Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 328 (2007) 429–437
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Semi-stratifiable spaces and the insertion of semi-continuous functions ☆

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Received 25 June 2005

Available online 19 June 2006

Submitted by B.S. Thomson

Abstract

In this paper, we investigate the relations between the stratifiable structure of spaces and the insertion of semi-continuous functions and give some characterizations of perfect spaces, semi-stratifiable spaces and K-semi-stratifiable spaces.

Keywords: Semi-stratifiable space; K-semi-stratifiable space; Perfect space; Semi-continuous function; K-semi-continuous function

1. Introduction

Let g and h be real-valued (non-continuous) functions defined on a space X and $g \leqslant h$ ($g(x) \leqslant h(x)$ for each $x \in X$). Under what conditions does there exist a continuous function f such that $g \leqslant f \leqslant h$? The problem has been investigated extensively. The resolution of the problem presents some characterizations of certain spaces, such as extremally disconnected spaces, stratifiable spaces, and others.

In 1949, Stone [6] proved that a space X is extremally disconnected if and only if for any real-valued functions g and h defined on X, g lower semi-continuous, h upper semi-continuous and $g \leqslant h$, there exists a continuous function f defined on X such that $g \leqslant f \leqslant h$. Dowker [2]

☆ Supported by the NNSF of China (10471084).
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.05.043

and Katětov [3] independently proved that a space X is normal and countably paracompact if and only if for any real-valued functions g and h defined on X, g upper semi-continuous, h lower semi-continuous and $g < h$, there exists a continuous function f defined on X such that $g < f < h$. Michael [5] proved that a T_1-space X is perfectly normal if and only if for every pair g, h of real-valued functions defined on X, where g is upper semi-continuous, h is lower semi-continuous and $g \leqslant h$, there exists a continuous function f defined on X such that $g \leqslant f \leqslant h$, and $g(x) < f(x) < h(x)$ whenever $g(x) < h(x)$. In 1999 it was shown in [4] that X is a stratifiable space if and only if for any real-valued functions g and h defined on X, g lower semi-continuous, h upper semi-continuous and $g \leqslant h$, there exists a continuous function f defined on X such that $g \leqslant f \leqslant h$, and if $g(x) < h(x)$, then $g(x) < f(x) < h(x)$. A special case of this theorem is that a space X is stratifiable if and only if for any lower semi-continuous function h, there exists a continuous function f defined on X such that $0 \leqslant f \leqslant h$, and if $h(x) > 0$, then $0 < f(x) < h(x)$. Enlightened by these theorems, we present some characterizations of semi-stratifiable spaces, K-semi-stratifiable spaces and perfect spaces which are remarkably similar to these characterizations, particularly the ones for perfectly normal and stratifiable spaces.

Before stating the main results of this paper, we shall introduce some notions. Throughout this paper, a space means a topological space and all spaces in this paper are assumed to be T_1.

A real-valued function f defined on a space X is lower (upper) semi-continuous if for any real number r, the set $\{x\colon f(x) > r\}$ (the set $\{x\colon f(x) < r\}$) is open.

Definition 1.1. A real-valued function f defined on a space X is K-lower (K-upper) semi-continuous if for every compact set K, f has a minimum (maximum) value on K.

Let X be a space. $R(X)$ represents the set of all real-valued functions on X, and we write $LSC(X)$ and $USC(X)$ for the set of all real-valued lower semi-continuous functions and upper semi-continuous functions on X into $I = [0, 1]$, respectively. $O(X)$ and $K(X)$ are the sets of all open and closed subsets of X, respectively. Also, denote by $UKL(X)$ the set of all real-valued upper and K-lower semi-continuous functions on X into $I = [0, 1]$.

Let X be a space. If $A \subset X$, we write χ_A for the characteristic function on A, that is, a function $\chi_A : X \to [0, 1]$ defined by

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

One easily verifies that if $A \in K(X)$, then $\chi_A \in USC(X)$; and $\chi_A \in LSC(X)$, if $A \in O(X)$.

Definition 1.2. A map $\phi : R(X) \to R(X)$ is called order-preserving if $\phi(g) \leqslant \phi(h)$ for every pair g, h of elements of $R(X)$ satisfying $g \leqslant h$.

Definition 1.3. [1] X is a K-semi-stratifiable space if, to each $U \in O(X)$, one can assign an increasing sequence $\{U_n\}_{n=1}^{\infty}$ of closed subsets of X such that

(a) $\bigcup_{n\in N} U_n = U$,
(b) $U_n \subset V_n$, whenever $U \subset V$,
(c) for every compact subsets K of U, there exists an $N \in \mathbf{N}$ such that $K \subset U_N$.

A space X is said to be a semi-stratifiable space [4] if to each $U \in O(X)$, one can assign an increasing sequence of closed subsets $\{U_n\}_{n=1}^{\infty}$ such that (a) and (b) above hold. X is said to be a perfect space [7] if to each $U \in O(X)$, one can assign an increasing sequence of closed subsets $\{U_n\}_{n=1}^{\infty}$ such that (a) above holds.

Lemma 1.4. [4] *X is a semi-stratifiable space if and only if there is a map* $\rho : \mathbf{N} \times O(X) \to K(X)$, *such that*

(a) $\bigcup_{n \in N} \rho(n, U) = U$ *for all* $U \in O(X)$,
(b) *for any* $U, V \in O(X)$, *if* $U \subset V$, *then* $\rho(n, U) \subset \rho(n, V)$ *for all* $n \in \mathbf{N}$,
(c) $\rho(n, U) \subset \rho(n+1, U)$ *for all* $U \in O(X)$ *and* $n \in \mathbf{N}$.

The above lemma holds for a K-semi-stratifiable space if ρ also satisfies

(d) *for every compact subset* K *of* U, *there exists an* $N \in \mathbf{N}$ *such that* $K \subset \rho(N, U)$.

2. Some properties of semi-continuous functions and K-semi-continuous functions

In this section, we shall introduce some properties of semi-continuous functions and K-semi-continuous functions that will be used in the proof of the primary results.

The following simple properties of semi-continuous functions can be found in [7].

A function $f : X \to R$ is lower (upper) semi-continuous if and only if for every real number r, the set $\{x\colon f(x) \leqslant r\}$ (the set $\{x\colon f(x) \geqslant r\}$) is closed. A function f is lower (upper) semi-continuous if and only if $-f$ is upper (lower) semi-continuous. The sum of finitely many lower (upper) semi-continuous functions is still a lower (upper) semi-continuous function.

The following Proposition 2.1 is easy to prove, so we omit the proof.

Proposition 2.1. *If* $f : X \to R^+$ *is a lower* (*an upper*) *semi-continuous function and* $g : X \to R^+$ *an upper* (*a lower*) *semi-continuous function, then* $\frac{f}{g}$ *is a lower* (*an upper*) *semi-continuous function on* X *into* R^+.

Proposition 2.2. *Let* $\{f_n\}_{n=1}^{\infty}$ *be a monotonically increasing* (*decreasing*) *sequence of lower* (*upper*) *semi-continuous functions defined on a space* X. *If* $\{f_n\}_{n=1}^{\infty}$ *is uniformly convergent to a function* f, *then* f *is also lower* (*upper*) *semi-continuous.*

Proof. We shall prove the case that all $f_n's$ are lower semi-continuous.

As the inequality $f(x) > f(x) - \varepsilon$ holds for each $x \in X$ and $\varepsilon > 0$, it suffices to show that there is a neighborhood U of x such that $f(\xi) > f(x) - \varepsilon$ for all $\xi \in U$ and $\varepsilon > 0$.

Since the sequence $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent to f, for every $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that $f_n(x) > f(x) - \varepsilon$ for all $x \in X$ and $n > N$. Thus, we have $f_{N+1}(x) > f(x) - \varepsilon$ for all $x \in X$ and $\varepsilon > 0$. Since f_{N+1} is lower semi-continuous, there exists a neighborhood U of x such that $f_{N+1}(\xi) > f(x) - \varepsilon$ for every $\xi \in U$. And $f(\xi) > f_{N+1}(\xi)$ for every $\xi \in U$, because the sequence $\{f_n\}_{n=1}^{\infty}$ is monotonically increasing. So we have $f(\xi) > f(x) - \varepsilon$ for every $\xi \in U$ and $\varepsilon > 0$. This concludes the proof. □

Proposition 2.3. *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of lower semi-continuous functions defined on a space* X. *If* $f_n(x) \geqslant 0$ *for every* $x \in X$ *and* $n \in \mathbf{N}$, *and* $\sum_{n=1}^{\infty} f_n = f$, *then* f *is also lower semi-continuous.*

Proof. Let $s_n = f_1 + f_2 + \cdots + f_n$ for every $n \in \mathbf{N}$. Since every f_n is lower semi-continuous, every s_n is also lower semi-continuous. Moreover, the sequence $\{s_n\}_{n=1}^{\infty}$ is monotonically increasing because $f_n(x) \geqslant 0$ for every $x \in X$ and $n \in \mathbf{N}$. With the condition that $\sum_{n=1}^{\infty} f_n = f$ which means that the sequence $\{s_n\}_{n=1}^{\infty}$ is uniformly convergent to f, one readily sees that f is lower semi-continuous by Proposition 2.2. □

Proposition 2.4. *Every lower* (*upper*) *semi-continuous function is a K-lower* (*upper*) *semi-continuous function.*

Proof. Let $f : X \to R$ be a lower semi-continuous function and K a compact subset of X. For each $n \in \mathbf{N}$, let $U_n = \{x\colon\ f(x) > -n\}$. Then the family $\{U_n\}_{n\in\mathbf{N}}$ is an open cover of K. Since K is compact, there exist finitely many numbers $1, 2, \ldots, m$ such that $K \subset \bigcup_{n=1}^{m} U_n$, which implies that f is lower bounded on K. Let α be the greatest lower bound of $f(K)$. Then α is the minimum value of f on K. Otherwise, we have $K \subset \bigcup_{n\in\mathbf{N}}\{x\colon\ f(x) > \alpha + \frac{1}{n}\}$. Thus there exists $m \in \mathbf{N}$ such that $K \subset \bigcup_{n=1}^{m}\{x\colon\ f(x) > \alpha + \frac{1}{n}\}$. A contradiction. Therefore, f is K-lower semi-continuous. □

The converse of Proposition 2.4 needs not be true, which can be seen from the following example.

Example 2.5. Let $X = \{x_n\colon\ n \in N\} \cup \{x\}$, where $\{x_n\colon\ n \in N\}$ is a converging sequence with the limit x. $f : X \to R$ is defined by $f(x_n) = -\frac{1}{n}$, $f(x) = 1$. One easily verifies that f is K-lower semi-continuous, but it is not lower semi-continuous.

Let f be a K-lower semi-continuous function. If $a, b \in R$ and $b > 0$, then $a + bf$ is also K-lower semi-continuous. Generally, the sum of two K-lower semi-continuous functions need not be K-lower semi-continuous.

Example 2.6. Let X be the space in Example 2.5. $f : X \to R$ and $g : X \to R$ are defined by $f(x_n) = -\frac{1}{n}$, $f(x) = 1$ and $g(x_n) = \frac{2}{n}$, $g(x) = 0$, respectively. It is easy to verify that f and g are both K-lower semi-continuous, but $f + g$ is not K-lower semi-continuous.

3. Primary results

Lemma 3.1. *A space X is semi-stratifiable if and only if for any partially-ordered set $(H, \leqslant)$ and any map $F : \mathbf{N} \times H \to K(X)$ that satisfy the following conditions*:

(i) *$F(n+1, h) \subset F(n, h)$ for all $h \in H$ and all $n \in \mathbf{N}$,*
(ii) *for any $h_1, h_2 \in H$, if $h_1 \leqslant h_2$ then $F(n, h_2) \subset F(n, h_1)$ for all $n \in \mathbf{N}$.*

There is a map $G : \mathbf{N} \times H \to O(X)$ such that (i) *and* (ii) *hold for G, $F(n, h) \subset G(n, h)$ for all $n \in \mathbf{N}$, $h \in H$, and $\bigcap_{n\in\mathbf{N}} F(n, h) = \bigcap_{n\in\mathbf{N}} G(n, h)$ for all $h \in H$.*

As for a K-semi-stratifiable space X, we need the following additional condition:

$$\text{For every compact subset } K \text{ of } X,\ \text{if } K \cap F(n, h) = \emptyset \text{ for some } n \in N, \text{ then there exists } m \in N \text{ such that } K \cap G(m, h) = \emptyset. \qquad (*)$$

Proof. Suppose that X is semi-stratifiable and $F:\mathbf{N}\times H\to K(X)$ is a map that satisfies conditions (i) and (ii) of the lemma. Let ρ be the map in Lemma 1.4. We shall show that the map $G:\mathbf{N}\times H\to O(X)$ defined by $G(n,h)=X-\rho(n,X-F(n,h))$ satisfies the conditions of the lemma. By the properties of ρ and F, one can easily verify that (i) and (ii) hold for G. Since, by Lemma 1.4, the equality $\bigcup_{n\in\mathbf{N}}\rho(n,U)=U$ holds for all $U\in O(X)$ and $n\in\mathbf{N}$, which shows that $\rho(n,U)\subset U$, we have $\rho(n,X-F(n,h))\subset X-F(n,h)$ and so $F(n,h)\subset G(n,h)$ for all $n\in\mathbf{N}$ and $h\in H$. The last inclusion also shows us that $\bigcap_{n\in\mathbf{N}}F(n,h)\subset\bigcap_{n\in\mathbf{N}}G(n,h)$.

We shall now show that the inclusion $\bigcap_{n\in\mathbf{N}}G(n,h)\subset\bigcap_{n\in\mathbf{N}}F(n,h)$ also holds. If $x\notin\bigcap_{n\in\mathbf{N}}F(n,h)$, then $x\notin F(N,h)$ for some $N\in\mathbf{N}$. Consequently, $x\in\rho(M,X-F(N,h))$ for some $M\in\mathbf{N}$ since $X-F(N,h)=\bigcup_{n\in\mathbf{N}}\rho(n,X-F(N,h))$. Let $m=\max\{M,N\}$. Then $x\in\rho(M,X-F(N,h))\subset\rho(m,X-F(N,h))\subset\rho(m,X-F(m,h))$. Thus $x\notin X-\rho(m,X-F(m,h))=G(m,h)$, which implies that $x\notin\bigcap_{n\in\mathbf{N}}G(n,h)$. □

Suppose now that X is a K-semi-stratifiable space. We shall show that the condition $(*)$ also holds. If $K\cap F(n,h)=\emptyset$ for some $n\in N$, then $K\subset X-F(n,h)$. Since X is K-semi-stratifiable, there exists $k\in\mathbf{N}$ such that $K\subset\rho(k,X-F(n,h))$. Fix $m\in\mathbf{N}$ such that $m>\max\{n,k\}$. Then $K\subset\rho(k,X-F(n,h))\subset\rho(m,X-F(n,h))\subset\rho(m,X-F(m,h))=X-G(m,h)$. Therefore, $K\cap G(m,h)=\emptyset$.

Conversely, for each $U\in O(X)$ consider the map $F:\mathbf{N}\times O(X)\to K(X)$ defined by $F(n,U)=X-U$. One can easily verify that F satisfies (i) and (ii) above. So there is a map $G:\mathbf{N}\times O(X)\to O(X)$ such that (i) and (ii) hold for G, $F(n,U)\subset G(n,U)$ for all $n\in\mathbf{N}$, $U\in O(X)$ and $\bigcap_{n\in\mathbf{N}}F(n,U)=\bigcap_{n\in\mathbf{N}}G(n,U)$. Let $\rho(n,U)=X-G(n,U)$. It is easy to verify that the last equality defines a map $\rho:\mathbf{N}\times O(X)\to K(X)$ that satisfies the conditions in Lemma 1.4. So X is semi-stratifiable.

Suppose now that the condition $(*)$ also holds. For every compact set K, if $K\subset U$, then $K\cap F(n,U)=\emptyset$ for each $n\in\mathbf{N}$. Thus there exists $m\in\mathbf{N}$ such that $K\cap G(m,U)=\emptyset$, which shows that $K\subset\rho(m,U)$. Therefore, X is K-semi-stratifiable.

Theorem 3.2. *A space X is perfect if and only if there is a map $\phi:LSC(X)\to USC(X)$ such that for any $h\in LSC(X)$, $0\leqslant\phi(h)\leqslant h$, and $0<\phi(h)(x)<h(x)$ whenever $h(x)>0$.*

Proof. Suppose that X is a perfect space and $h\in LSC(X)$. By letting $U_i=\{x\colon h(x)>\frac{1}{2^i}\}$, we obtain an open set U_i for each $i\in\mathbf{N}$. Since X is perfect, for each $i\in\mathbf{N}$, there exist a sequence of closed subsets F_{ij} of X satisfying $F_{ij}\subset F_{ij+1}$ such that $U_i=\bigcup_{j\in\mathbf{N}}F_{ij}$. For each $i\in\mathbf{N}$, define an upper semi-continuous function $g_i:X\to R$ by letting $g_i(x)=0$ for all $x\notin U_i$, $g_i(x)=\frac{1}{2^{i+1}}$ for all $x\in F_{i1}$, and $g_i(x)=\frac{1}{2^{i+j+1}}$ for all $x\in F_{ij+1}-F_{ij}$, where $j\geqslant 1$. Then For each $i\in\mathbf{N}$, $g_i\in USC(X)$. Let

$$\phi(h)(x)=\sum_{i=1}^{\infty}g_i(x)=\frac{1}{2}-\sum_{i=1}^{\infty}\frac{1}{2^{i+1}}-g_i(x)\quad\text{for all } x\in X \text{ and } h\in LSC(X).$$

By Proposition 2.3, one easily sees that the equality above defines a map $\phi:LSC(X)\to USC(X)$.

We shall now show that the map ϕ has the required properties. Suppose $x\in X$. If $h(x)=0$, then $x\notin U_i$ and so $g_i(x)=0$ for each $i\in\mathbf{N}$. Thus $\phi(h)(x)=0$. If $h(x)>0$, then $h(x)>\frac{1}{2^i}$ and so $x\in U_i$ for some $i\in\mathbf{N}$. Let

$$m=\min\{i\colon x\in U_i\}.$$

We have then

$$0 < \phi(h)(x) = \sum_{i=1}^{m-1} g_i(x) + \sum_{i=m}^{\infty} g_i(x) = \sum_{i=m}^{\infty} g_i(x) \leqslant \sum_{i=m}^{\infty} \frac{1}{2^{i+1}} = \frac{1}{2^m} < h(x).$$

Conversely, for each $U \in O(X)$, let $h_U = \chi_U$. Then $h_U \in LSC(X)$, and so $\phi(h_U) \in USC(X)$. By putting $F_n = \rho(n, U) = \{x\colon\ \phi(h_U)(x) \geqslant \frac{1}{2^n}\}$, we obtain an increasing sequence of closed subsets F_n of X and it is easy to check that $U = \bigcup_{n\in\mathbf{N}} F_n$. Therefore, X is perfect. □

Theorem 3.3. *A space X is semi-stratifiable if and only if there is an order-preserving map $\phi : LSC(X) \to USC(X)$ such that for any $h \in LSC(X)$, $0 \leqslant \phi(h) \leqslant h$, and $0 < \phi(h)(x) < h(x)$ whenever $h(x) > 0$.*

Proof. Suppose that X is semi-stratifiable. For each $n \in \mathbf{N}$ and $h \in LSC(X)$, let $F(n,h) = \{x\colon\ h(x) \leqslant \frac{1}{2^{n-1}}\}$. This defines a map $F : \mathbf{N} \times LSC(X) \to K(X)$ and it is easy to verify that F satisfies (i) and (ii) in Lemma 3.1. Since X is semi-stratifiable, by Lemma 3.1, there exists a map $G : \mathbf{N} \times LSC(X) \to O(X)$ such that (i) and (ii) hold for G, $F(n,h) \subset G(n,h)$ for all $n \in \mathbf{N}$ and $h \in LSC(X)$, and $\bigcap_{n\in\mathbf{N}} G(n,h) = \bigcap_{n\in\mathbf{N}} F(n,h)$. Thus

$$\bigcap_{n\in\mathbf{N}} G(n,h) = \bigl\{x\colon\ h(x) = 0\bigr\}. \tag{$*$}$$

Let $\alpha(n,h) = \chi_{G(n,h)}$ and

$$\phi(h)(x) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n}\alpha(n,h)(x) \quad \text{for all } x \in X.$$

By Proposition 2.3, we have $\phi(h) \in USC(X)$. We shall now show that ϕ is order-preserving. Suppose that $h_1 \leqslant h_2$. Then, for each $n \in \mathbf{N}$, we have $G(n,h_2) \subset G(n,h_1)$, and so $\chi_{G(n,h_2)} \leqslant \chi_{G(n,h_1)}$ which shows that $\alpha(n,h_2) \leqslant \alpha(n,h_1)$. By the definition of the map ϕ, one easily sees that $\phi(h_1) \leqslant \phi(h_2)$.

It remains to show that the map ϕ defined above satisfies the necessary conditions in the theorem. Suppose that $x \in X$. If $h(x) = 0$, then $x \in G(n,h)$ and so $\alpha(n,h)(x) = 1$ for all $n \in \mathbf{N}$ by $(*)$. Therefore

$$\phi(h)(x) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n}\alpha(n,h)(x) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} = 0.$$

If $h(x) > 0$, then $x \notin \bigcap_{n\in N} G(n,h)$. Let

$$N = \min\bigl\{n\colon\ x \notin G(n,h)\bigr\}.$$

Then $x \in G(n,h)$ and so $\alpha(n,h)(x) = 1$ for all $n < N$. But $x \notin G(N,h)$, and so $x \notin F(N,h)$, since $F(N,h) \subset G(N,h)$. This implies that $h(x) > \frac{1}{2^{N-1}}$, Hence, we have

$$\sum_{n=1}^{\infty} \frac{1}{2^n}\alpha(n,h)(x) = \sum_{n=1}^{N-1} \frac{1}{2^n} + \sum_{n=N}^{\infty} \frac{1}{2^n}\alpha(n,h)(x) = 1 - \frac{1}{2^{N-1}} + \sum_{n=N}^{\infty} \frac{1}{2^n}\alpha(n,h)(x).$$

Consequently,

$$1 - \frac{1}{2^{N-1}} \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n}\alpha(n,h)(x) < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

By the definition of ϕ, we have $0 < \phi(h)(x) \leqslant \frac{1}{2^{N-1}} < h(x)$.

Conversely, suppose there is an order-preserving map $\phi : LSC(X) \to USC(X)$ that satisfies the conditions given in the theorem. For any fixed $U \in O(X)$, let $h_U = \chi_U$. Then $\phi(h_U) \in USC(X)$. For each $n \in \mathbf{N}$, let $\rho(n,U) = \{x\colon \phi(h_U)(x) \geqslant \frac{1}{2^n}\}$. This defines a map $\rho : \mathbf{N} \times O(X) \to K(X)$. To prove that X is semi-stratifiable, it suffices to show that ρ satisfies (a) through (c) in Lemma 1.4. It can be easily checked that the map ρ satisfies (b) and (c) in Lemma 1.4. So we shall show that ρ also satisfies (a).

For each $n \in \mathbf{N}$, if $x \in \rho(n,U)$, then we have $\frac{1}{2^n} \leqslant \phi(h_U)(x) \leqslant h_U(x)$. So $\chi_U(x) = h_U(x) \geqslant \frac{1}{2^n} > 0$. Hence, $x \in U$. This implies that $\rho(n,U) \subset U$ for each $n \in \mathbf{N}$ and so $\bigcup_{n\in N} \rho(n,U) \subset U$. Conversely, for each $x \in U$, we have $h_U(x) = \chi_U(x) = 1 > 0$, and so $\phi(h_U)(x) > 0$. Hence, there is an $N \in \mathbf{N}$ such that $\phi(h_U)(x) \geqslant \frac{1}{2^N}$, which implies that $x \in \rho(N,U)$. Therefore, $U \subset \bigcup_{n\in N} \rho(n,U)$. □

Theorem 3.4. *X is a K-semi-stratifiable space if and only if there is an order-preserving map $\phi : LSC(X) \to UKL(X)$ such that for any $h \in LSC(X)$, $0 \leqslant \phi(h) \leqslant h$, and $0 < \phi(h)(x) < h(x)$ whenever $h(x) > 0$.*

Proof. Suppose that X is K-semi-stratifiable. Define a map ϕ as that in the proof of Theorem 3.3 (necessity) with G satisfying the additional condition in Lemma 3.1 for K-semi-stratifiable. Then we need only to show that for each $h \in LSC(X)$, $\phi(h)$ is K-lower semi-continuous. Suppose that K is a compact set. If $K \cap G(n,h) = \emptyset$ for some $n \in \mathbf{N}$, let

$$N = \min\bigl\{n\colon\ K \cap G(n,h) = \emptyset\bigr\}.$$

Then $K \cap G(n,h) = \emptyset$ for all $n \geqslant N$ and $K \cap G(n,h) \neq \emptyset$ for all $n < N$. Thus $K \cap \bigcap_{n<N} G(n,h) = K \cap G(N-1,h) \neq \emptyset$. Take $x_0 \in K \cap \bigcap_{n<N} G(n,h)$. Then for each $x \in K$,

$$\begin{aligned} \phi(h)(x) &= 1 - \sum_{n=1}^{\infty} \frac{1}{2^n}\alpha(n,h)(x) = 1 - \sum_{n=1}^{N-1} \frac{1}{2^n}\alpha(n,h)(x) \\ &\geqslant 1 - \sum_{n=1}^{N-1} \frac{1}{2^n} = \frac{1}{2^{N-1}} = \phi(h)(x_0). \end{aligned}$$

If $K \cap G(n,h) \neq \emptyset$ for each $n \in \mathbf{N}$, then, by Lemma 3.1, $K \cap F(n,h) \neq \emptyset$ for each $n \in N$, and so $K \cap \bigcap_{n\in N} F(n,h) \neq \emptyset$ because of the compactness of K. Thus by the equality $\bigcap_{n\in\mathbf{N}} F(n,h) = \bigcap_{n\in\mathbf{N}} G(n,h)$, we have $K \cap \bigcap_{n\in N} G(n,h) \neq \emptyset$. Take $x_0 \in K \cap \bigcap_{n\in\mathbf{N}} G(n,h)$. Then for each $x \in K$, $\phi(h)(x) \geqslant 0 = \phi(h)(x_0)$.

From the discussion above, we see that $\phi(h)$ has a minimum value on K. Therefore, for each $h \in LSC(X)$, $\phi(h)$ is K-lower semi-continuous.

Conversely, let ρ be the same map as that in the proof of Theorem 3.3 (sufficiency). We need only to show that ρ also satisfies condition (d) of Lemma 1.4.

Suppose that U is an arbitrary open subset of X and K a compact subset of X satisfying $K \subset U$. Then $\phi(h_U)$ is K-lower semi-continuous, and so there exists $x_0 \in K$ such that $0 < \phi(h_U)(x_0) \leqslant \phi(h_U)(x)$ for all $x \in K$. Fix $N \in \mathbf{N}$ such that $\phi(h_U)(x_0) \geqslant \frac{1}{2^N}$. Then $\phi(h_U)(x) \geqslant \frac{1}{2^N}$ and so $x \in \rho(N,U)$ for all $x \in K$, which implies that $K \subset \rho(N,U)$. This concludes the proof. □

The following corollaries are similar to Urysohn' lemma.

Corollary 3.5. *A space X is semi-stratifiable if and only if for each pair of (A, U) of subsets of X, A closed, U open and $A \subset U$, there is a lower semi-continuous function $f_{U,A} : X \to [0,1]$ such that $A = f_{U,A}^{-1}(0)$, $X - U = f_{U,A}^{-1}(1)$, and $f_{U,A} \geqslant f_{V,B}$ whenever $A \subset B$ and $U \subset V$.*

Proof. Suppose that X is a semi-stratifiable space. Then, by Theorem 3.3, there is an order-preserving map $\phi : LSC(X) \to USC(X)$ such that for any $h \in LSC(X)$, $0 \leqslant \phi(h) \leqslant h$ and $0 < \phi(h)(x) < h(x)$ whenever $h(x) > 0$. Let $f_A = 1 - \chi_A$ and $g_U = \phi(\chi_U)$. Since A is closed and U is open, We have then $f_A \in LSC(X)$, $g_U \in USC(X)$. Define $f_{U,A} : X \to [0,1]$ by letting $f_{U,A}(x) = \frac{f_A(x)}{1+g_U(x)}$ for all $x \in X$. Then $f_{U,A}$ is lower semi-continuous by Proposition 2.1, and one easily verifies that $f_{U,A} \geqslant f_{V,B}$ when $A \subset B$ and $U \subset V$.

From the equality $f_{U,A} = \frac{f_A}{1+g_U}$, one can see that $f_{U,A}(x) = 0$ and $x \in A$ are equivalent, which implies that $A = f_{U,A}^{-1}(0)$. Similarly, one can verify that $X - U = f_{U,A}^{-1}(1)$.

Conversely, for each open subset U of X, let $g_U = 1 - f_{U,\phi}$, where ϕ is the empty set. Then $g_U \in USC(X)$, and $g_U \leqslant g_V$ when $U \subset V$, and it is easy to verify that $g_U(x) = 0$ if and only if $x \notin U$. As in the proof of Theorem 3.3 (sufficiency), put $\rho(n, U) = \{x\colon\ g_U(x) \geqslant \frac{1}{2^n}\}$. Then ρ satisfies the conditions in Lemma 1.4. So X is semi-stratifiable. □

Corollary 3.6. *A space X is semi-stratifiable if and only if for each open subset U of X, we can assign an upper semi-continuous function $g_U : X \to [0,1]$ such that $X - U = g_U^{-1}(0)$, and $g_U \leqslant g_V$ whenever $U \subset V$.*

Proof. Suppose that X is semi-stratifiable, and let $g_U = 1 - f_{U,\phi}$ as above. As we have seen just now, g_U is as desired. The sufficiency can be proved in the same manner as in Corollary 3.5. □

Corollary 3.7. *A space X is perfect if and only if for each pair (A, U) of subsets of X, A closed, U open and $A \subset U$, there is a lower semi-continuous function $f_{U,A} : X \to [0,1]$ such that $A = f_{U,A}^{-1}(0)$, $X - U = f_{U,A}^{-1}(1)$.*

Proof. Similar to Corollary 3.5. □

Corollary 3.8. *A space X is perfect if and only if for each open subset U of X, we can assign an upper semi-continuous function $f_U : X \to [0,1]$ such that $X - U = f_U^{-1}(0)$.*

Proof. Similar to Corollary 3.6. □

The proofs of the following two theorems are similar to Theorem 3.3 in [4].

Theorem 3.9. *A space X is semi-stratifiable space if and only if every lower semi-continuous function f on X is the limit of a monotonically increasing sequence of upper semi-continuous functions $\{\delta_n(f)\}$ such that $\delta_n(f) \leqslant \delta_n(g)$ for each $n \in N$ whenever $f, g \in LSC(X)$ and $f \leqslant g$.*

Theorem 3.10. *A space X is perfect if and only if every lower semi-continuous function f on X is the limit of a monotonically increasing sequence of upper semi-continuous functions.*

Acknowledgment

The authors thank the referee for his kind suggestions.

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