On the Existence of a Solution in a Domain Identification Problem

DENISE CHENAIS

Institut de Mathématiques et Sciences Physiques, Université de Nice,
06034 Nice, France

Submitted by J. L. Lions

NOTATIONS

$x \in \mathbb{R}^n$: $x = (x^1, \ldots, x^n)$.

For $\varphi: \mathbb{R}^n \to \mathbb{R}$, $j \in \{1, \ldots, n\}$, $D^j \varphi = \partial \varphi / \partial x^j$.

For $\varphi: \mathbb{R}^n \to \mathbb{R}$,

$$\alpha = (\alpha_1, \ldots, \alpha_n); D^\alpha \varphi = \frac{\partial^{\alpha_1}}{(\partial x^1)^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{(\partial x^n)^{\alpha_n}} \varphi.$$  

$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

$\Omega \subset \mathbb{R}^n$: $\varphi |_\Omega$ = restriction of $\varphi$ to $\Omega$.

$\Omega \subset \mathbb{R}^n$, $\Omega$ open set, $m \in \mathbb{N}$:

$$H^m(\Omega) = \{\varphi \in L^2(\Omega); D^\alpha \varphi \in L^2(\Omega), |\alpha| \leq m\}.$$  

For $\varphi, \psi \in L^2(\Omega)$:

$$(\varphi, \psi)_{L^2(\Omega)} = \int_{\Omega} \varphi(x) \psi(x) \, dx.$$  

For $\varphi, \psi \in H^m(\Omega)$:

$$((\varphi, \psi))_{H^m(\Omega)} = \sum_{|\alpha| \leq m} (D^\alpha \varphi, D^\alpha \psi)_{L^2(\Omega)}.$$  

$\gamma_\Omega$: solution of the Neuman problem in $\Omega$.

$A$ and $B$ being two subsets of $\mathbb{R}^n$:

$$A \setminus B = A \cap (\overline{B}).$$  

$\partial \Omega$ is the boundary of $\Omega$,

$\hat{\Omega}$ is the interior of $\Omega$,

$\overline{\Omega}$ is the closure of $\Omega$.
\( \chi_\Omega = \omega \) is the characteristic function of \( \Omega \).

\( B(x, r) \): open ball with radius \( r \) and center \( x \).

\( \mu \) is the Lebesgue measure on \( \mathbb{R}^n \).

For \( \theta \in [0, \pi/2[, \ h > 0, \ \xi \in \mathbb{R}^n(\|\xi\| = 1) \):

\[
C(\xi, \theta, h) = \{x \in \mathbb{R}^n; \|x\| < h, (x, \xi) > \|x\| \cos \theta \}.
\]

If \( r \) is given \((2r \leq h)\), \( \Omega \) satisfies the cone property if and only if:

\[
\forall x \in \partial \Omega, \quad \exists C_x = C(\xi_x, \theta, h)
\]

such that

\[
\forall y \in B(x, r) \cap \Omega, \quad y + C_x \subseteq \Omega.
\]

\( \Pi(\theta, h, r) \) is the set of open subsets of the given subset \( D \) of \( \mathbb{R}^n \) satisfying the cone property.

\( \Pi_0(\theta, h, r) \) is the set of elements of \( \Pi(\theta, h, r) \) containing the given subset \( \Omega_0 \) of \( \mathbb{R}^n \).

I. INTRODUCTION

Let us consider, as an example, the following problem:

Let \( \Omega_0 \) and \( D \) be 2 bounded sets in \( \mathbb{R}^n \), such that \( \Omega_0 \subseteq D \), and let \( f \) be given in \( L^2(D) \) and \( y_d \) be given in \( L^2(\Omega_0) \). \( y_d \) is suspected to be the restriction to \( \Omega_0 \) of the solution in an unknown set \( \Omega(\Omega_0 \subseteq \Omega \subseteq D) \) of the Neuman problem:

\[
-A_y = f \quad \text{on} \quad \Omega,
\]

\[
\frac{\partial y}{\partial n} = 0 \quad \text{on} \quad \partial \Omega.
\]

The question is: knowing only \( \Omega_0, D, y_d \), is it possible to find \( \Omega \) such that:

\[ y_\Omega |_{\Omega_0} = y_d , \]

where \( y_\Omega |_{\Omega_0} \) denotes the restriction of \( y_\Omega \) to \( \Omega_0 \)?

This problem can also be formulated in the following way. For an open set \( \Omega(\Omega_0 \subseteq \Omega \subseteq D) \), we consider:

\[
J(\Omega) = \| y_\Omega |_{\Omega_0} - y_d \|_{L^2(\Omega_0)} .
\]

This defines a functional on a certain set \( I \) of subsets of \( \mathbb{R}^n \) which we shall define precisely later. The question now is to minimize \( J(\Omega) \) on \( I \).
This problem appears to be an example of optimal control problem where the control set is a set of domains in $\mathbb{R}^n$ and where the system is governed by a partial differential equation.

In this paper, we shall work on the existence of an optimal control for such a kind of problem. This will lead us to study properties of continuity of $J$ and compactness of the set of controls.

Therefore, the crux of our problem is to find a set of open subsets of $\mathbb{R}^n$ and to put on it a topology, so that we have in the same time compactness of $\Pi$ and continuity of the functional $J$ on $\Pi$.

In order to prove the continuity of $J$ in these problems, the following property for $\Pi$ appears to be of great help:

Let $m$ be a positive integer. There exists a positive constant $K$ such that:

$\forall \Omega \in \Pi$, there exists a linear, continuous extension operator $p_{\Omega}$ from $H^m(\Omega)$ to $H^m(\mathbb{R}^n)$ such that:

$$\|p_{\Omega}\| \leq K$$

($H^m(\Omega)$ denotes the usual Sobolev spaces).

This will be called the "uniform extension property."

We shall prove that a certain set $\Pi$ of open subsets of $\mathbb{R}^n$ satisfying this "uniform extension property" is compact for an appropriate topology. Then we shall use these results in the previous example. We shall prove that the functional $J$ defined above is continuous on the compact set $\Pi$, which means that there exists an optimal control in $\Pi$.

General definitions are given in this introduction.

The "uniform extension property" is studied in Section II.

The compactness property is studied in Section III.

The example is studied in Section IV.

**Definition 1.** Let $h > 0$ and $\theta \in [0, \pi/2]$ be 2 given numbers, and $\xi$ be a given element in $\mathbb{R}^n$ such that $\|\xi\| = 1$. We shall call cone of angle $\theta$, height $h$, and axis $\xi$ the set:

$$C(\xi, \theta, h) = \{x \in \mathbb{R}^n; (x, \xi) > \|x\| \cos \theta, \|x\| < h\}$$

($\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$).

**Definition 2.** Let $\theta \in ]0, \pi/2[$, $h > 0$, $r > 0$ ($2r \leq h$) be 3 given numbers. A subset $\Omega$ of $\mathbb{R}^n$ is said to satisfy the "cone property" if and only if:

$$\forall x \in \partial \Omega, \quad \exists C_x = C(\xi_x, \theta, h),$$
cone of angle \( \theta \) and height \( h \) such that:

\[
\forall y \in B(x, r) \cap \Omega, \quad y + C_x \subset \Omega
\]

(\( \partial \Omega \) denotes the boundary of \( \Omega \), \( B(x, r) \) denotes the open ball of radius \( r \) and center \( x \) in \( \mathbb{R}^n \)).

**Definition 3.** Let \( D \) be a given bounded open set in \( \mathbb{R}^n \). \( \Pi(\theta, h, r) \) is the set of all the open subsets of \( D \) satisfying the cone property of definition 2.

**Remark 1.** The cone property of definition 2 is the "restricted cone property" given by Agmon in [1]. Yet, we can make the following remark. The definition given by Agmon is:

Let \( \Omega \) be an open set in \( \mathbb{R}^n \). \( \Omega \) satisfies the "restricted cone property" if \( \partial \Omega \) has a locally finite open covering \( (O_i) \) and corresponding cones \( (C_i) \) such that:

\[
\forall x \in O_i \cap \Omega: x + C_i \subset \Omega.
\]

Agmon does not a priori prescribe the height and angle of the cone, and a special type for the sets \( O_i \). Nevertheless, for a given bounded open set, it can be proved that it is possible to find \( \theta, h \) and \( r \) such that both definitions are equivalent.

What makes the distinction important here is that we work on an infinite set of domains.

We require \( \theta, h \) and \( r \) to be uniform for all the domains.

**Topology on** \( \Pi(\theta, h, r) \). The topology we shall use on \( \Pi(\theta, h, r) \) is the strong \( L^2(D) \) topology of the characteristic functions in \( D \) of the elements of \( \Pi(\theta, h, r) \). It means that we shall say that \( \Omega_1 \) converges to \( \Omega_2 \) in \( \Pi(\theta, h, r) \) if and only if:

\[
\omega_1 \rightharpoonup \omega_2 \quad \text{in} \quad L^2(D) \text{ (strong topology)},
\]

where \( \omega_1 \) and \( \omega_2 \) denote the characteristic functions of \( \Omega_1 \) and \( \Omega_2 \) in \( D \).

**Remark 2.** This is more precisely a topology on the set of equivalence classes for the relation:

\[
\omega_1 \sim \omega_2 \iff \omega_1 = \omega_2 \text{ a.e.}
\]

Nevertheless, it is possible to prove that there is at most one element of \( \Pi(\theta, h, r) \) in each equivalence class.

In the following pages, \( U(\theta, h, r) \) denotes the set of the characteristic functions in \( L^2(D) \) of the elements of \( \Pi(\theta, h, r) \). \( \omega \) denotes the characteristic function of \( \Omega \).
We shall prove in the next sections that if \( \theta, h \) and \( r \) are fixed, then \( n(\theta, h, r) \) satisfies the "uniform extension property" and is a compact set for the strong \( L^2(D) \) topology on \( U(\theta, h, r) \).

We shall deduce from that an existence result for the problem stated at the beginning of the paper.

II. The Uniform Extension Property

Let \( \theta, h, r \) be 3 fixed numbers (\( 0 < \theta < \pi/2 \), \( h > 0 \), \( 2r < h \)).

We prove in this section that \( n(\theta, h, r) \) satisfies the "uniform extension property", i.e.,

\[
\forall m \in \mathbb{N}, \quad \exists K > 0
\]

such that:

\[
\forall \Omega \in \Pi(\theta, h, r), \quad \exists p_\Omega: H^m(\Omega) \to H^m(\mathbb{R}^n),
\]

linear and continuous extension operator, such that:

\[
\| p_\Omega \| \leq K.
\]

We shall use for that Calderon's extension theorem which can be found in Agmon [1]. Agmon proves that if a domain \( \Omega \) satisfies a cone property, then there exists a linear continuous extension operator from \( H^m(\Omega) \) to \( H^m(\mathbb{R}^n) \). We shall refine this proof to show that the norm of such an operator depends on the domain only through \( \theta, h \) and \( r \).

Stein also builds an extension operator in [5], which is different from Calderon's.

We first prove a local result, then a global one using a partition of unity.

I. Local Result.

**Proposition II.1.** Let \( \Omega \) be an open set in \( \mathbb{R}^n \) whose boundary contains the origin, and such that there exists a cone \( C \) with axis \( \xi \), angle \( \theta \), height \( h \) such that:

\[
\forall x \in B(0, r) \cap \Omega, \quad x + C \subset \Omega.
\]

Moreover, let \( u \) be an element of \( C^m(\Omega) \cap H^m(\Omega) \) (\( m \in \mathbb{N} \)) whose support is included in the ball \( B(0, r/2) \). Then, \( u \) can be extended to an element \( v \in H^m(\mathbb{R}^n) \). Moreover, there exists a constant \( K \) such that:

\[
\| v \|_{H^m(\mathbb{R}^n)} \leq K \| u \|_{H^m(\Omega)},
\]

and where \( K \) depends on \( \Omega \) through \( \theta, h \) and \( r \) only.
Proof.

Notations. \( B = B(0, r) \);
\( B' = B(0, r/2) \);
\( \Sigma = \{ x \in \mathbb{R}^n; \| x \| = 1 \}; \)
\( \Gamma = \{ x \in \mathbb{R}^n; (x, \xi) > \| x \| \cos \theta \}. \)

For simplicity's sake, we prove the result for \( m = 1 \). It can be proved in the same way for \( m > 1 \). A few remarks about it can be found at the end of the paragraph.

Let us consider an element \( \varphi \) of \( \mathcal{C}^\infty(\Omega) \) the support of which is contained in \( \Sigma \cap \Gamma \) and such that:

\[
\int_{\Sigma} \varphi(\sigma) \, d\sigma = -1.
\]

Then, for each \( j \) belonging to \( \{1, \ldots, n\} \), and \( y \in \mathbb{R}^n \) (\( y \neq 0 \)), we set:

\[
\phi_j(y) = \frac{y^j}{|y|^n} \varphi\left(\frac{y}{|y|}\right) \quad (y^j \text{ is the } j\text{th component of } y).
\]

Each of these functions is positively homogeneous of degree \( 1 - n \) with support contained in \( \Gamma \). From Sobolev representation formula (cf. [1]), we have:

\[
\forall x \in \Omega \setminus B: u(x) = \sum_{j=1}^{m} \int_{C} \phi_j(y) \, D^j u(x + y) \, dy
\]

(notice that this formula holds in \((B \setminus B') \cap \Omega \) where \( u = 0 \)).

Let us then set:

\[
\omega^j(y) = \begin{cases} D^j u(y), & \text{for } y \in D, \\ 0, & \text{for } y \notin D. \end{cases}
\]

As \( \Omega \) satisfies the cone property at the origin, if \( x \in \Omega \cap B \), then for each \( y \) in \( C \), \( x + y \) belongs to \( \Omega \). So:

\[
\forall x \in \Omega \cap B, \quad \forall y \in C, \quad \omega^j(x + y) = D^j u(x + y),
\]

and:

\[
\forall x \in \Omega \cap B: u(x) = \sum_{j=1}^{m} \int_{C} \phi_j(y) \, \omega^j(x + y) \, dy.
\]

Let us now consider a function \( \psi \in \mathcal{C}^\infty(\mathbb{R}^n) \) such that:

\[
\psi(x) = \begin{cases} 1, & \text{for } x \in B', \\ 0, & \text{for } x \notin B, \end{cases}
\]

\( 0 \leq \psi(x) \leq 1, \quad \forall x \in B \setminus B' \).
We define:
\[ v(x) = \psi(x) \sum_{j=1}^{m} \int_{C} \phi_j(y) w(x + y) \, dy, \quad \forall x \in \mathbb{R}^n. \]

We shall prove that this function satisfies the required conditions: It is obviously an extension of \( u \); we only have to prove that it belongs to \( H^1(\mathbb{R}^n) \) and to estimate its norm in \( H^1(\mathbb{R}^n) \).

(a) Estimation of \( \| v \|_{L^2(\mathbb{R}^n)} \). Considering that:
\[ |\psi(x)| \leq 1, \quad \forall x \in \mathbb{R}^n, \]
and:
\[ |\phi_j(y)| = |y|^{-n} |\phi_j(\sigma)| \leq |y|^{-n} \max_{\sigma \in \Sigma} |\phi_j(\sigma)| = K(\theta) |y|^{-n}, \]
where \( y = |y| \sigma \), we have:
\[ \| v \|_{L^2(\mathbb{R}^n)} \leq K(\theta) \left[ \int_{\mathbb{B}} \left( \sum_{j=1}^{m} \int_{C} |y|^{-n} |w(x + y)| \, dy \right)^2 \, dx \right]^{1/2} \]
\[ \leq K(\theta) \sum_{j=1}^{m} \left\| \int_{C} |y|^{-n} |w(x + y)| \, dy \right\|_{L^1(\mathbb{B})} \]
\[ \leq K(\theta) \sum_{j=1}^{m} \int_{C} \left\| |y|^{-n} |w(x + y)| \right\|_{L^2(\mathbb{B})} \, dy \]
\[ \leq K(\theta) \sum_{j=1}^{m} \int_{C} |y|^{-n} \left( \int_{\mathbb{B}} |w(x + y)|^2 \, dx \right)^{1/2} \, dy. \]

Now, for each \( y \):
\[ \int_{\mathbb{B}} |w(x + y)|^2 \, dx \leq \| D^1 u \|^2_{L^2(\mathbb{G})} \]
so:
\[ \| v \|_{L^2(\mathbb{R}^n)} \leq K(\theta) \sum_{j=1}^{m} \| D^1 u \|_{L^2(\mathbb{G})} \int_{C} |y|^{-n} \, dy \]
and as the integral \( \int_{C} |y|^{-n} \, dy \) is convergent:
\[ \| v \|_{L^2(\mathbb{R}^n)} \leq K(\theta, \mathcal{H}) \sum_{j=1}^{m} \| D^1 u \|_{L^2(\mathbb{G})}. \]
(b) Estimation of \( \| D^i v \|_{L^2(\mathbb{R}^n)} \). Let us set:
\[
W^j(x) = \int_C \phi_j(y) \psi^j(x + y) \, dy
\]
so that:
\[
v(x) = \sum_{j=1}^m \psi(x) W^j(x).
\]

Let us take the weak derivatives of \( v \):
\[
D^k v = D^k \psi \sum_{j=1}^m W^j + \psi \sum_{j=1}^m D^k W^j.
\]

First, as \( \psi \in C^\infty(\mathbb{R}^n) \), we know that:
\[
\sup_{x \in \mathbb{R}^n} |D^k \psi(x)| \leq K(r),
\]
and from the same arguments as in (a):
\[
\| T_1 \|_{L^1(\mathbb{R}^n)} \leq K(\theta, h, r) \sum_{i=1}^m \| D^i u \|_{L^2(\Omega)}.
\]

Let us now work on \( T_2 \). We notice that for each \( x \in \mathbb{R}^n \), we have:
\[
\psi(x) \int_C \phi_j(y) \psi^j(x + y) \, dy = \psi(x) \int_{\mathbb{R}^n} \phi_j(y) \psi^j(x + y) \, dy
\]
then, if we set \( \phi_j'(y) = \phi_j(-y) \), we get:
\[
T_2 = \psi \sum_{j=1}^m D^k (\phi_j' \ast \psi^j)
\]

(* denotes the usual convolution).

As \( \phi_j' \) is positively homogeneous of degree \( 1 - n \) and as \( \psi^j \) is bounded, we can use the Calderon–Zygmund theorem (cf. Agmon [1]). We know that
\[
D^k (\phi_j' \ast \psi^j)
\]
belongs to \( L^2(\mathbb{R}^n) \) and that there exists a constant \( K(\theta) \) such that:
\[
\| D^k (\phi_j' \ast \psi^j) \|_{L^2(\mathbb{R}^n)} \leq K(\theta) \| \psi^j \|_{L^2(\mathbb{R}^n)}
\]
hence:
\[
\| T_2 \|_{L^2(\mathbb{R}^n)} \leq K(\theta) \sum_{j=1}^m \| D^j u \|_{L^2(\Omega)}
\]
and:

$$\| D^{h}v \|_{L^2(\mathbb{R}^n)} \leq K(\theta, h, r) \sum_{j=1}^{m} \| D^{j}u \|_{L^2(\Omega)}.$$ 

Now, from (a) and (b) we have:

$$\| v \|_{H^1(\mathbb{R}^n)}^2 \leq K(\theta, h, r) \sum_{j=1}^{m} \| D^{j}u \|_{L^2(\Omega)}^2,$$

which is the expected result.

Remarks. To prove the result for $m > 1$, we should use a function $v \in H^m(\mathbb{R}^n)$ whose support is included in $\Sigma \cap \Gamma$ such that:

$$s = v(a) \, du = \frac{(-1)^m}{(m - 1)!},$$

for each multiindex $\alpha$ such that $|\alpha| = m$, and:

$$w_\alpha(y) = \begin{cases} D^\alpha u(y), & \text{for } y \in \Omega, \\ 0, & \text{for } y \notin \Omega, \end{cases}$$

$$v(x) = \psi(x) \sum_{|\alpha|=m} \int_C \phi_\alpha(y) w_\alpha(x+y) \, dy.$$ 

The estimation of $\| D^{h}v \|$ can be carried out as in (a) for $|\beta| < m$. Arguments of singular integrals used in (b) are necessary for $|\beta| = m$.

II. Global Result

To prove the desired extension theorem, for each $\Omega \in \Pi(\theta, h, r)$ we shall need a special type of partition of unity. We first prove the existence of these partitions of unity.

Proposition II.2. There exists an integer $N$ and a constant $M$ depending on $\Omega \in \Pi(\theta, h, r)$ through $r$ only, such that for each $\Omega \in \Pi(\theta, h, r)$, $\bar{\Omega}$ has an open covering $(B_i')_{i=0,...,\nu}$, where $\nu$ is smaller than $N$, and such that:

- $B_0' \subset \Omega$;
- for $i \geq 1$, $B_i' = B(x_i, r/2)$, $x_i \in \partial \Omega$;
- there exists $\nu + 1$ positive functions $(\zeta_t)_{t=0,...,\nu}$.
belonging to $\mathcal{C}^\infty(\mathbb{R}^n)$ the supports of which are, respectively, contained in $B'_i$, and such that for a given $m$ and for each multiindex $|\alpha| \leq m$:

$$\sup_{x \in \mathbb{R}^n} |D^\alpha \zeta_i(x)| \leq M, \quad \forall i = 0, \ldots, v.$$  

**Proof.** It can be proved that there is an integer $N$ and a constant $\epsilon$ depending on $r$ only, such that for each $\Omega$ in $\Pi(\theta, h, r)$, there exists an open covering $(B'_i, i = 0, \ldots, v)$ of $\Omega$ and 2 other coverings $(K_i)$ and $(\tilde{K}_i, i = 0, \ldots, v)$ of $\Omega$ by compact sets, such that:

$$\nu \leq N;$$
$$B'_0 \subset \Omega; \quad B'_i = B(x_i, r/2), \quad x_i \in \partial \Omega \text{ for } i \geq 1;$$
$$K_i \subset \tilde{K}_i \subset B'_i, \quad i = 0, \ldots, v;$$
$$d(K_i, \partial \tilde{K}_i) > \epsilon, \quad i = 0, \ldots, v;$$
$$d(\tilde{K}_i, \partial B'_i) > \epsilon, \quad i = 0, \ldots, v.$$

We now build a partition of unity by a classical method [1]: Let us consider a positive function $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ with support included in $B(0, \epsilon)$ and such that:

$$\int_{\mathbb{R}^n} \psi(x) \, dx = 1.$$  

Let us then denote by $\tilde{x}_i$ the characteristic function of $K_i$. We set:

$$\psi_i = \tilde{x}_i \ast \psi, \quad i = 0, \ldots, v.$$  

These functions belong to $\mathcal{C}^\infty(\mathbb{R}^n)$, their supports are, respectively, included in $B'_i$, and:

$$\psi_i(x) = 1, \quad \text{for } x \in K_i,$$
$$0 \leq \psi_i(x) \leq 1, \quad \text{for } x \in B'_i \setminus K_i.$$  

Moreover, their derivatives can be bounded above:

$$|D^\alpha \psi_i(x)| = \left| \int_{\mathbb{R}^n} \tilde{x}_i(y) D^\alpha \psi(x - y) \, dy \right|$$

$$\leq \int_{\mathbb{R}^n} |\tilde{x}_i(y)| \left| D^\alpha \psi(x - y) \right| \, dy$$

$$\leq \mu(D) \operatorname{Max}_{y \in \mathbb{R}^n} |D^\alpha \psi(y)|.$$  

This bound depends on $\Omega \in \Pi(\theta, h, r)$ only through $\epsilon$, therefore only through $r$.  

Now, if we set:

\[ \zeta_0 = \psi_0, \]
\[ \zeta_i = \psi_i \prod_{k=0}^{i-1} (1 - \psi_k), \quad i = 1, \ldots, \nu, \]

these functions fulfill the required conditions.

Let us now prove the uniform extension result.

**Theorem II.1.** Let \( \theta, h, r \) be 3 real numbers \((\theta \in ]0, \pi/2[, 2r \leq h)\) and \( m \) a positive integer. There exists a constant \( K(\theta, h, r) \) depending on \( \Omega \in \Pi(\theta, h, r) \) through \( \theta, h \) and \( r \) only, and such that:

\[ \forall \Omega \subset \Pi(\theta, h, r), \quad \exists p_\Omega: H^m(\Omega) \rightarrow H^m(\mathbb{R}^n), \]

linear and continuous extension operator, such that:

\[ \| p_\Omega \| \leq K(\theta, h, r). \]

**Proof.** For an element \( \Omega \) of \( \Pi(\theta, h, r) \), we first consider a covering \( (B_i', i = 0, \ldots, \nu) \) of \( \overline{\Omega} \) fulfilling the conditions of Proposition II.2 and the associated partition of unity \( (\zeta_i, i = 0, \ldots, \nu) \), that is:

\[ B_0' \subset \Omega, \]
\[ B_i' = B(x_i, r/2), \quad x_i \in \partial \Omega, \quad i = 1, \ldots, \nu; \]
\[ \nu \leq N \quad (N \text{ independent of } \Omega \in \Pi(\theta, h, r)); \]
\[ \zeta_i \in \mathcal{C}^\infty(\mathbb{R}^n) \text{ with support included in } B_i'; \]

\[ \sum_{i=0}^{\nu} \zeta_i(x) = 1, \quad \forall x \in \Omega; \]
\[ \sup_{x \in \mathbb{R}^n} |D^\alpha \zeta_i(x)| \leq M, \quad \forall \ |\alpha| \leq m, \quad i = 0, \ldots, \nu. \]

We shall use the following notations:

\( \xi \) is an element of \( \mathbb{R}^n \) with norm 1, \( C \) is the cone with axis \( \xi \), angle \( \theta \), height \( h \);
\( \xi_i \) is the axis of the cone \( C_i \) such that:

\[ \forall y \in B(x_i, r) \cap \Omega: y + C_i \subset \Omega \quad (i = 1, \ldots, \nu). \]

Moreover, for each index \( i \in \{1, \ldots, \nu\} \):

\( A_i \) is the rotation operator in \( \mathbb{R}^n \) such that \( A_i \xi = \xi_i \);
\( T_i \) is the operator from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) defined by \( T_i x = A_i x + x_i \);

\( \overline{\Omega}_i = T_i^{-1}(\Omega). \)
We notice that $\tilde{Q}_i$ obviously satisfies:

$$\forall y \in \tilde{Q}_i \cap B(0, r): y + C \subset \tilde{Q}_i.$$ 

And if $u \in H^1(\Omega)$, then $\tilde{u} = u \circ T_i$ belongs to $H^1(\tilde{Q}_i)$ and:

$$\| \tilde{u} \|_{H^1(\tilde{Q}_i)} = \| u \|_{H^1(\Omega)}.$$ 

Let us now consider an element $u \in C^m(\Omega) \cap H^m(\Omega)$ that we want to extend to an element $v \in H^m(\mathbb{R}^n)$.

We set:

$$u_i(x) = u(x) \cdot \xi_i(x), \quad \forall x \in \Omega,$$

and we extend each $u_i$.

For $i = 0$, $u_0$ is trivially extended by $v_0$ defined by:

$$v(x) = \begin{cases} u_0(x), & \forall x \in \Omega, \\
0, & \forall x \notin \Omega. \end{cases}$$

$v_0$ belongs to $H^m(\mathbb{R}^n)$ and

$$\| v_0 \|_{H^m(\mathbb{R}^n)} = \| u_0 \|_{H^m(\Omega)}.$$ 

For $i \geq 1$, we notice that the couple $(\tilde{Q}_i, \tilde{u}_i = u_i \circ T_i)$ fulfills the conditions of the local extension theorem (Proposition 11.1). Therefore, there exists an element $\tilde{\varepsilon}_i \in H^m(\mathbb{R}^n)$ which is an extension of $\tilde{u}_i$ and such that:

$$\| \tilde{\varepsilon}_i \|_{H^m(\mathbb{R}^n)} \leq K(\theta, h, r) \| \tilde{u}_i \|_{H^m(\tilde{Q}_i)}.$$ 

Now, the function $v_i = \tilde{\varepsilon}_i \circ T_i^{-1}$ also belongs to $H^m(\mathbb{R}^n)$ and it is an extension of $u_i$. Moreover:

$$\| v_i \|_{H^m(\mathbb{R}^n)} = \| \tilde{\varepsilon}_i \|_{H^m(\mathbb{R}^n)} \leq K(\theta, h, r) \| u_i \|_{H^m(\Omega)}.$$ 

From that, we derive that $v = \sum_{i=0}^{\nu} v_i$ is an extension of $u$ and that:

$$\| v \|_{H^m(\mathbb{R}^n)} \leq \sum_{i=0}^{\nu} K(\theta, h, r) \| u_i \|_{H^m(\Omega)}.$$ 

Now, considering that $v$ and the derivatives of the functions $\xi_i$ are bounded uniformly on $\Pi(\theta, h, r)$, using the Leibnitz’s rule on the identities:

$$u_i = u \cdot \xi_i, \quad i = 0, \ldots, \nu,$$

we get:

$$\| u_i \|_{H^m(\Omega)} \leq K(r) \| u \|_{H^m(\Omega)}.$$
Hence:
\[ \| v \|_{H^m(\mathbb{R}^n)} \leq K(\theta, h, r) \| u \|_{H^m(\Omega)} . \]

We have thus found an extension of any element of \( C^m(\Omega) \cap H^m(\Omega) \). Now, as \( \Omega \) satisfies the cone property, we know that \( C^m(\Omega) \) is dense in \( H^m(\Omega) \) (cf. [1]). So, we can also extend any element of \( H^m(\Omega) \). And the norm of the extension operator we have built is bounded by the constant \( K(\theta, h, r) \). The expected result is now proved.

III. COMPACTNESS OF \( \Pi(\theta, h, r) \)

We prove in this section that \( \Pi(\theta, h, r) \) is compact for the strong \( L^2(D) \) topology of the characteristic functions of its elements.

Remark. We notice that for each \( p \in \mathbb{N}, 1 \leq p < +\infty \), the \( L^p(D) \) topology on the set of characteristic functions of the elements of \( \Pi(\theta, h, r) \) is the same as the \( L^2(D) \) topology. We prove here the compactness result for the \( L^2(D) \) topology. The same result follows for any \( L^p(D) \) topology \((1 \leq p < +\infty)\).

In the first paragraph, we prove that the open sets satisfying the "cone property" are the "uniform Lipschitz sets" (a notion we shall define precisely later).

This result is of some help to prove the compactness of \( \Pi(\theta, h, r) \). In the second paragraph, we prove that \( \Pi(\theta, h, r) \) is relatively compact. In the third paragraph, we prove that it is closed.

Let us now give a few definitions and notations we shall use in this section.

Notations. Let \( x \) be an element of \( \mathbb{R}^n \). In a given system of coordinates, we denote by:

- \( x^1, \ldots, x^n \) the coordinates of \( x \);
- \( x = (\hat{x}, x^n) \) where \( \hat{x} \in \mathbb{R}^{n-1} \), \( x^n \in \mathbb{R} \) (last coordinate of \( x \));
- if \( \delta \) and \( \delta' \) are 2 given positive numbers;
- \( P_{\delta}(x) = \{ y \in \mathbb{R}^n; | y^i - x^i | < \delta, i = 1, \ldots, n - 1, | y^n - x^n | < \delta' \} \);
- \( \bar{P}_{\delta}(x) = \{ y \in \mathbb{R}^{n-1}; | x^i - y^i | < \delta, i = 1, \ldots, n - 1 \} \).

DEFINITION III.1. Let \( k \) and \( \delta \) be 2 given positive numbers. We denote by \( \text{Lip}(k, \delta) \) the set of all open subsets \( \Omega \) of \( D \) such that: \( \forall x \in \partial \Omega \), there exists a local coordinate system and a function \( \varphi_x: \bar{P}_{\delta}(x) \to \mathbb{R} \), which is a Lipschitz function of constant \( k \), such that:

\[ y \in P_{\delta}(x) \cap \Omega \Rightarrow \begin{cases} y \in P_{\delta}(x), \\ y^n > \varphi_x(y), \end{cases} \]

with \( \delta' = k\delta(n - 1)^{1/2} \).
I. The Connection between $\Pi(\theta, h, r)$ and $\text{Lip}(k, \delta)$

We prove here that:

1. $\forall \theta, h, r, \exists k, \delta$ such that $\Pi(\theta, h, r) \subset \text{Lip}(k, \delta)$,
2. $\forall k, \delta, \exists \theta, h, r$ such that $\text{Lip}(k, \delta) \subset \Pi(\theta, h, r)$.

We first prove the following lemma.

**Lemma III.1.** Let $\Omega$ be an element of $\Pi(\theta, h, r)$, $x$ an element of $\partial \Omega$, and $C_x$ the cone with axis $\xi_x$ associated to $x$ by the "cone property." Then:

(i) $\forall y \in B(x, r) \cap \partial \Omega$ we have:

$$y + C_x \subset \Omega$$

$$(y - C_x) \cap B(x, r) \subset \Omega;$$

(ii) $\forall y \in B(x, r)$, the set:

$$\{y + \lambda \xi_x; \lambda \in \mathbb{R}\}$$

contains at most one point of $\partial \Omega \cap B(x, r)$.

**Proof.**

(i) Let us recall that if $t \in B(x, r) \cap \Omega$, then $t + C_x$ is included in $\Omega$, which means that if an element $z$ of $\mathbb{R}^n$ verifies:

$$(z - t, \xi_x) - \| z - t \| \cos \theta > 0,$$

$$\| z - t \| < h,$$

where $t \in B(x, r) \cap \Omega$, then $z$ belongs to $\Omega$.

Now, if $y$ is an element of $\partial \Omega \cap B(x, r)$ and $z$ is an element of the cone $y + C_x$, using the definition of $\partial \Omega$ and the continuity of the functions of $t$ defined by:

$$(z - t, \xi_x) - \| z - t \| \cos \theta, \quad \| z - t \|,$$

it follows that there is an element $t$ in $\Omega \cap B(x, r)$ such that $z$ belongs to $t + C_x$. Therefore, $z$ belongs to $\Omega$. So:

$$\forall y \in B(x, r) \cap \partial \Omega, \quad y + C_x \subset \Omega.$$
If \( z \) would belong to \( \Omega \), from the cone property, we should have:

\[
z + C_x \subseteq \Omega,
\]

which is impossible because \( z + C_x \) contains \( y \) which is a point of the boundary of the open set \( \Omega \). So:

\[
(y - C_x) \cap B(x, r) \subseteq \overset{\circ}{\Omega}
\]

and even, as \( (y - C_x) \cap B(x, r) \) is an open set:

\[
(y - C_x) \cap B(x, r) \subseteq \overset{\bullet}{\Omega}.
\]

(ii) Now, let \( y \) be an element of the set \( \{y + \lambda \xi_x; \lambda \in \mathbb{R}\} \). If a point \( y + \lambda \xi_x \) of this set belongs to \( \partial \Omega \cap B(x, r) \) then, as \( h > 2r \) and from (i) of this lemma:

If

\[
y + \lambda \xi_x \in B(x, r) \quad \text{and} \quad \lambda > \lambda_0,
\]

then

\[
y + \lambda \xi_x \in \Omega.
\]

If

\[
y + \lambda \xi_x \in B(x, r) \quad \text{and} \quad \lambda < \lambda_0,
\]

then

\[
y + \lambda \xi_x \in \overset{\circ}{\Omega}.
\]

This proves that \( y + \lambda \xi_x \) is the only point belonging to \( \partial \Omega \cap B(x, r) \). This ends the proof of Lemma III.1.

We now prove the following.

**Proposition III.1.** \( \forall \theta, h, r, \exists k, \delta \) such that \( \Pi(\theta, h, r) \subseteq \text{Lip}(k, \delta) \).

**Proof.** We have to find 2 positive numbers \( k \) and \( \delta \), and for each \( x \in \partial \Omega \) \( (\Omega \in \Pi(\theta, h, r)) \) we have to build the function \( \varphi_x \) of Definition III.1.

So for a given \( \Omega \) in \( \Pi(\theta, h, r) \) and \( x \in \partial \Omega \), let us consider a coordinate system where the axis \( \xi_x \) of the cone \( C_x \) is the last coordinate axis. It is easy to prove that if an element \( y \in \mathbb{R}^n \) is such that \( \| \hat{y} - \hat{\xi} \| < r \sin \theta \), then both intersections of the set:

\[
A_y = \{ y \mid \lambda \xi_x; \lambda \in \mathbb{R} \}
\]

with \( (x + C_x) \cap B(x, r) \) and \( (x - C_x) \cap B(x, r) \) are not empty. Moreover, from Lemma III.1, we know that the first is a part of \( \Omega \) and the second is a part of \( \overset{\circ}{\Omega} \).

Therefore, there is at least one point of \( \partial \Omega \) in \( A_y \cap B(x, r) \) (theorem of boundary crossing); and, from Lemma III.1, we know that there cannot be more than 1. Consequently, there is a function:

\[
\varphi_x: \{ \hat{y} \in \mathbb{R}^{n-1}; \| \hat{\xi} - \hat{y} \| < r \sin \theta \} \rightarrow \mathbb{R}
\]
such that:

\[
\begin{align*}
\{ y \in \partial \Omega \cap B(x, r) \} &\Rightarrow \{ \| \hat{y} - \hat{x} \| < r \sin \theta \} \\
\{ \| \hat{y} - \hat{x} \| < r \sin \theta \} &\Rightarrow \{ y^n = \varphi_x(\hat{y}) \}
\end{align*}
\]

and

\[
\begin{align*}
\{ y \in \Omega \cap B(x, r) \} &\Rightarrow \{ y \in B(x, r), \| \hat{y} - \hat{x} \| < r \sin \theta \} \\
\{ \| \hat{x} - \hat{y} \| < r \sin \theta \} &\Rightarrow \{ y^n > \varphi_n(\hat{y}) \}.
\end{align*}
\]

We now prove that this is a Lipschitz function: let \( \hat{y} \) and \( \hat{x} \) be 2 points of \( \mathbb{R}^{n-1} \) such that:

\[
\| \hat{y} - \hat{x} \| < r \sin \theta,
\]

\[
\| \hat{x} - \hat{y} \| < r \sin \theta.
\]

The elements \( y = (\hat{y}, \varphi_x(\hat{y})) \) and \( z = (\hat{x}, \varphi_x(\hat{x})) \) of \( \mathbb{R}^n \) belong to \( \partial \Omega \cap B(x, r) \) (by definition of \( \varphi_x \)). And from Lemma III.1, we know that:

\[
(y + C_x) \cap B(x, r) \subset \Omega
\]

\[
(y - C_x) \cap B(x, r) \subset \partial \Omega.
\]

So \( z \) does not belong to these sets. From that and the condition \( 2r \leq h \), we derive that:

\[
|\langle z - y, \xi_x \rangle| \leq \| z - y \| \cos \theta
\]

that is:

\[
| \varphi_x(z) - \varphi_x(y) | \leq \| \hat{z} - \hat{y} \| \cot \theta.
\]

So \( \varphi_x \) is a Lipschitz function of constant \( k = \cot \theta \).

Now, if we set:

\[
\delta = \frac{r \sin \theta}{(n - 1)^{1/2}}, \quad \delta' = r \cos \theta = h\delta(n - 1)^{1/2}
\]

we have

\[
y \in P_{\delta'}(x) \cap \partial \Omega \Rightarrow \{ y \in P_{\delta'}(x), \| y^n > \varphi_x(\hat{y}) \},
\]

where \( \varphi_x \) is a Lipschitz function of constant \( k \). Constants \( k \) and \( \delta \) are both uniform on \( \Pi(\theta, h, r) \).

This ends the proof of Proposition III.1. \( \blacksquare \)

**Proposition III.2.** \( \forall k, \delta, \exists \theta, h, r \) such that \( \text{Lip}(k, \delta) \subset \Pi(\theta, h, r) \).
Proof. Let $\Omega$ be an element of $\text{Lip}(k, \delta)$ and $x$ an element of $\partial \Omega$. We know that there is a local coordinate system and a Lipschitz function $\varphi_x$ such that:

$$y \in P_{\delta'}(x) \cap \Omega \iff \begin{cases} y \in P_{\delta'}(x), \\ y^n > \varphi_x(y) \end{cases}.$$

Let $\xi_x$ denote the unit vector of the last axis, and let:

$$\theta = \arctg k \quad (\theta \in ]0, \pi/2[),$$

$$\Gamma = \{ y \in \mathbb{R}^n ; (y, \xi_x) > \| y \| \cos \theta \}.$$

We first prove that if $y \in \partial \Omega \cap P_{\delta'}(x)$, then $(y + \Gamma) \cap P_{\delta'}(x) \subset \Omega$. To achieve that, we just have to prove that if $z$ belongs to $(y + \Gamma) \cap P_{\delta'}(x)$, then:

$$z^n > \varphi_x(z).$$

If $z$ is an element of $(y + \Gamma) \cap P_{\delta'}(x)$, we know that:

$$(z - y, \xi_x) > \| z - y \| \cos \theta,$$

which means:

$$z^n - y^n > k \| z - y \|.$$

And as $y$ belongs to $\partial \Omega \cap P_{\delta'}(x)$, we know that:

$$y^n = \varphi_x(y).$$

So:

$$z^n - \varphi_x(y) > k \| z - y \|.$$

On the other hand, we know that $\varphi_x$ is a Lipschitz function, so that:

$$\varphi_x(z) - \varphi_x(y) \leq k \| y - z \|.$$

Thus:

$$\varphi_x(z) < z^n,$$

which is the expected result.

Now, we set:

$$r = \frac{\min(\delta, \delta')}{3}, \quad h = 2r$$

and prove that $\theta, h, r$ fulfill the requirements: let $y$ be an element of $B(x, r) \cap \Omega$. The element $Y = (y, \varphi_x(y))$ of $\mathbb{R}^n$ belongs to $P_{\delta'}(x) \cap \partial \Omega$. So, by the previous result, we know that:

$$(Y + \Gamma) \cap P_{\delta'}(x) \subset \Omega,$$
and as \( y^n > \varphi_{\omega}(\hat{y}) \), \( y \in Y + \Gamma \); from geometrical properties of cones:

\[
(y + \Gamma) \cap P_{\delta^n}(x) \subset (Y + \Gamma) \cap P_{\delta^n}(x) \subset \Omega.
\]

Now, as \( |y^n - x^n| \) is less than \( \delta'/2 \):

\[
y + C(\xi, \theta, h) \subset (y + \Gamma) \cap P_{\delta^n}(x) \subset \Omega.
\]

This ends the proof of Proposition III.2.

II. \( \Pi(\theta, h, r) \) is Relatively Compact

Let us recall that the topology we study on \( \Pi(\theta, h, r) \) is the strong \( L^2(D) \) topology for the characteristic functions of its elements. So, considering that

\[
P(W) \rightarrow L^2(D): U \rightarrow UD
\]

is continuous, to prove that \( \Pi(\theta, h, r) \) is compact, we only have to prove that the set of characteristic functions of the elements of \( \Pi(\theta, h, r) \) in \( \mathbb{R}^n \) is relatively compact in \( L^p(\mathbb{R}^n) \). Since no confusion is possible, \( \omega \) will denote both characteristic functions of an element \( \Omega \) of \( \Pi(\theta, h, r) \) in \( D \) and in \( \mathbb{R}^n \).

Now, we use the Rellich characterization of the compact subsets of \( L^p(\mathbb{R}^n) \) (cf. [2]):

Let \( p \) be a positive integer. A subset \( \Phi \) of \( L^p(\mathbb{R}^n) \) such that all its elements have their supports included in a same compact, is relatively compact if and only if:

(i) \( \Phi \) is bounded;

(ii) \( \forall f \in \Phi: \int_{\mathbb{R}^n} |f(x + t) - f(x)|^p \, dx \to 0 \)

uniformly on \( \Phi \) when \( t \to 0 \).

In our problem, the condition on the supports of the functions and the condition (i) are trivially satisfied.

We just have to prove that (ii) also holds. To achieve that, we prove the following result.

**Proposition III.3.** Let \( \theta, h, r \) be given. There exists a positive number \( M \) such that:

\[
\forall p \text{ positive integer, } \forall \Omega \in \Pi(\theta, h, r) \text{ with characteristic function } \omega:
\]

\[
\|t\| \leq \delta = \frac{r \sin \theta}{(n - 1)^{1/2}} \Rightarrow \int_{\mathbb{R}^n} |\omega(x + t) - \omega(x)|^p \, dx \leq M \|t\|.
\]
Proof. We know (Proposition III.1) that there are 2 positive numbers $k$ and $\delta$ such that $\Pi(\theta, h, r) \subset \text{Lip}(k, \delta)$. It is then sufficient to prove the result on $\text{Lip}(k, \delta)$.

First, it is possible to prove that there is an integer $N$ such that for each $\Omega \in \text{Lip}(k, \delta)$, $\partial \Omega$ can be covered by $v \leq N$ sets $P_{\delta^*}(x_j)$ ($j = 1, \ldots, v$), where $x_j \in \partial \Omega$ ($j = 1, \ldots, v$) and where:

$$x \in \Omega \cap P_{\delta^*}(x_j) \implies \begin{cases} x \in P_{\delta^*}(x_j), \\ |x^\mu| > \varphi_{\delta}(\xi). \end{cases}$$

So, let $\Omega$ be an element of $\text{Lip}(k, \delta)$ and $(P_{\delta^*}(x_j), j = 1, \ldots, v)$ be such a covering of $\partial \Omega$. For a given $t \in \mathbb{R}^n$, we seek an upper bound for

$$I(t) = \int_{\mathbb{R}^n} |\omega(x + t) - \omega(x)|^p \, dx$$

independently of $\rho$.

We first notice that:

$$\forall p \in \mathbb{N} \quad (p \neq 0), \quad I(t) = \mu(\Delta_t(\Omega))$$

where:

$$\Delta_t(\Omega) = (\Omega \cup (t + \Omega)) \setminus (\Omega \cap (t + \Omega)),$$

$\mu$ is the Lebesgue measure on $\mathbb{R}^n$.

And as:

$$\Delta_t(\Omega) \subset \bigcup_{x \in \Omega^\mu} \overline{B(x, \|t\|)}$$

and

$$\partial \Omega \subset \bigcup_{j=1}^{v} P_{\delta^*}(x_j)$$

we have:

$$\mu(\Delta_t(\Omega)) \leq \sum_{j=1}^{v} \mu(S_j(t))$$

where:

$$S_j(t) = \bigcup_{x \in P_{\delta^*}(x_j) \cap \partial \Omega} \overline{B(x, \|t\|)}.$$

Let us now compute $\mu(S_j(t))$ for a fixed $j$.

We know that there is a coordinate system and a Lipschitz function $\varphi_{\lambda} : P_{\delta}(\xi_j) \to \mathbb{R}$ such that:

$$x \in P_{\delta^*}(x_j) \cap \partial \Omega \iff \begin{cases} \xi \in P_{\delta}(\xi_j), \\ x^\mu = \varphi_{\lambda}(\xi). \end{cases}$$
Let us consider 2 elements \( y \) and \( y' \) of \( S_j(t) \) such that \( y' = y' \). First, we know that:
\[
\| y^i - x_j \| \leq \delta + \| t \|, \quad \forall i = 1, \ldots, n - 1.
\]

Let us seek an upper bound for \( | y^n - y' | \):

From the definition of \( S_j(t) \), we know that there are 2 elements \( x \) and \( x' \) of \( \partial \Omega \cap P_{B_1}(x_j) \) such that:
\[
| x - y_i | \leq \| t \|, \quad \| x' - y_i' \| \leq \| t \|.
\]

Hence:
\[
| y^n - y' | = \| y - y' \| \leq 2 \| t \| + \| x - x' \|.
\]

On the other hand, as \( x \) and \( x' \) belong to \( \partial \Omega \):
\[
x^n = \varphi_{x_j}(\delta), \quad x'^n = \varphi_{x_j}(\delta')
\]
\[
\| x - x' \| < \| \delta - \delta' \| (1 + h^2)^{1/2}
\]
and:
\[
\| \delta' - \delta \| \leq \| \delta - \delta' \| + \| y' - y' \| + \| y' - \delta' \| \leq 2 \| t \|.
\]

Then:
\[
| y^n - y'^n | \leq 2 \| t \| (1 + (1 + h^2)^{1/2}).
\]

Now, computing \( \mu(S_j(t)) \) by integration, we get:
\[
\mu(S_j(t)) - \int_{S_j(t)} dx \leq (\delta + \| t \|)^{n-1} 2(1 + (1 + h^2)^{1/2}) \| t \|
\]
and when \( \| t \| < \delta \):
\[
\mu(\Delta t(\Omega)) < N(2\delta)^{n-1} 2(1 + (1 + h^2)^{1/2}) \| t \|,
\]
that is:
\[
\int_{\mathbb{R}^n} | \omega(x + t) - \omega(x) |^p dx \leq M \| t \|,
\]
where
\[
M = 2N(2\delta)^{n-1} (1 + (1 + h^2)^{1/2}).
\]

This ends the proof of Proposition III.3. \( \blacksquare \)

This proposition shows that the required conditions for the Rellich compactness theorem are fulfilled.

**Theorem III.1.** \( \Pi(\theta, h, r) \) is relatively compact for the strong \( L^2(D) \) topology on the set of the characteristic functions of its elements.
III. \( \Pi(\theta, h, r) \) is Closed

In this paragraph, we shall prove that \( \Pi(\theta, h, r) \) is closed for the topology we are concerned with.

For this we consider a sequence \( (\Omega_n)_{n \in \mathbb{N}} \) of elements of \( \Pi(\theta, h, r) \) such that the sequence \( (\omega_n)_{n \in \mathbb{N}} \) of their characteristic functions converges to an element \( \omega \in L^2(D) \). We shall prove that there is an element \( \Omega \in \Pi(\theta, h, r) \) whose characteristic function is almost everywhere \( \omega \). The method used for this is the following. We build a subset \( \Omega \) of \( D \) such that for each \( x \in \partial \Omega \) there exists a subsequence still denoted by \( (\Omega_n)_{n \in \mathbb{N}} \) of \( (\Omega_n)_{n \in \mathbb{N}} \) such that:

(i) \( \forall n, \exists x_n \in \partial \Omega \) such that \( x_n \to x \) when \( n \to +\infty \).

(ii) The sequence \( (\xi_n) \) of axis of cones \( C_n \) associated to the points \( x_n \) in \( \Omega_n \) converges to an element \( \xi \) of \( \mathbb{R}^n \) with norm 1, which we take as the axis of a cone \( C \).

(iii) \( \forall y \in B(x, r) \cap \partial \Omega, \exists y_n \in B(x_n, r) \cap \Omega_n \) such that \( y_n \to y \).

(iv) \( \forall y \in B(x, r) \cap \Omega, y + C \subseteq \Omega \).

We shall then derive from these properties that \( \Omega \) satisfies the cone condition, and that \( \omega \) is a.e. its characteristic function.

We first prove a few propositions we shall need to establish the closure theorem.

**Definition III.2.** A subset of \( D \) is said to be clean if:

(i) \( \forall \epsilon > 0, \forall x \in \Omega, \mu[B(x, \epsilon) \cap \Omega] \neq 0 \);

(ii) \( \forall \epsilon > 0, \forall x \in D \setminus \Omega, \mu[B(x, \epsilon) \cap (D \setminus \Omega)] \neq 0 \) (\( \mu \) denotes the Lebesgue measure on \( \mathbb{R}^n \)).

**Proposition III.4.** Let \( G \) be a subset of \( D \). There is a subset \( X \) of \( G \) and a subset \( Y \) of \( D \), both of measure 0, such that:

\[ \Omega = (G \cup Y) \setminus X \]

is clean. Moreover, \( X \) and \( Y \) are both parts of \( \partial G \), \( \partial \Omega \) is contained in \( G \), and if \( G \) contains an open set \( \Omega_0 \), then \( \Omega \) also contains \( \Omega_0 \).

**Proof.** For a positive integer \( p \), consider:

\[ X_p = \left\{ x \in G; \mu \left[ B \left( x, \frac{1}{p} \right) \cap G \right] = 0 \right\}, \]

\[ Y_p = \left\{ y \in D \setminus G; \mu \left[ B \left( y, \frac{1}{p} \right) \cap (D \setminus G) \right] = 0 \right\}. \]
We prove that $X_p$ and $Y_p$ are both of measure 0: $X_p$ is a compact set, and:

$$X_p \subseteq \bigcup_{x \in X_p} B\left(x, \frac{1}{p}\right).$$

So there is a finite family $(x_i)_{i=1,...,p}$ of elements of $X_p$ such that:

$$X_p \subseteq X_p \subseteq \bigcup_{i=1}^p B\left(x_i, \frac{1}{p}\right).$$

Moreover, as $X_p$ is a part of $G$:

$$X_p \subseteq \bigcup_{i=1}^p \left[B\left(x_i, \frac{1}{p}\right) \cap G\right]$$

therefore:

$$\mu(X_p) \leq \sum_{i=1}^p \mu\left[B\left(x_i, \frac{1}{p}\right) \cap G\right] = 0.$$

From the same argument, we have:

$$\mu(Y_p) = 0.$$ 

We now consider:

$$X = \bigcup_{p \in \mathbb{N}} X_p, \quad Y = \bigcup_{p \in \mathbb{N}} Y_p,$$

which are both of measure 0. One can see that:

$$\Omega = (G \cup Y) \setminus X$$

is clean, and that $X$ and $Y$ are both parts of $\partial G$.

Now, $\partial \Omega$ is included in $\partial G \cup \partial X \cup \partial Y = \partial G$.

If $G$ contains an open set $\Omega_0$, since $X$ is a part of $\partial G$, it contains no point of $\Omega_0$. So, $\Omega$ also contains $\Omega_0$.

**Proposition III.5.** Let $(\Omega_n)_{n \in \mathbb{N}}$ be a sequence of subsets of $D$ such that the sequence $(\omega_n)_{n \in \mathbb{N}}$ of the characteristic functions of $\Omega_n$ converges a.e. to $\omega \in L^2(D)$, $\omega$ is almost everywhere the characteristic function of:

$$G = \bigcap_{m \in \mathbb{N}} G_m \quad \text{where} \quad G_m = \bigcup_{n \geq m} \Omega_n.$$ 

**Proof.** We know that there is a part $E$ of $D$ of measure $D$ such that:

$$\forall x \in D \setminus E, \quad \omega_n(x) \rightarrow \omega(x).$$
So, for each \( x \) in \( D \setminus E \):

\[
\omega(x) = \lim_{n \to \infty} \omega_n(x) = \lim_{m \to \infty} \sup_{n \geq m} \omega_n(x) = \lim_{m \to \infty} \chi_{G_m}(x),
\]

where \( \chi_{G_m} \) is the characteristic function of \( G_m \). Therefore:

\[
\omega(x) = \lim_{m \to \infty} \chi_{G_m}(x) = \chi_{G}(x) = \chi_{G}(x),
\]

and:

\[
\omega = \chi_{G} \quad \text{a.e.}
\]

**Proposition III.6.** Under the hypothesis of Proposition III.5, if we set:

\[
G = \bigcap_{m \in \mathbb{N}} G_m = \bigcup_{n \geq m} \Omega_n,
\]

for each \( x \in \partial G \), there is a subsequence \((\Omega_{n_k}, k \in \mathbb{N})\) of \((\Omega_n)_n\) and elements \( x_k \in \partial \Omega_{n_k} \) such that \( x_k \to x \) when \( k \to \infty \).

**Proof.** Let \( x \) be an element of \( \partial G \). We shall prove that for each \( k \in \mathbb{N} \) \((k \neq 0)\), it is possible to approximate \( x \) by an element \( x_k \) belonging to the boundary of a certain \( \Omega_{n_k} \) and such that \( \| x - x_k \| \leq 1/k \).

To achieve that, we first approximate \( x \) with an element \( x_k \) of the boundary of a certain \( \Omega_{n_k} \).

(1) \( \forall k \in \mathbb{N}, (k \geq 0), \exists m_k \in \mathbb{N} \text{ and } y_{k} \in \partial G_{m_k} \text{ such that } \| x - y_{k} \| \leq 1/2k. \)

We just have to prove that the distance from \( x \) to \( \partial G_m \) vanishes when \( m \to \infty \).

If it were not true, since the sequence \((G_m)_m\) is decreasing, there would exist \( \delta > 0 \) such that:

\[
\forall m > 0, \quad B(x, \delta) \cap \partial G_m = \emptyset.
\]

But, as \( x \in \partial G \):

\[
\forall m, B(x, \delta) \cap G_m \neq \emptyset.
\]

So, for any \( m \), we would have:

\[
B(x, \delta) \subset G_m.
\]

\( G \) would then be a neighborhood of \( x \), which contradicts the assumption that \( x \in \partial G \).

Now the boundaries of the sets \( G_m \) are compact. So \( x \) has a best approximation in every \( \partial G_m \). And there exists an index \( m_n \) and an element \( y_k \) in \( \partial G_{m_k} \) such that:

\[
\| x - y_k \| \leq \frac{1}{2k}.
\]

We now approximate this \( y_k \).
(2) \( \exists n_k \) and \( x_k \in \partial \Omega_{n_k} \) such that \( \| x_k - y_k \| \leq 1/2k \).

As \( y_k \in \partial G_{m_k} \), there is an element \( v_k \) in \( G_{m_k} \) such that:

\[
y_k - v_k \| \leq \frac{1}{2k},
\]

and from the definition of \( G_{m_k} \), \( v_k \) belongs to a certain \( \Omega_{n_k} \) (\( n_k \geq m_k \)). So that:

\[
d(y_k, \Omega_{n_k}) \leq \| y_k - v_k \| \leq \frac{1}{2k}.
\]

Now, as \( G_{m_k} \) is an open set, \( y_k \) does not belong to \( G_{m_k} \), therefore, it does not belong to \( \Omega_{n_k} \). So its projection \( x_k \) on \( \Omega_{n_k} \) belongs to \( \partial \Omega_{n_k} \). And:

\[
\| y_k - x_k \| = d(y_k, \Omega_{n_k}) \leq \frac{1}{2k}.
\]

Thus, we have found an index \( n_k \) and an element \( x_k \in \partial \Omega_{n_k} \) such that:

\[
\| x - x_k \| \leq \frac{1}{k}.
\]

**Proposition III.7.** Given:

a sequence \( (\Omega_n)_{n \in \mathbb{N}} \) of subsets of \( D \) such that the associated \( (\omega_n)_{n \in \mathbb{N}} \) converges almost everywhere to \( \omega \in L^2(D) \);

\( \Omega \) a clean subset of \( D \) with characteristic function \( \omega \) almost everywhere;

an element \( x \in \partial \Omega \) supposed to be limit of a sequence \( (x_n)_{n \in \mathbb{N}} \) of elements of \( \partial \Omega_n \);

an element \( y \in B(x, r) \cap \Omega \)

then there exists a subsequence \( (\Omega_{n_p}, p \in \mathbb{N}) \) of \( (\Omega_n)_{n \in \mathbb{N}} \) and elements \( y_p \in B(x_{n_p}, r) \cap \Omega_{n_p} \) such that \( y_p \to y \) when \( p \to \infty \).

**Proof.** Since \( B(x, r) \) is an open set, it is possible to find \( \rho > 0 \) such that \( B(y, \rho) \) is included in \( B(x, r) \). Let us consider a sequence \( (\rho_p)_{p \in \mathbb{N}} \) of positive numbers, decreasing to 0, and such that \( \rho_0 < \rho \). As \( (\omega_n) \) converges a.e. to \( \omega \) and is bounded above in \( L^2(D) \), it converges in \( L^2(D) \). So, for a fixed \( p \), we have:

\[
\left| \left[ \int_{B(y, \rho_p)} |\omega_n(x)|^2 \, dx \right]^{1/2} - \left[ \int_{B(y, \rho_p)} |\omega(x)|^2 \right]^{1/2} \right| \\
\leq \left[ \int_{B(y, \rho_p)} |\omega_n(x) - \omega(x)|^2 \, dx \right]^{1/2} \to 0,
\]
when \( n \to \infty \), which also means:

\[
\mu[B(y, \rho_p) \cap \Omega_n] \to \mu[B(y, \rho_p) \cap \Omega] \quad \text{when} \quad n \to \infty.
\]

And from the hypothesis that \( \Omega \) is clean, we know that \( \mu[B(y, \rho_p) \cap \Omega] \neq 0 \).

So, there is an index \( n_p \) for which \( B(y, \rho_p) \cap \Omega_{n_p} \) is not empty. Let us then choose for each \( \rho \in \mathbb{N} \) an element \( y_{n_p} \) in \( B(y, \rho_p) \cap \Omega_{n_p} \). We prove that the sequences \( \Omega_{n_p}, \rho \in \mathbb{N} \), and \( (y_{n_p})_{\rho \in \mathbb{N}} \) answer the question:

As \( \rho_p \to 0 \), \( y_{n_p} \to y \). And when \( \rho \) is large enough, \( y_{n_p} \) belongs to \( B(x_{n_p}, r) \) because:

\[
\| x_{n_p} - y_{n_p} \| \leq \| x_{n_p} - x \| + \| y - y_{n_p} \| - \rho + \| x_{n_p} - x \| + \| y - y_{n_p} \|.
\]

As \( \| x_{n_p} - x \| \to 0 \) and \( \| y - y_{n_p} \| \to 0 \) when \( \rho \to \infty \), for a large enough \( \rho \) we have:

\[
\| x_{n_p} - y_{n_p} \| < r.
\]

This ends the proof of Proposition III.7.

We can now prove the closure theorem itself.

**Theorem III.2.** \( \Pi(\theta, h, r) \) is closed for the strong \( L^2(D) \) topology of the characteristic functions of its elements.

**Proof.** Let us consider a sequence \( (\Omega_n)_{n \in \mathbb{N}} \) of elements of \( \Pi(\theta, h, r) \) such that the sequence \( (\omega_n)_{n \in \mathbb{N}} \) of the characteristic functions of the sets \( \Omega_n \) converge to \( \omega \in L^2(D) \). We have to prove that \( \omega \) is a.e. the characteristic function of an element of \( \Pi(\theta, h, r) \).

First, there is a subsequence still denoted by \( (\omega_n)_{n \in \mathbb{N}} \) of \( (\omega_n)_{n \in \mathbb{N}} \) which converges to \( \omega \) almost everywhere. We shall now work on this sequence.

From Proposition III.5 \( \omega \) is a.e. the characteristic function of the set:

\[
G = \bigcap_{m \in \mathbb{N}} G_m \quad \text{where} \quad G_m = \bigcup_{n \geq m} \Omega_n.
\]

Next, from Proposition III.4, we can modify \( G \) by a set of measure 0, to build a set \( \Omega \) which is clean. \( \omega \) is a.e. its characteristic function.

We now prove that this set (which is not necessarily open) satisfies the cone property, that is:

\[
\forall x \in \partial \Omega, \exists C_x \quad \text{such that} \quad \forall y \in B(x, r) \cap \Omega, \quad y + C_x \subset \Omega.
\]

To prove that, for each \( x \in \partial \Omega \), we build a subsequence still denoted by \( (\Omega_n)_{n \in \mathbb{N}} \) of \( (\Omega_n)_{n \in \mathbb{N}} \) such that:
there is a sequence \((x_n)_{n\in\mathbb{N}}\) of elements of \(\partial \Omega_n\) converging to \(x\);
the sequence of axis of cones \(C_n\) associated to \(x_n\) in \(\Omega_n\) converges to an element \(\xi \in \mathbb{R}^n\) with norm 1;
for each \(y \in B(x, r) \cap \Omega\), there is a sequence \((y_n)_{n\in\mathbb{N}}\) of elements of \(B(x_n, r) \cap \Omega_n\) converging to \(y\);
the cone \(C\) with axis \(\xi\) verifies:
\[
\forall y \in B(x, r) \cap \Omega, \quad y + C \subseteq \Omega.
\]
So, let \(x\) be an element of \(\partial \Omega\). From Proposition III.4, \(x\) also belongs to \(\partial \Omega\), and from Proposition III.6, there is a subsequence \((\Omega_n)_n\) and elements \(x_n \in \partial \Omega_n\) such that:
\[
x_n \to x \quad \text{when} \quad n \to \infty.
\]
Now, as the sets \(\Omega_n\) belong to \(\Pi(\theta, h, r)\), for each \(n\) there is a cone \(C_n\) with axis \(\xi_n\) such that:
\[
\forall y \in B(x_n, r) \cap \Omega_n, \quad y + C_n \subseteq \Omega_n.
\]
\((\xi_n)_n\) is a sequence in the unit ball in \(\mathbb{R}^n\). So there is a subsequence still denoted by \((\xi_n)_n\) converging to an element \(\xi \in \mathbb{R}^n\) with norm 1.
Let us consider an element \(y \in B(x, r) \cap \Omega\). From Proposition III.7, there is a subsequence \((\Omega_n)_n\) of the last sequence \((\Omega_n)_n\) and elements \(y_n\) belonging to \(B(x_n, r) \cap \Omega_n\) such that:
\[
y_n \to y \quad \text{when} \quad n \to \infty.
\]
\(C\) denoting the cone with axis \(\xi\), we prove that:
\[
y + C \subseteq \Omega.
\]
We denote by:
\[
\chi_n \text{ the characteristic function of the cone } C_n;
\]
\(\chi\) the characteristic function of the cone \(C\);
\(\mu(C)\) the measure of the cone \(C\), which is equal to \(\mu(C_n)\).

First, notice that:
\[
\int_{\mathbb{R}^n} \chi(t+y) \omega(t) \, dt = \mu(C) \Leftrightarrow \{y + C \subseteq \Omega\} \text{ except may be for a set of measure 0}
\]
and as \(\Omega\) is clean, we even have:
\[
\int_{\mathbb{R}^n} \chi(t+y) \omega(t) \, dt = \mu(C) \Leftrightarrow y + C \subseteq \Omega.
\]
So we just have to prove that this equality holds:
We know that \( y_n + C_n \) is included in \( \Omega_n \), so:

\[
\int_{\mathbb{R}^n} \chi_n(t + y_n) \omega_n(t) \, dt = \mu(C_n).
\]

And if we consider the functions:

\[
\varphi_n(t) = \chi_n(t + y_n), \quad \varphi(t) = \chi(t + y),
\]

it follows from the convergence of \( y_n \) to \( y \) and \( \xi_n \) to \( \xi \) that \( v_n \) converges to \( v \) in \( L^2(\mathbb{R}^n) \) (strong topology). Moreover, we know that \( \omega_n \to \omega \) in \( L^2(\mathbb{R}^n) \). So:

\[
(\varphi_n, \omega_n)_{L^2(\mathbb{R}^n)} \rightharpoonup (\varphi, \omega)_{L^2(\mathbb{R}^n)}.
\]

Thus:

\[
\int_{\mathbb{R}^n} \chi(t + y) \omega(t) \, dt = \mu(C),
\]

and \( y + C \) is included in \( \Omega \).

Now the proof of the theorem is not complete because we do not know whether \( \Omega \) is open or not. So, let us consider the open set \( \hat{\Omega} \). One can see that it satisfies the cone property (since \( S_\zeta \) does). It will answer the question if and only if:

\[
\mu(\Omega \setminus \hat{\Omega}) = 0.
\]

To prove it, we first notice that \( \mu(\partial(\hat{\Omega})) = 0 \) (this because \( \hat{\Omega} \in \text{Lip}(\kappa, \delta) \) and \( \partial(\hat{\Omega}) \) is a finite union of Lipschitz function graphs).

Then if \( x \in \Omega \setminus (\hat{\Omega}) \), as \( \Omega \) satisfies the cone property, we know that \( x + C_x \) is included in \( \Omega \). So \( x \in \partial(\hat{\Omega}) \). And:

\[
\Omega \setminus (\hat{\Omega}) \subseteq \partial(\hat{\Omega}) \Rightarrow \mu(\Omega \setminus (\hat{\Omega})) = \mu(\partial(\hat{\Omega})) = 0.
\]

So \( \hat{\Omega} \) is an open set satisfying the cone property, and \( \omega \) is a.e. its characteristic function. It follows that the sequence \( (\Omega_n)_n \) converges to \( \hat{\Omega} \), and that \( \Pi(\theta, h, r) \) is closed.

We have proved that \( \Pi(\theta, h, r) \) is relatively compact and closed. The expected result is now established: \( \Pi(\theta, h, r) \) is compact for the strong \( L^2(D) \) topology on the set of characteristic functions of its elements. It is also compact for the \( L^p(D) \) topology for each \( p \in \mathbb{N}, 1 \leq p < +\infty \).

IV. An Application to an Identification in a Neuman Problem

Let us go back to the particular problem given in Section I: As previously \( D \) is a given bounded open set in \( \mathbb{R}^n \). Moreover, let it be given:
a not empty open set \( \Omega_0 \) in \( \mathbb{R}^n \), sufficiently smooth, included in \( D \);
an element \( f \in L^2(D) \);
an element \( y_d \in L^2(\Omega_0) \).

Now, for an open set \( \Omega \subset \mathbb{R}^n \) (\( \Omega_0 \subset \Omega \subset \mathbb{R}^n \)), we consider the solution \( y_\Omega \) of the following variational Neuman problem:

\[
y_\Omega \in H^1(\Omega); \quad ((y_\Omega, \varphi))_{H^1(\Omega)} = (f, \varphi)_{L^2(\Omega)}, \quad \forall \varphi \in H^1(\Omega).
\]

We would like to find an open set \( \Omega_{\text{opt}} \subset \mathbb{R}^n \) such that:

\[
\Omega_0 \subset \Omega_{\text{opt}} \subset D, \quad y_{\Omega_{\text{opt}}} |_{\Omega_0} = y_d.
\]

To achieve this, we consider the functional:

\[
J(\Omega) = \| y_\Omega |_{\Omega_0} - y_d \|_{L^2(\Omega_0)}.
\]

We want to minimize \( J \) on a convenient set of open subsets of \( \mathbb{R}^n \). Using the results of the preceding chapters, we get:

Let \( \Pi_0(\theta, h, r) \) be the set of all the elements of \( \Pi(\theta, h, r) \) containing \( \Omega_0 \).

Then:

**Proposition IV.1.** The functional \( J \) is continuous on \( \Pi_0(\theta, h, r) \) for the strong \( L^2(D) \) topology of the characteristic functions of its elements.

**Proof.** Considering that \( L^2(D) \) is a metric space, we just have to prove that if a sequence \( (\Omega_n)_{n \in \mathbb{N}} \) of elements of \( \Pi(\theta, h, r) \) converges to \( \Omega \in \Pi_0(\theta, h, r) \) (that is if \( \omega_n \rightarrow \omega \) for the strong topology of \( L^2(D) \)), then \( J(\Omega_n) \rightarrow J(\Omega) \).

So, let us consider such a sequence. We define \( y_n \) (\( n \in \mathbb{N} \)) and \( y \) by:

\[
y_n \in H^1(\Omega_n); \quad ((y_n, \varphi))_{H^1(\Omega_n)} = (f, \varphi)_{L^2(\Omega_n)}, \quad \forall \varphi \in H^1(\Omega_n),
\]

\[
y \in H^1(\Omega); \quad ((y, \varphi))_{H^1(\Omega)} = (f, \varphi)_{L^2(\Omega)}, \quad \forall \varphi \in H^1(\Omega).
\]

\( J(\Omega_n) \) and \( J(\Omega) \) are defined by:

\[
J(\Omega_n) = \| y_n |_{\Omega_0} - y_d \|_{L^2(\Omega_0)} , \quad J(\Omega) = \| y |_{\Omega_0} - y_d \|_{L^2(\Omega_0)}.
\]

From the "uniform extension property" (Theorem II.1), we know that for each \( n \), \( y_n \) has an extension \( \tilde{y}_n \in H^1(D) \) and that:

\[
\| \tilde{y}_n \|_{H^1(D)} \leq K(\theta, h, r) \| y_n \|_{H^1(\Omega_n)}.
\]

Now, if we denote by \( z_n \) the restriction of \( y_n \) to \( \Omega \), we also have:

\[
J(\Omega_n) = \| z_n |_{\Omega_0} - y_d \|_{L^2(\Omega_0)}.
\]
A DOMAIN IDENTIFICATION PROBLEM

So, to prove that \( J(\Omega_n) \) converges to \( J(\Omega) \), it is sufficient to prove that \( z_n \) converges to \( y \) for the strong topology of \( L^2(\Omega) \). Now, the extension property in \( \Omega \) is sufficient to imply the compactness of the injection from \( H^1(\Omega) \) to \( L^2(\Omega) \). Therefore, we only have to prove the convergence of \( z_n \) to \( y \) for the weak topology of \( H^1(\Omega) \). We shall now prove this.

First, considering that:

\[
\| y_n \|_{H^1(\Omega_n)} \leq \| f \|_{L^2(\Omega_n)} \leq \| f \|_{L^2(\Omega)},
\]

and:

\[
\| \bar{y}_n \|_{H^1(\Omega)} \leq K(\theta, h, r) \| y_n \|_{H^1(\Omega_n)},
\]

we know that the sequence \((\bar{y}_n)_n\) is bounded in \( H^1(\Omega) \). So, from the weak compactness of the closed subsets of \( H^1(\Omega) \), we only have to prove that the restriction \( z \) to \( \Omega \) of any cluster point \( y \) of the sequence \((\bar{y}_n)_n\) satisfies the equation defining \( y \), that is:

\[
((z, \varphi))_{H^1(\Omega)} = (f, \varphi)_{L^2(\Omega)}, \quad \forall \varphi \in H^1(\Omega).
\]

So, let \((\bar{y}_{n_k})_{k \in \mathbb{N}}\) be a subsequence of \((y_n)_n\) converging to an element \( y \) for the weak topology of \( H^1(\Omega) \). It has a subsequence still denoted by \((\bar{y}_{n_k})_k\) such that the associated sequence \((\omega_{n_k})_k\) converges to \( \omega \) almost everywhere. We now work on this subsequence. We know that:

\[
((y_{n_k}, \varphi))_{H^1(\Omega_{n_k})} = (f, \varphi)_{L^2(\Omega_{n_k})}, \quad \forall \varphi \in H^1(\Omega_{n_k}),
\]

and from the extension property, this is equivalent to:

\[
\int_\Omega \omega_{n_k}(x) \bar{y}_{n_k}(x) \varphi(x) \, dx + \sum_{j=1}^n \int_\Omega \omega_{n_k}(x) D^j \varphi(x) D^j \bar{y}_{n_k}(x) \, dx
\]

\[
= \int_\Omega \omega_{n_k}(x) f(x) \varphi(x) \, dx, \quad \forall \varphi \in H^1(\Omega).
\]

We set:

\[
I_0 = \int_\Omega \omega_{n_k}(x) \varphi(x) \bar{y}_{n_k}(x) \, dx,
\]

\[
I_j = \int_\Omega \omega_{n_k}(x) \varphi(x) D^j \bar{y}_{n_k}(x) \, dx, \quad j = 1, \ldots, n,
\]

\[
I_{n+1} = \int_\Omega \omega_{n_k}(x) f(x) \varphi(x) \, dx,
\]

and we study the limit of each of these terms.
Limit of $I_n$ and $I_{n+1}$. \((\omega_n)_{k}\) converges a.e. to \(\omega\), and:
\[
|\omega_n \varphi| \leq |\varphi|
\]
with \(\varphi \in L^2(D)\). So, from the dominated convergence theorem:
\[
\omega_n \varphi \to \omega \varphi
\]
for the strong topology of \(L^2(D)\), and:
\[
I_n = (\omega_n \varphi, y_n)_{L^2(D)} \to (\omega \varphi, y)_{L^2(D)}.
\]
Likewise:
\[
I_{n+1} = (\omega_n \varphi, f)_{L^2(D)} \to (\omega \varphi, f)_{L^2(D)}.
\]

Limit of $I_j$, \(j = 1, \ldots, n\). As previously:
\[
\omega_n D^i \varphi \to \omega D^i \varphi \quad \text{(strong – } L^2(D)).
\]
Now, the mapping \(y \mapsto D^i y\) is a linear and continuous mapping from \(H^1(D)\) into \(L^2(D)\) for the strong topology on both spaces and therefore for the weak topology on both spaces. So
\[
D^i y_{n_k} \to D^i \bar{y} \quad \text{(weak – } L^2(D)),
\]
and:
\[
(\omega_n D^i \varphi, D^i \bar{y}_{n_k})_{L^2(D)} \to (\omega D^i \varphi, D^i \bar{y})_{L^2(D)}.
\]
From all that, we derive:
\[
(\omega \varphi, y)_{L^2(D)} + \sum_{j=1}^{n} (\omega D^i \varphi, D^i \bar{y})_{L^2(D)} = (\omega \varphi, f)_{L^2(D)}, \quad \forall \varphi \in H^1(D).
\]
That is:
\[
((y |_{\Omega}, \varphi))_{H^1(\Omega)} = (f, \varphi)_{L^2(\Omega)}, \quad \forall \varphi \in H^1(\Omega).
\]
It shows that:
\[
y |_{\Omega} = y.
\]
So all the cluster points \(\bar{y}\) of the sequence \((y_{n_k})_k\) verify:
\[
\bar{y} |_{\Omega} = y.
\]
Consequently:
\[
J(\Omega_n) \to J(\Omega) \quad \text{when} \quad n \to \infty,
\]
and \(J\) is continuous on \(\Pi_\phi(\theta, h, r)\).
We now have the following.

**Theorem IV.1.** The functional $J$ reaches at least once its minimum in $\Pi_0(\theta, h, r)$.

*Proof.* We just have to put the previous results together: We have seen that $J$ is continuous on $\Pi_0(\theta, h, r)$. On the other hand, we have seen that $\Pi(\theta, h, r)$ is compact (Section III). So, we just need to verify that $\Pi_0(\theta, h, r)$ is a closed subset of $\Pi(\theta, h, r)$. This is a direct consequence of Section III, specially Theorem III.2 and Proposition III.4. So, $J$ is a continuous functional on a compact set. It reaches at least once its minimum.

Now, to conclude, we just say that the existence of a minimum for $J$ allows us to use a classical algorithm to minimize $J$ (cf. [3]). Of course, we know nothing about the fact that this minimum is 0 or not. We shall have to compute it. If the computed minimum is 0, then the corresponding domain is a solution. If it is not 0, the original problem has no solution in $\Pi_0(\theta, h, r)$.

**References**