star in } L^\infty(\Omega x]0, T[)^3 \tag{3.36}$$

$$\hat{G}_{2m} \to G_2 \quad \text{weakly star in } L^\infty(0, T; L^2(\Omega)^3). \tag{3.37}$$

As the inclusion $H^1(0, T, L^2(\mathbb{R}^3)^3) \cap L^\infty(0, T; V) \subset L^2(\Omega; C^0([0, T])^3)$ is compact, we have

$$M_m(x, \cdot) \to M(x, \cdot) \quad \text{strongly in } C^0([0, T])^3, \text{ a.e. in } \Omega,$$

whence by (3.13)

$$\mathcal{G}_1(M_m(x, \cdot), H_1^0(x)) \to \mathcal{G}_1(M(x, \cdot), H_1^0(x)) \quad \text{strongly in } C^0([0, T])^3, \text{ a.e. in } \Omega,$$

then also

$$\hat{G}_{1m}(x, \cdot) \to \mathcal{G}_1(M(x, \cdot), H_1^0(x)) \quad \text{strongly in } C^0([0, T])^3, \text{ a.e. in } \Omega$$

and by (3.36)

$$G_1(x, t) = [\mathcal{G}_1(M(x, \cdot), H_1^0(x))](t) \quad \forall t \in [0, T], \text{ a.e. in } \Omega. \tag{3.38}$$

For any $v \in H^1(0, T; V)$, setting $\hat{v}(t) := v(nk)$ for $(n-1)k < t \le nk$, we have

$$0 \le k \sum_{n=1}^{m} {}_{V'}\langle \mathcal{G}_2(M_m(nk)) - \mathcal{G}_2(v(nk)), M_m(nk) - v(nk)\rangle_V$$

$$= \int_0^T {}_{V'}\langle \hat{G}_{2m} - \mathcal{G}_2(\hat{v}), \hat{M}_m - \hat{v}\rangle_V \, dt \to \int_0^T {}_{V'}\langle G_2 - \mathcal{G}_2(v), M - v\rangle_V \, dt,$$

whence as $\mathcal{G}_2$ is maximal monotone

$$G_2 = \mathcal{G}_2(M) \quad \text{in } V', \forall t \in [0, T]. \square \tag{3.39}$$

Now we introduce a second weak formulation corresponding to Maxwell's equations with no displacement current term, coupled with (2.15). Let $(V', R, \mathcal{F})$ be a distributed memory function in the sense of (2.1), . . . , (2.4), and let (3.5) hold.

(P2) Find $H \in C^0([0, T]; V') \cap L^2(0, T; W)$ with

$$H(0) = H^0 \qquad \text{in } V' \tag{3.40}$$

such that, setting

$$M(t) := [\mathcal{F}(H, M^0)](t) \qquad \text{in } V, \forall t \in [0, T], \tag{3.41}$$

then

$$\frac{\partial}{\partial t}(H + M) + \nabla \mathrm{x} \nabla \mathrm{x} H = f \qquad \text{in } W', \text{ a.e. in } [0, T]. \tag{3.42}$$

See remark 1.

THEOREM 2. Assume that (3.10) holds, that $(V', R, \mathcal{F})$ is a piecewise monotone distributed memory functional in the sense of (2.1), . . . , (2.4), (2.7) and that moreover

$$\forall (v, \xi) \in \mathrm{Dom}(\mathcal{F}), \| \mathcal{F}(v, \xi) \|_{L^\infty(Q)^3} \leq C \qquad \text{(positive constant)} \tag{3.43}$$

$$\left\{ \begin{array}{l} \forall \{(v_n, \xi) \in \mathrm{Dom}(\mathcal{F})\}_{n \in \mathbb{N}}, \text{if } v_n \to v \text{ weakly star in } H^1(0, T; L^2(\mathbb{R}^3)^3) \cap L^\infty(0, T; W), \\ \text{then } (v, \xi) \in \mathrm{Dom}(\mathcal{F}) \text{ and } \mathcal{F}(v_n, \xi) \to \mathcal{F}(v, \xi) \text{ strongly in } C^0([0, T]; V). \end{array} \right. \tag{3.44}$$

Then problem (P2) has at least one solution such that moreover

$$H \in H^1(0, T; L^2(\mathbb{R}^3)^3) \cap L^\infty(0, T; W); M \in L^\infty(Q)^3 \cap C^0([0, T]; V). \tag{3.45}$$

Proof.

(i) *Approximation.*

Let $m \in \mathbb{N}$, $k := \dfrac{T}{m}$

(P2)$_m$ Find $H^n_m \in W$ ($n = 1, \ldots, m$) such that, setting

$$H_m := \text{linear interpolate in } [0, T] \text{ of } H_m(nk) := H^n_m \; (H^0_m \equiv H^0) \tag{3.46}$$

$$M^n_m := [\mathcal{F}(H_m, M^0)](nk) \qquad \text{in } V', n = 0, \ldots, m, \tag{3.47}$$

then

$$\frac{1}{k}(H^n_m - H^{n-1}_m + M^n_m - M^{n-1}_m) + \nabla \mathrm{x} \nabla \mathrm{x} H^n_m = f^n_m \quad \text{in } W', n = 1, \ldots, m, \tag{3.48}$$

where f^n_m is as above. This problem has one and only one solution, which can be evaluated step by step as for $(\mathrm{P1})_m$; also in this case a standard numerical method can be used, once an approximation procedure for $\mathcal{F}$ has been provided.

(ii) *Estimates*

As before, we multiply (3.48) by $H^n_m - H^{n-1}_m$ and sum for $n = 1, \ldots, l$, for a generic $l \in \{1, \ldots, m\}$; as $\mathcal{F}$ is piecewise monotone we have

$$\frac{1}{k} \sum_{n=1} \int_\Omega (M^n_m - M^{n-1}_m) \cdot (H^n_m - H^{n-1}_m) dx \geq 0;$$

by a stanard procedure we get (using the same notations as in the previous proof)

$$\| H_m \|_{H^1(0,T;L^2(\mathbb{R}^3)^3) \cap L^\infty(0,T;W)} \leq C; \tag{3.49}$$

moreover by (3.43)

$$\| M_m \|_{L^\infty(Q)^3} \leq C. \tag{3.50}$$

(iii) *Limit*
By the previous a priori estimates there exist H, M such that, possibly taking subsequences,

$$H_m \rightarrow H \quad \text{weakly in } H^1(0, T; L^2(\mathbb{R}^3)^3) \text{ and weakly star in } L^\infty(0, T, W) \tag{3.51}$$

$$M_m \rightarrow M \quad \text{weakly star in } L^\infty(Q)^3; \tag{3.52}$$

hence by (3.44)

$$\mathcal{F}(H_m, M^0) \rightarrow \mathcal{F}(H, M^0) \quad \text{strongly in } C^0([0, T]; V), \tag{3.53}$$

then by (3.47) we get (3.41). Taking $m \rightarrow \infty$ in the approximate equation we obtain (3.42).□

Now we consider the problem obtained coupling the equations of magnetostatics with the constitutive relation (1.18). We assume that

$$(M^0, H^0) \in S, H^0 \in W', X \in L^2(\mathbb{R}^3)^3. \tag{3.54}$$

(P3) Find $M \in C^0([0, T]; V)$ with $M(t) \in \text{Dom}(S)\ \forall t \in [0, T]$,

$$M(0) = M^0 \qquad \text{a.e. in } \Omega, \tag{3.55}$$

such that, setting

$$H(t) := [\mathcal{G}(M, H^0)](t) \qquad \text{in } V', \forall t \in [0, T], \tag{3.56}$$

then $H \in L^2(\mathbb{R}^3 \text{x}]0, T[)$ and

$$\nabla \text{x}(H - X) = 0 \qquad \text{in } W', \text{ a.e. in }]0, T[\tag{3.57}$$

$$\nabla \cdot (H + M) = 0 \qquad \text{in } H^{-1}(\mathbb{R}^3), \text{ a.e. in }]0, T[. \tag{3.58}$$

THEOREM 3. Assume that (3.11), . . . , (3.19), (3.54) hold and that

$$X \in H^1(0, T; L^2(\mathbb{R}^3)). \tag{3.59}$$

Then problem (P3) has at least one solution such that moreover

$$M \in H^1(0, T; W), H \in H^1(0, T; L^2(\mathbb{R}^3)). \tag{3.60}$$

Proof. (i) Approximation. Let $m \in \mathbb{N}$, $k = \dfrac{T}{m}$.

(P3)$_m$ Find $M_m^n \in V$ $(n = 1, \ldots, m)$ such that, setting (3.21) and (3.22), then $H_m^n \in L^2(\mathbb{R}^3)^3$ and

$$\nabla \text{x}(H_m^n - X_m^n) = 0 \qquad \text{in } W' \tag{3.61}$$

$$\nabla \cdot (H_m^n + M_m^n) = 0 \qquad \text{in } H^{-1}(\mathbb{R}^3), \tag{3.62}$$

where $X_m^n := \int_{(n-1)nk}^{nk} X(t)dt$, $(n = 1, \ldots, m)$.

(3.61) corresponds to the existence of $\psi_m^n: \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla\psi_m^n = H_m^n - X_m^n$ in $\mathcal{D}'(\mathbb{R}^3)^3$. As for $(P1)_m$, we have $H_m^n = \phi_m^n(M_m^n)$hence (3.62) can be rewritten in the form

$$\nabla \cdot [\nabla\psi_m^n + X_m^n + (\phi_m^n)^{-1}(\nabla\psi_m^n + X_m^n)] = 0 \quad \text{in } H^{-1}(\mathbb{R}^3) \tag{3.63}$$

or also, if $\partial\Phi_m^n = \phi_m^n$,

$$\int_\Omega (\nabla\psi_m^n + X_m^n)(v - \nabla\psi_m^n)dx + \int_\Omega [\Phi_m^n(v) - \Phi_m^n(\nabla\psi_m^n + X_m^n)]dx \geqq 0 \quad \forall\, v \in L^2(\mathbb{R}^3)^3. \tag{3.64}$$

This variational inequality has one and only one solution $\psi_m^n \in H^1(\mathbb{R}^3)$; this can be also numerically approximated by standard techniques, once an approximation of $\mathcal{G}$ has been provided. (3.61) and (3.62) yield

$$\int_{\mathbb{R}^3} (H_m^n - H_m^{n-1} - X_m^n + X_m^{n-1}) \cdot (H_m^n - H_m^{n-1})dx + {}_{V'}\langle H_m^n - H_m^{n-1} - X_m^n + X_m^{n-1}, M_m^n - M_m^{n-1} \rangle_V = 0 \tag{3.65}$$

whence by (3.24), using notations introduced above

$$\| H_m \|_{H^1(0,T;L^2(\mathbb{R}^3)^3)} \leq C \text{ (constant independent of } m) \tag{3.66}$$

$$\| M_m \|_{H^1(0,T;W)} \leq C. \tag{3.67}$$

These estimates allow to take the limit in (3.22) as in the proof of theorem 1.□

4. FAST EVOLUTION

If the evolution is fast, the relaxation dynamics of the elementary magnets cannot be neglected and accordingly a dissipative term has to be inserted into the macroscopic constitutive relation. This allows to study Maxwell's equations taking account also of the displacement current term.

We shall denote the characteristic function of Ω (equal to 1 in Ω and vanishing outside) by χ_Ω; it represents a normalized electric conductivity. Let $(V, S, \mathcal{G})$ be a (distributed) memory functional, in the sense of (2.1), . . . , (2.4); let

$$E^0, H^0 \in L^2(\mathbb{R}^3)^3, (M^0, H^0) \in S, f \in L^2(\mathbb{R}^3 x]0, T[)^3. \tag{4.1}$$

We introduce a weak problem:

(P4) Find $E \in L^2(\mathbb{R}^3 x]0, T[)^3$ and $M \in C^0([0, T]; V) \cap H^1(0, T; V')$ with $M(t) \in \mathrm{Dom}(S)\ \forall t \in [0, T]$ and

$$M(0) = M^0 \quad \text{a.e. in } \Omega, \tag{4.2}$$

such that, setting

$$H: = \mathcal{G}(M, H^0) + \frac{\partial M}{\partial t} \quad \text{in } W', \text{ a.e. in }]0, T[, \tag{4.3}$$

then $H \in L^2(\mathbb{R}^3 x]0, T[)^3$ and

$$\nabla x H = \frac{\partial E}{\partial t} + \chi_\Omega(E + f) \qquad \text{in } W', \text{ a.e. in }]0, T[\tag{4.4}$$

$$\nabla x E = -\frac{\partial}{\partial t}(H + M) \qquad \text{in } W', \text{ a.e. in }]0, T[\tag{4.5}$$

$$E(0) = E^0 \qquad \text{in } W' \tag{4.6}$$

$$(H + M)|_{t=0} = H^0 + M^0 \qquad \text{in } W'. \tag{4.7}$$

Remark. (4.4) and (4.5) yield $E, H + M \in H^1(0, T; W')$ and this gives a meaning to (4.6) and (4.7).

THEOREM 4. Assume that (4.1), (3.11), . . . , (3.19) hold and that

$$\begin{cases} \mathcal{G}_2 = \partial\psi, \text{ where } \psi: V \to \mathbb{R} \cup \{+\infty\} \text{ is convex and lower semi-continuous} \\ (\partial \text{ denoting the subdifferential}). \end{cases} \tag{4.8}$$

Then problem (P4) has at least one solution such that moreover

$$M \in H^1(0, T; L^2(\mathbb{R}^3)^3) \cap L^\infty(0, T; V). \tag{4.9}$$

Proof.

(i) *Approximation.*

Let $m \in \mathbb{N}$, $k: = \dfrac{T}{m}$.

(P4)$_m$ Find $E^n_m \in W$ and $M^n_m \in V \cap W (n = 1, \ldots, m)$ such that, setting

$$M_m: = \text{linear interpolate of } M_m(nk): = M^n_m \text{ in } [0, T] \; (M^0_m: = M^0) \tag{4.10}$$

$$H^n_m: = [\mathcal{G}(M_m, H^0)](nk) + \frac{1}{k}(M^n_m - M^{n-1}_m) \text{ in } V', \; n = 1, \ldots, m, \tag{4.11}$$

then $H^n_m \in W \; \forall n$ and (setting $H^0_m: = H^0$ and $E^0_m: = E^0$)

$$\nabla x H^n_m = \frac{1}{k}(E^n_m - E^{n-1}_m) + \chi_\Omega(E^n_m + f^n_m) \qquad \text{a.e. in } \mathbb{R}^3, \; n = 1, \ldots, m, \tag{4.12}$$

$$\nabla x E^n_m = -\frac{1}{k}(H^n_m - H^{n-1}_m + M^n_m - M^{n-1}_m) \qquad \text{a.e. in } \mathbb{R}^3, \; n = 1, \ldots, m, \tag{4.13}$$

where $f^n_m(x): = 1/k \int_{(n-1)k}^{nk} f(x, t)dt$ a.e. in Ω. As for (P1)$_m$, this problem can be solved step by step; at every step it is equivalent to a standard minimization problem and has one and only one solution, which can be numerically approximated by standard methods once an approximation of $\mathcal{G}$ has been provided.

(ii) *Estimates.*

We multiply (4.12) by E^n_m, (4.13) by $-kH^n_m$ and sum for $n = 1, \ldots, l$, for a generic $l \in$

$\{1, \ldots, m\}$. Note that

$$k \sum_{n=1}^{l} {}_{V'}\langle \frac{M_m^n - M_m^{n-1}}{k}, H_m^n \rangle_V = k \sum_{n=1}^{l} \int_\Omega \frac{M_m^n - M_m^{n-1}}{k} \cdot [\mathcal{G}_1(M_m, H^0)](nk)dx$$
$$+ {}_V\langle \frac{M_m^n - M_m^{n-1}}{k}, \mathcal{G}_2(M_m^n) \rangle_{V'} + \left\| \frac{M_m^n - M_m^{n-1}}{k} \right\|^2_{L^2(\Omega)^3} \geq$$
$$-C_1 \left(k \sum_{n=1}^{l} \left\| \frac{M_m^n - M_m^{n-1}}{k} \right\|^2_{L^2(\Omega)^3} \right)^{1/2} + \psi(M_m^l) - \psi(M_m^0)$$
$$+ k \sum_{n=1}^{l} \left\| \frac{M_m^n - M_m^{n-1}}{k} \right\|^2_{L^2(\Omega)^3};$$

$$\int_{\mathbb{R}^3} (\nabla \mathrm{x} H_m^n \cdot E_m^n - \nabla \mathrm{x} E_m^n \cdot H_m^n)dx = \int_{\mathbb{R}^3} \nabla \cdot (H_m^n \times E_m^n)dx = 0.$$

Therefore by a standard procedure, using the notations introduces in the proof of theorem 1 and defining E_m, $\tilde{E}_m$ similarly to H_m, $\tilde{H}_m$, we get

$$\| E_m \|_{L^\infty(0,T;L^2(\mathbb{R}^3)^3)} \leq C \text{ (constant independent of } m) \tag{4.14}$$

$$\| H_m \|_{L^\infty(0,T;L^2(\mathbb{R}^3)^3)} \leq C \tag{4.15}$$

$$\| M_m \|_{H^1(0,T;L^2(\Omega)^3) \cap L^\infty(0,T;V)} \leq C; \tag{4.16}$$

moreover as in the proof of theorem 1 we get (3.32) and (3.33).

(iii) *Limit*
By the previous a priori estimates there exist E, H, M, G_1 and G_2 such that, possibly taking subsequences,

$$E_m \to E \quad \text{weakly in } L^\infty(0, T; L^2(\mathbb{R}^3)^3) \tag{4.17}$$

$$H_m \to H \quad \text{weakly in } L^\infty(0, T; L^2(\mathbb{R}^3)^3) \tag{4.18}$$

$$M_m \to M \quad \text{weakly in } H^1(0, T; L^2(\mathbb{R}^3)^3), \text{ weakly star in } L^\infty(0, T; V) \tag{4.19}$$

$$\hat{G}_{1m} \to G_1 \quad \text{weakly star in } L^\infty(Q)^3 \tag{4.20}$$

$$\hat{G}_{2m} \to G_2 \quad \text{weakly star in } L^\infty(0, T; L^2(\Omega)^3). \tag{4.20}$$

Taking $m \to \infty$ in (4.12) and (4.13) we get (4.4) and (4.5); (4.3) can be proved as for (P1).□

Finally we notice that Theorems 1 and 3 hold also if in (P1) and (P3) (3.7) and (3.57) are replaced by (4.3); moreover the presence of the dissipative term $\frac{\partial M}{\partial t}$ yields further regularity for M.

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