JOURNAL OF COMPUTER AND SYSTEM SCIENCES **16**, 362–399 (1978)

On the Algebraic Structure of Rooted Trees

CALVIN C. ELGOT

Mathematical Sciences Department, IBM T. J. Watson Research Center, Yorktown Heights, New York

STEPHEN L. BLOOM AND RALPH TINDELL

Department of Mathematics, Stevens Institute of Technology, Hoboken, New Jersey 07030

Received October 13, 1976; revised November 15, 1977

Many kinds of phenomena are studied with the aid of (rooted) digraphs such as those indicated by Figs. 1.1 and 1.2.

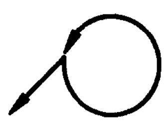


Figure 1.1

Figure 1.2

These two digraphs, while different, usually represent the same phenomenon, say, the same "computational process." Our interest in rooted trees stems from the fact that these two digraphs "unfold" into the SAME infinite tree. In some cases at least it is also true that different (i.e. non-isomorphic) trees represent different phenomena (of the same kind). In these cases the unfoldings (i.e. the trees) are surrogates for the phenomena.

1. INTRODUCTION

1.1 *Outside influence*

Many kinds of phenomena are studied with the aid of (rooted) digraphs such as those indicated by Figs. 1.1 and 1.2.

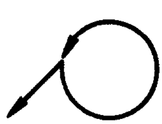


Figure 1.1

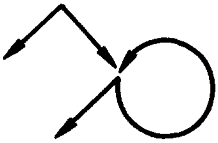


Figure 1.2

These two digraphs, while different, usually represent the same phenomenon, say, the

0022-0000/78/0163-0362$02.00/0

same "computational process." Our interest in rooted trees stems from the fact that these two digraphs "unfold" into the SAME infinite tree viz. that of Fig. 2.1.8. In some cases at least it is also true that different (i.e. non-isomorphic) trees represent different phenomena (of the same kind; for example, see Theorem 4.7 [5]). In these cases the unfoldings (i.e. the trees) are surrogates for the phenomena.

We give a specific example. Flowchart schemes, in the sense of [5], are appropriately labelled, rooted, diagraphs. Two such flowchart schemes are "strongly equivalent" (cf. [5]) iff they unfold into the same tree. Moreover, as indicated in [5], two flowchart schemes are strongly equivalent iff they are "semantically equivalent" i.e. equivalent under all interpretations.

It may be remarked that a "flowchart scheme" is a variant of the notion of a "primitive normal description." This latter notion plays a central role in [1, 5], and is exploited in Sections 3 and 4.

The abstract development relies heavily on the notion algebraic theory" introduced by [14] (cf. also [9]). Generalizations of this notion, (including one used here), are discussed in ([8], cf., in particular, p. 113).

1.2. *On Section 2*

The longest section, Section 2, deals, in concrete fashion with rooted trees (locally finite and ordered). We adopt a Peano-like characterization of this kind of tree as definition (and in Appendix 1 relate this characterization to appropriate graph theoretic concepts). Sufficient properties of appropriately labelled trees, on which we define "composition," "source-tupling," and "iteration," are indicated to assert that they form an "iterative algebraic theory" (without, however, using the concept). In this connection we find it convenient to faithfully represent trees by matrices of sets of words and to introduce the notion "profile" of a tree. The final subsection shows that every tree is a component of a (possibly infinite) vector iteration of a "primitive tree," the trees of "finite index" being obtained by finite vector iterations.

1.3 *On Sections 3 and 4*

While an appreciation of Section 2 requires very little background, an appreciation of Sections 3 and 4 requires rather more. These two sections axiomatically characterize a subcollection $\Gamma\, tr$ of the collection $\Gamma\, Tr$ of trees nearly on the basis of tree composition alone. The subcollection $\Gamma\, tr$ is the iterative subtheory of $\Gamma\, Tr$ generated by Γ. This characterization is our main result: $\Gamma\, tr$ is the collection of trees of finite index in $\Gamma\, Tr$ and is the iterative algebraic theory freely generated by Γ. The argument for Theorem 3.4, which is the heart of the proof of the main result, depends upon a new insight concerning trees viz. (3.4) of Theorem 3.4. The setting of the main result involves the "base category" $\mathscr{N}$, (the skeletal category of finite sets). If one replace $\mathscr{N}$ by $\mathscr{S}$, (the category of (all) sets), one obtains an analogue of the main result: the collection of trees called $\Gamma\, Tr(\mathscr{S})$ is the "completely iterative algebraic theory over $\mathscr{S}$ freely generated by Γ." The trees in $\Gamma\, Tr(\mathscr{S})$ differ from those in $\Gamma\, Tr = \Gamma\, Tr(\mathscr{N})$ in that the outdegree (or 'rank") of a vertex is not restricted to be finite and in that the local order is replaced by local indexing.

If, however, the "rank" of each γ in Γ is a finite set, each tree in $\Gamma\ Tr(\mathscr{S})$ is locally finite. In this case we have: the collection of all trees in $\Gamma\ Tr(\mathscr{S})$ whose singly rooted components have finite index may be described as the "(finitely) iterative algebraic theory over $\mathscr{S}$ freely generated by Γ" or the "scalar iterative algebraic theory over $\mathscr{S}$ freely generated by Γ."

1.4 *Historical background*

In the above we have introduced "iterative theory" as a convenient summery for a collection of facts concerning the trees $\Gamma\ Tr$ and "iterative theory freely generated by Γ" as an axiomatic description of $\Gamma\ tr$. Actually the notion "iterative theory" preceded in time [3] the recognition of $\Gamma\ Tr$ as a particular instance of the notion and the existence of free iterative theories [1] preceded in time the recognition of $\Gamma\ tr$ as the iterative theory freely generated by Γ. From a purely mathematical point of view the usefulness of this result stems from the fact that (a) to establish certain assertions, e.g. identities, for all iterative theories, it is sufficient to establish these assertions for free ones and (b) many true assertions concerning iterative theories are transparently true in the tree theory $\Gamma\ tr$.

The suggestion that a suitable collection of trees might provide a concrete description of "free iterative theories" was first made by Goguen *et al.* [11]. A proof of this fact was first offered by Ginali (cf. [13]). In her readable thesis, [13], Ginali characterizes the trees involved as "regular" and relates the material to studies of Mitchell Wand, Erwin Engeler, the above-mentioned authors, and others.

2

2.1 *Unlabelled Rooted Trees*

According to many texts on graph theory a "tree" is an (undirected) graph which is connected and acyclic. Our concern is with certain kinds of "rooted trees," i.e. trees equipped with distinguished vertices, called roots. Furthermore, the rooted trees discussed here have the property that the set of "immediate successors" of any vertex is finite and linearly ordered; we call these trees "locally finite" and "locally ordered."

Before giving a formal definition, we indicate some examples of rooted, locally finite, locally ordered trees. See Figs. 2.1.0–2.1.11.

The singly rooted, locally finite, locally ordered tree (briefly, "tree") represented by Fig. 2.1.0 has three *vertices*; one vertex, the *root*, has *rank* 2; the *first* and *second* successors of the root have rank zero. The fact that we regard the phrases "first successor" and "second successor" as meaningful, suggests that Figs. 2.1.0 and 2.1.2 represent the same tree, and Figs. 2.1.4 and 2.1.5 represent different (i.e. "non-isomorphic") trees. (The phrase "ordered tree" is sometimes used in this connection, but we prefer to say "locally ordered.") The vertices of rank 0 in a tree are called the *leaves* of the tree. Thus, the tree of Fig. 2.1.3 has four leaves, and the tree of Fig. 2.1.7 has no leaves. Figure 2.1.10 indicates a *doubly rooted* tree.

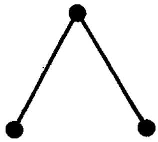


Figure 2.1.0

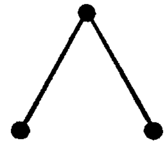


Figure 2.1.1 (B_1)

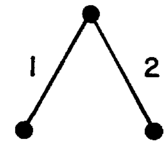


Figure 2.1.2

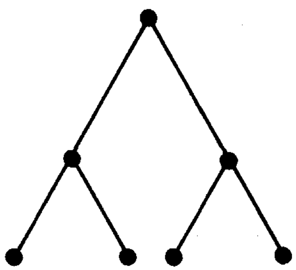


Figure 2.1.3 (B_2)

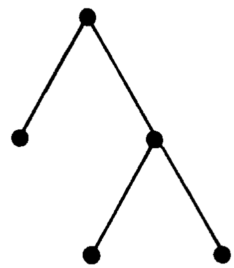


Figure 2.1.4

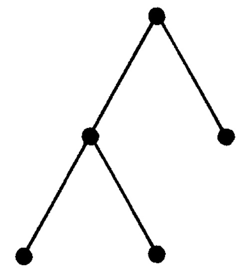


Figure 2.1.5

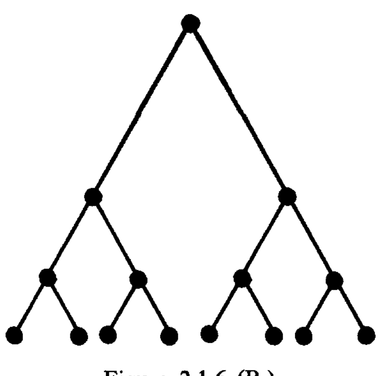


Figure 2.1.6 (B_3)

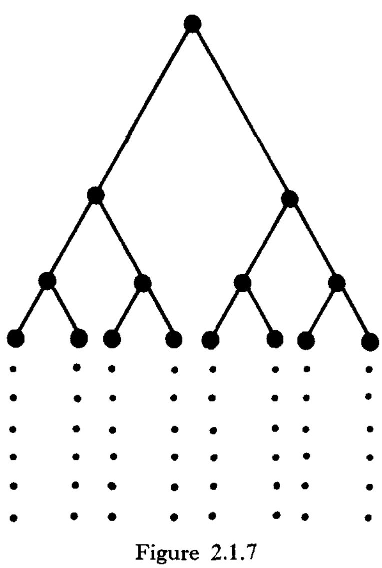


Figure 2.1.7

One might say that the trees represented by Figs. 2.1.0 and 2.1.1 are "isomorphic" or "the same." In our technical discussion we do not identify isomorphism with equality. We now give our formal definition.

DEFINITION 2.1.1. A *ranked set* is a set V together with a function $\rho\colon V \to N$, where N

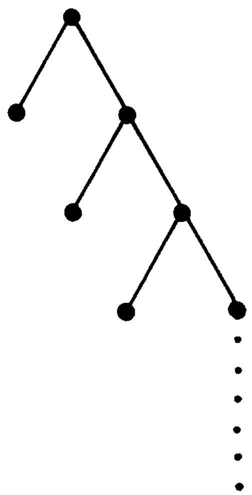


Figure 2.1.8

Figure 2.1.9

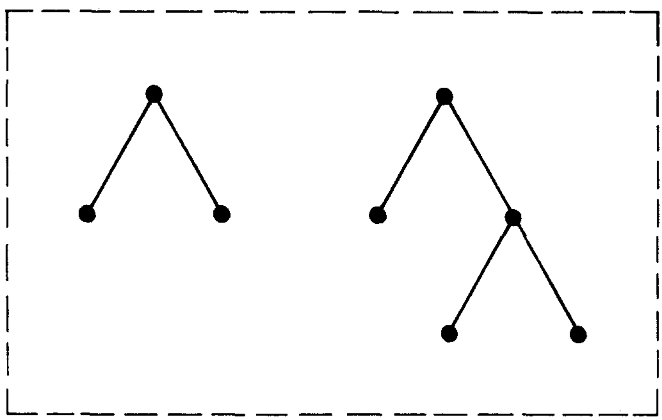


Figure 2.1.10

is the set of non-negative integers. An *edge* of the ranked set (V, ρ) is a pair (v, i) where $v \in V$ and $i \in [v\rho]$.[1] Let n be a non-negative integer.

DEFINITION 2.1.2. An *n-rooted* (*locally finite, locally ordered, unlabelled*) *tree* T is a ranked set (V, ρ) equipped with a function $\sigma: E \to V$ (where E is the set of edges of

[1] For $n \in N$, $[n]$ denotes the set $\{1, 2, \ldots, n\}$. In particular, $[0]$ is the empty set $\varnothing$, $[1] = \{1\}$, etc. If $f: X \to Y$ is a function and $x \in X$, we write the value of f at x variously as xf of $f(x)$. The composition of $f: X \to Y$ with $g: Y \to Z$ is written $fg: X \to Z$ or $X \overset{f}{-} Y \xrightarrow{g} Z$ (note the missing arrowhead).

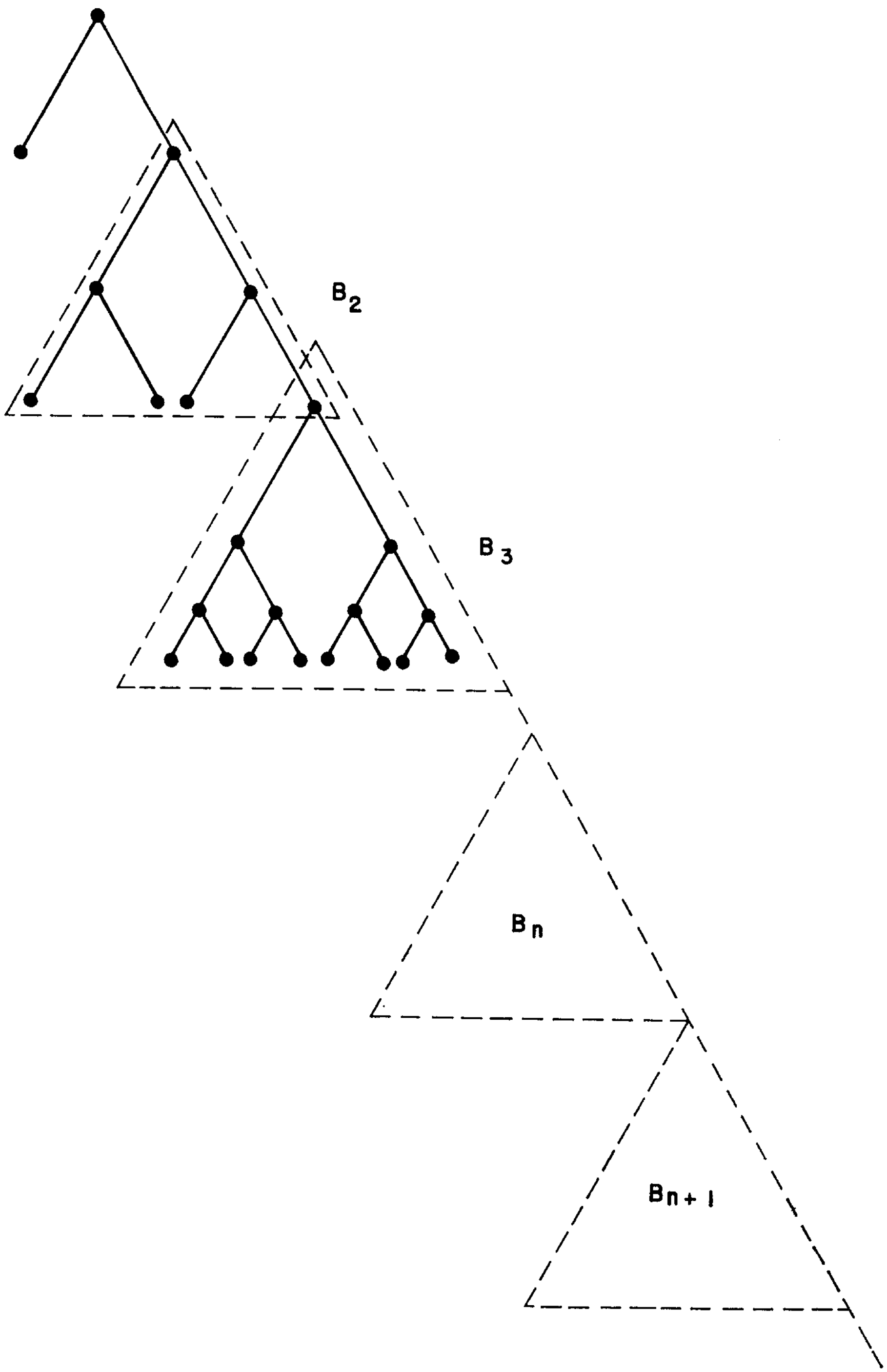


Figure 2.1.11

$(V, \rho))$ and an ordered set of n distinct elements $\epsilon_1, \ldots, \epsilon_n$ of V satisfying the following (Peano-like) conditions:

$$\sigma \text{ is injective; i.e. } \sigma(v, i) = \sigma(v', i') \Rightarrow (v, i) = (v', i'). \tag{2.1.1}$$

$$\text{No element } \epsilon_i,\ i \in [n], \text{ is in the range of } \sigma. \tag{2.1.2}$$

If V' is any subset of V which contains $\epsilon_1, \ldots, \epsilon_n$, and is "closed under σ," then $V' = V$. (V' *is closed under* σ if $\sigma(v, i) \in V'$ whenever $v \in V'$ and $i \in [v\rho]$.) (2.1.3)

The elements of V are called the *vertices* of T. The elements $\epsilon_1, \ldots, \epsilon_n$ are the first, second,..., nth *roots* of T; σ is called the *successor function* of T.

The above definition might be labelled "Proposition" or "Theorem." The terminology reflects our view that this is more appropriately a definition. Certainly the definition is the result of some analysis of the subject. A discussion of the relation between the common notion of tree and the special case of our singly rooted trees is given in Appendix I.

We list some elementary properties of n-rooted trees.

PROPOSITION 2.1.3. *The set V of vertices of a* 0*-rooted tree is empty.*

Indeed, let $V' = \varnothing$ in the "induction clause" (2.1.3) of Definition 2.1.2.

If $T = ((V, \rho), \sigma, \epsilon_1, \ldots, \epsilon_n)$, abbreviated $T = (V, \rho, \sigma, \epsilon_1, \ldots, \epsilon_n)$, is an n-rooted tree and $v, v' \in V$ we say v' is an *immediate successor* of v if $v' = \sigma(v, i)$, some $i \in [v\rho]$. We say v' is a *descendant* of v if there is a finite sequence $v_1, v_2, \ldots, v_k$, $k \geqslant 1$, of vertices such that $v = v_1$, $v' = v_k$ and v_{i+1} is an immediate successor of v_i, for $1 \leqslant i < k$. In particular, for each v, v is a descendant of v.

PROPOSITION 2.1.4. *Let $T = (V, \rho, \sigma, \epsilon_1, \ldots, \epsilon_n)$ be an n-rooted tree.*

(a) *For any vertex $v \in V$, the collection of all descendants of v, denoted vD_T, is a* 1*-rooted tree, where the rank function on vD_T is ρ restricted to vD_T, the successor function on vD_T is σ restricted to vD_T, and the root of vD_T is v.* (*This* tree, *as well as its set of vertices, is denoted vD_T.*)

(b) *If neither $v \in v'D_T$ nor $v' \in vD_T$, then $vD_T \cap v'D_T = \varnothing$.*

(c) *V is the set of all descendants of the roots $\epsilon_1, \ldots, \epsilon_n$. More specifically V is the union $\epsilon_1 D_T \cup \cdots \cup \epsilon_n D_T$ of the disjoint sets $\epsilon_1 D_T, \ldots, \epsilon_n D_T$.*

These facts follow easily from Definition 2.1.2.

We see that D_T, defined in 2.1.4(a) above is a function mapping V into the set of singly rooted "subtrees" of T. For $v \in V$, the tree vD_T is called the "tree of descendants of v," or the "descendency tree of T at v."

DEFINITION 2.1.5. Let $T = (V, \rho, \sigma, \epsilon_1, \ldots, \epsilon_n)$ and $T' = (V', \rho', \sigma', \epsilon_1', \ldots, \epsilon_n')$ be n-rooted trees. An *isomorphism* θ: $T \to T'$ is a bijective function $V \to V'$ such that

(a) $\epsilon_i\theta = \epsilon_i'$, each i, $1 \leqslant i - n$.

(b) For each $v \in V$, $v\rho = v\theta\rho'$ (i.e. θ preserves rank).

(c) For each edge (v, i) in T, $\sigma(v, i)\theta = \sigma'(v\theta, i)$.

The conditions (b) and (c) may be expressed by saying the following two diagrams commute,

$$\begin{tikzcd} V \arrow[r, "\theta"] \arrow[dr, "\rho"'] & V' \arrow[d, "\rho'"] \\ & N \end{tikzcd}$$

$$\begin{tikzcd} & E \arrow[r, "\sigma"] \arrow[d] & V \arrow[d, "\theta"] \\ (v, i) \arrow[d, mapsto] & {} & {} \\ (v\theta, i) & E' \arrow[r, "\sigma'"'] & V' \end{tikzcd}$$

where E and E' are the set of edges of T and T' respectively.

PROPOSITION 2.1.6. *Let T and T' be n-rooted trees. If θ and θ' are isomorphisms $T \to T'$, then $\theta = \theta'$.*

Proof. We use the notation of Definition 2.1.5. Let $X \subseteq V$ be the set of vertices v of T such that $v\theta = v\theta'$. Clearly the roots of T belong to X. But if $v \in X$ and $(v, i) \in E$, then $\sigma(v, i)\theta = \sigma'(v\theta, i) = \sigma'(v\theta', i) = \sigma(v, i)\theta'$. Thus X is closed under σ, proving $X = V$.

Thus *if two n-rooted trees are isomorphic, they are "uniquely isomorphic."*

A tree is *m-homomgeneous* if $v\rho = m$, for each vertex v of T. Clearly a singly rooted, 1-homogeneous tree may be identified with the natural numbers N, with $\epsilon_1 = 0$, and $\sigma(n, 1) = n + 1$. The proof of the following proposition is straightforward.

PROPOSITION 2.1.7. *If T_1 and T_2 are m-homogeneous n-rooted trees, then T_1 is isomorphic to T_2.*

By virtue of 2.1.7, one may speak of "the" m-homogeneous n-rooted tree; for example Fig. 2.1.7 depicts "the" 2-homogeneous singly rooted tree.

Note. m-homogeneous trees are defined for the sake of example only.

At this point, we want to select a "canonical" representative for each isomorphism class of n-rooted trees. First we treat the case $n = 1$. The selection may be done in a number of ways, of course. The following choice[2] appears in the literature. Let $[\omega] = \{1, 2, \ldots\}$.

[2] Knuth [6, p. 558, 15] attributes (essentially) this idea to Francis Galton, "Natural Inheritance" (Macmillan, 1889, p. 249).

Let T be a singly rooted tree. Using (2.1.3), one shows that there is a unique map from V to $[\omega]^*$, $v \in V \Rightarrow \bar{v} \in [\omega]^*$, satisfying

$$\text{the root } \epsilon \text{ goes to the null sequence } \Lambda; \tag{2.1.4}$$

$$\text{if } v \in V,\ i \in [v\rho], \text{ then } \overline{\sigma(v, i)} = \bar{v}i. \text{ Let } \bar{T} \text{ be the image of } T. \tag{2.1.5}$$

A rank and successor function may be defined on $\bar{T}$ in exactly one way to make $T \rightarrow \bar{T}$ and isomorphism. Trees of the form $\bar{T}$ are called (singly rooted) *normal* trees. We note the following facts:

PROPOSITION 2.1.8. *If $V \subseteq [\omega]^*$, ρ: $V \rightarrow N$ a function, then $T = (V, \rho)$ is a singly rooted normal tree iff*

$$\Lambda \in V \tag{2.1.6}$$

$$\textit{if } v \in V, \textit{ then } vi \in V \textit{ iff } i \in [v\rho]; \tag{2.1.7}$$

$$\textit{if } V' \subseteq V,\ \Lambda \in V', \textit{ and for all } v \in V',\ i \in [v\rho], \textit{ we have } vi \in V', \tag{2.1.8}$$

then $V' = V$.

PROPOSITION 2.1.9. *If (V, ρ) is a normal tree, then $\sigma(v, i) = vi$, and ρ is determined by V. Thus we may identify a single rooted normal tree with its set of vertices.*

PROPOSITION 2.1.10. *If $V_i \subseteq [\omega]^*$, $i = 1, 2$ are isomorphic singly rooted normal trees, then $V_1 = V_2$.*

Now we will define the notion of a normal n rooted tree, $n > 1$. By Proposition 2.1.4(c), the set of vertices of such a tree is the union of the sets of vertices of n disjoint singly rooted trees.

DEFINITION 2.1.11. A subset V of $[n] \times [\omega]^*$ is (the set of vertices of) a *normal n-rooted tree* if for each $i \in [n]$, the set $V_i = \{v \in [\omega]^* \mid (i, v) \in V\}$ is (the set of vertices of) a normal singly rooted tree.

[By identifying $[1] \times [\omega]^*$ with $[\omega]^*$, we may allow $n = 1$ in Definition 2.1.11.]

If $V \subseteq [n] \times [\omega]^*$ is a normal n-rooted tree, its ith root is the vertex (i, Λ); ρ and σ are determined by V. Note too that an edge of a normal n-rooted tree is of the form $((i, v), j)$.

The following is obvious.

PROPOSITION 2.1.12. *Let T be an n-rooted tree. There is a unique n-rooted normal tree $\bar{T}$ isomorphic to T. Thus if T_1 and T_2 are isomorphic normal n-rooted trees, $T_1 = T_2$.*

Remark 2.1.13. The characterization of rooted trees embodied in Definition 2.1.2 permits the formulation of the *principle of (mathematical) tree induction*, the analogue of the principle of mathematical induction. Namely, assume $P(v)$ is a proposition which depends on a vertex v of a rooted tree. The principle asserts: if $P(\epsilon_i)$ for $i \in [n]$ and $P(v) \Rightarrow P(\sigma(v,j))$, $j \in [\rho(v)]$, then $P(v)$ for all v.

2.2. *Trees* $T: n \to p$

A tree $T: n \to p$ (also written $n \xrightarrow{T} p$) consists of an n-rooted tree T' as in Section 2.1 together with a function τ from a subset (called the set of ***termini*** of T) of the leaves of T' into $[p]$. The function τ is called the ***termini*** function of T. We say the tree $T = (T', \tau): n \to p$ is normal if T' is normal. By 2.1.11, the normal tree T is fully specified by (V, τ), where V is the set of vertices of T'.

A tree T in the sense of Section 2.1 may be regarded as a tree $T: n \to 1$ by taking $\rho^{-1}(0)$, the set of all leaves of T, as the set of termini, and letting $\tau: \rho^{-1}(0) \to [1]$ be the constant function. The tree T may also be regarded as a tree $T: n \to 0$ by taking the empty set $\varnothing = [0]$ as the set of termini, and taking the unique function $[0] \to [0]$ as τ.

Given trees $T_i: n \to p$, $i = 1, 2$, and a bijection $\theta: V_1 \to V_2$ between their sets of vertices, we say $\theta: T_1 \to T_2$ is an ***isomorphism*** if θ preserves ρ, σ, the roots, the property of being a terminus, and τ. It should be clear that if T_1, $T_2: n \to p$ are isomorphic, they are uniquely isomorphic, generalizing Proposition 2.1.6.

For each p, and each $j \in [p]$, there is a ***root-tree*** $\mathbf{j}_p: 1 \to p$ (alternatively $1 \xrightarrow{j} p$) determined up to isomorphism by the following description. The tree $\mathbf{j}_p$ consists of a single vertex ϵ whose rank is 0. The root ϵ is also a terminus, and $\tau(\epsilon) = j$. These root trees play a significant role in our discussion.

We now define an operation of ***composition*** on trees $n \to p$. Strictly speaking, this "operation" is really an operation "up to isomorphism." Given trees $T: n \to p$, $U: p \to q$, ***composition*** produces the tree $T \cdot U: n \to q$ defined (up to isomorphism) as follows. Let $U_i = \epsilon_i D_U$, the tree of descendants of the ith root of U, $i \in [p]$. The tree $T \cdot U$ is obtained from T by attaching a copy of U_i to each terminus v of T such that $\tau(v) = i$. For example if $T = \mathbf{j}_p: 1 \to p$, $T \cdot U$ is isomorphic to U_j.

As a second example, let $T: 2 \to 3$ be the tree indicated in Fig. 2.2.1. Let $U: 3 \to 2$ be the tree indicated in Fig. 2.2.2. Then $T \cdot U: 2 \to 2$ is the tree indicated by Fig. 2.2.3. Let $T: n \to p$ be a normal tree. We associate with T an "augmented matrix" $\bar{T} = (A; a)$, where A is an $n \times p$ matrix, and a is a $n \times 1$ matrix. $A_{ij} \subseteq [\omega]^*$, $i \in [n]$, $j \in [p]$, consists of those words $v \in [\omega]^*$ such that (i, v) is a terminus of T and $\tau(i, v) = j$; $a_i \subseteq [\omega]^*$, $i \in [n]$,

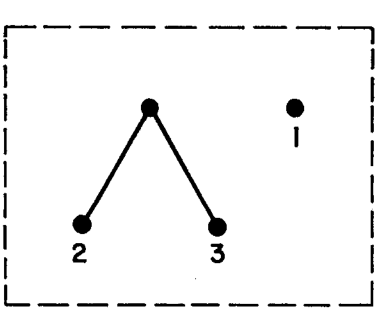


Figure 2.2.1

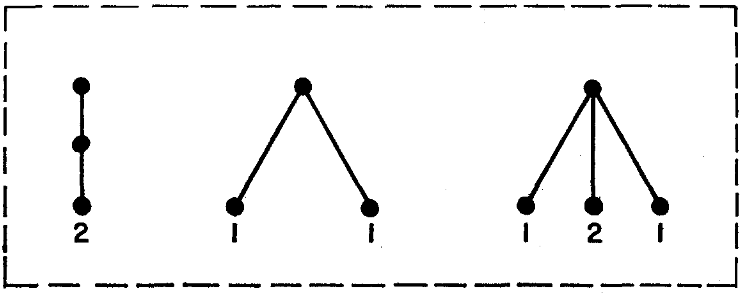


Figure 2.2.2

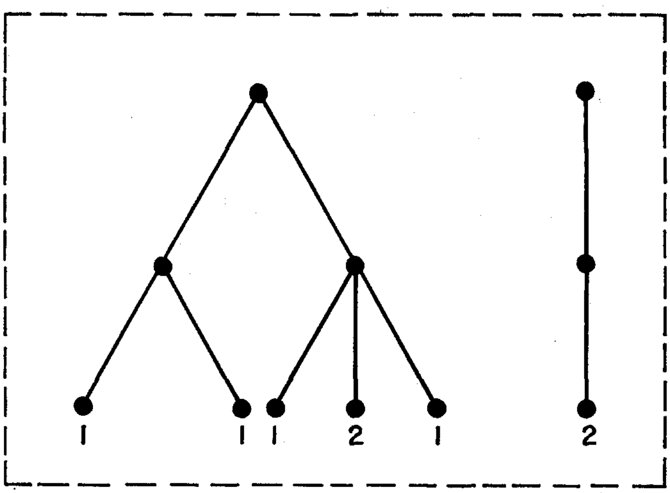


Figure 2.2.3

consists of all $v \in [\omega]^*$ such that (i, v) is a vertex of T which is *not* a terminus. For example, in the case $n = 1$ and $T = \mathbf{j}_p$, $A_{1j} = \{\Lambda\}$, $A_{ik} = \varnothing$, $k \neq j$; $a = a_1 = \varnothing$. [When $n = 1$, we identify $(1, v)$ with v.]

In the case of finite trees $T: n \to p$, a more efficient representation as an augmented $n \times p$ matrix $\bar{T} = (A; a)$ is available (but we will not use this alternative); viz. $a_i \subseteq [\omega]^*$ is taken as the set of words $v \in [\omega]^*$ such that (i, v) is a non-terminus *leaf* of T; A_{ij} is unchanged. Thus this representation takes into account only the "successful paths" i.e. the paths from a root to a leaf.

These augmented matrices are useful to show the operation of composition is associative.

Now let $\bar{U} = (B; b)$ be an augmented $p \times q$ matrix, and define

$$(A; a) \cdot (B; b) = (AB; a + Ab) \tag{2.2.1}$$

where AB is the $n \times q$ matrix obtained by ordinary matrix multiplication (addition of matrix entries being union, and multiplication of matrix entities being (complex) concatenation of sets of words), Ab is the n column vector obtained by multiplying the $n \times p$ matrix A with the $p \times 1$ column vector b; $a + Ab$ is the n column vector whose ith component is $a_i \cup (Ab)_i$.

It is possible to define the augmented matrix $\bar{T} = (A; a)$ for an arbitrary (i.e. not necessarily normal) tree $T: n \to p$. First, define the *label of the path* from the ith root ϵ_i of T to any vertex v in $\epsilon_i D_T$ to be the word $w \in [\omega]^*$, where $\theta(v) = (i, w)$ and where $\theta: T \to T'$ is the isomorphism between T and a normal tree T'. Then define A_{ij} to be the set of all labels of paths from ϵ_i to a terminus v of T such that $\tau(v) = j$; a_i is the set of all labels of paths from ϵ_i to a nonterminus. Clearly, if T itself is normal, this definition of $\bar{T}$ agrees with the previous one.

PROPOSITION 2.2.1. *If $n \to^T p \to^U q$ are trees, then*

$$\overline{T \cdot U} = \bar{T} \cdot \bar{U} \tag{2.2.2}$$

where the multiplication on the right is given by (2.2.1).
Furthermore, if T is not isomorphic to T', $\bar{T} \neq \bar{T}'$.

By an $n \times p$ *surmatrix* we mean an $n \times p$ augmented matrix of the form $\bar{T}$, where $T: n \to p$ is a tree. If T is normal, with the set $V \subseteq [n] \times [\omega]^*$ of vertices, then each set $V_i = \{v \mid (i, v) \in V\}, i \in [n]$ is (the set of vertices of) a normal tree T_i: $1 \to p$. We note that

$$V_i = A_{i1} \cup A_{i2} \cup \cdots \cup A_{in} \cup a_i; \tag{2.2.3}$$

$$\operatorname{dom} \tau_i = A_{i1} \cup A_{i2} \cup \cdots \mathrm{K}\, A_{in}; \tag{2.2.4}$$

$$\tau_i(v) = j \Rightarrow v \in A_{ij}. \tag{2.2.5}$$

Thus $n \times p$ surmatrices serve as simple "surrogates" or representations for normal trees $n \to p$.

The following is immediate from Proposition 2.1.1.

COROLLARY 2.2.2. *The set of surmatrices is closed under multiplication; i.e. if $(A; a)$ is an $n \times p$ surmatrix and $(B; b)$ is a $p \times q$ surmatrix then the product $(A; a) \cdot (B; b) = (AB; a + Ab)$ is an $n \times q$ surmatrix.*

Remark. In the notation for surmatrices, a semicolon, rather than a comma, is used to avoid any possible confusion with "source pairing" in algebraic theories.

2.3. *Tr*

Let AUG be the collection of all augmented matrices $(A; a)$ where $A_{ij} \subseteq [\omega]^*$, $a_i \subseteq [\omega]^*$ and let SUR be the subcollection of all surmatrices. If f_i: $1 \to p$, $i \in [n]$, are augmented "row matrices" then define $(f_1, f_2, \ldots, f_n)$: $n \to p$ to mean the $n \times p$ augmented matrix whose ith augmented row is f_i. We call this operation *source-tupling*. Let $\mathbf{j}_p$ be the surmatrix which is surrogate for the normal root tree $\mathbf{j}_p$ and let $1_p = (\mathbf{1}_p, \mathbf{2}_p, \ldots, \mathbf{p}_p)$ be the $p \times p$ surmatrix $(A; a)$ where A is the identity matrix and a is empty. We have in AUG and in SUR (cf. Corollary 2.2.2) for $n \to^f p \to^g q \to^d r$

$$(f \cdot g) \cdot h = f \cdot (g \cdot h) \tag{2.3.1}$$

$$1_n \cdot f = f = f \cdot 1_p \tag{2.3.2}$$

$$f = (\mathbf{1}_n \cdot f, \mathbf{2}_n \cdot f, \ldots, \mathbf{n}_n \cdot f) \tag{2.3.3}$$

$$\mathbf{i}_n \cdot (f_1, f_2, \ldots, f_n) = f_i, \qquad \text{where} \quad f_i\colon 1 \to p,\ i \in [n]. \tag{2.3.4}$$

By virtue of satisfying (2.3.1)–(2.3.4), AUG and SUR are "algebraic theories," SUR being a subtheory of AUG (see [3] for the definition of "algebraic theory" as used here). We note the obvious fact

$$\text{if } f = (A; a)\colon 1 \to p \text{ is a surmatrix, } f \neq \mathbf{j}_p, \text{ for } j \in [p], \text{ then } \Lambda \notin \bigcup_j A_{ij}. \tag{2.3.5}$$

From (2.3.5) it follows that

$$\text{in SUR, if } f\colon 1 \to p \text{ is not } \mathbf{j}_p \text{ for any } j \in [p], \text{ then } f \cdot g \text{ is not } \mathbf{j}_q \text{ for any } g\colon p \to q,\ j \in [q]. \tag{2.3.6}$$

By virtue of satisfying (2.3.6) (as well as (2.3.1)–(2.3.4)) SUR is an "ideal theory" [3]. The next property of SUR we wish to note concerns solutions to equations in SUR. Let $f = (A; a)\colon n \to p + n$ be in AUG. We decompose the $n \times (p + n)$ matrix A into "blocks" $A = [BC]$ where B is an $n \times p$ matrix, and C is an $n \times n$ matrix; specifically $B_{ij} = A_{ij}$, $i \in [n], j \in [p]$; $C_{ij} = A_{i(p+j)}$, $i, j \in [n]$.

In AUG, consider the equation in the "unknown" $\xi\colon n \to p$

$$\xi = f \cdot (1_p, \xi) \tag{2.3.7}$$

where $(1_p, \xi) = (\mathbf{1}, \mathbf{2}_p, \ldots, \mathbf{p}_p, \xi_1, \ldots, \xi_n)$ and $\xi_i = \mathbf{i}_n \cdot \xi$. Using

$$f = (A; a) = ([BC]; a) \text{ and}$$
$$\xi = (Y; \nu),$$

equation (2.3.1) becomes

$$(Y; v) = ([B \quad C]; a)\left(\begin{bmatrix} 1 \\ Y \end{bmatrix}; \begin{bmatrix} 0 \\ v \end{bmatrix}\right) = (B + CY; Cv + a) \tag{2.3.8}$$

where the "1" on the right is the $p \times p$ matrix with $\{\Lambda\}$ along the diagonal and $\varnothing$ elsewhere. The "0" is a $p \times 1$ matrix with $\varnothing$ everywhere. The equation (2.3.8) is equivalent to

$$\begin{aligned} Y &= CY + B, \\ v &= Cv + a. \end{aligned} \tag{2.3.9}$$

By repeated substitution we obtain $Y = CY + B = C(CY + B) + B = C^2Y + CB + B = C^3Y + C^2B + CB + B = \cdots$.

Thus (2.3.9) is equivalent to

$$\begin{aligned} Y &= C^{r+1}Y + \sum_{i=0}^{r} C^iB, \\ v &= C^{r+1}v + \sum_{i=0}^{r} C^ia, \qquad \text{all} \quad r \geqslant 0. \end{aligned} \tag{2.3.10}$$

If we let

$$\begin{aligned} Y &= \sum_{i=0}^{\infty} C^iB, \\ v &= \sum_{i=0}^{\infty} C^ia, \end{aligned} \tag{2.3.11}$$

then a simple calculation shows (2.3.11) satisfies (2.3.7), and hence (2.3.8) and (2.3.9) as well.

Call a matrix *positive* if the union of all entries consists only of words of positive length; i.e. Λ is not in the union. We claim that if C is positive, the solution (2.3.11) to (2.3.7) is *unique*. To establish the uniqueness, it is helpful to introduce the following operations $\mathbf{s}_r$ on $n \times p$ matrices D, where $D_{ij} \subseteq [\omega]^*$.

$$\begin{aligned} &\mathbf{s}_r(D_{ij}) = \{w \in D_{ij} \mid \text{length } w \leqslant r\} \\ &\mathbf{s}_r(D) \text{ is the } n \times p \text{ matrix whose } (i,j)\text{th entry is } \mathbf{s}_r(D_{ij}). \end{aligned} \tag{2.3.12}$$

Now if C is positive, i.e. $\mathbf{s}_0(C) = 0$, then for all $r \geqslant 0$, $\mathbf{s}_r(C^{r+1}) = 0$, and $\mathbf{s}_r(C^{r+1}Y) = 0$, so that from (2.3.10), $\mathbf{s}_r(Y) = \mathbf{s}_r(\sum_{i=0}^{r} C^iB)$, for all $r \geqslant 0$. This uniquely characterizes Y. The argument is identical for ν. In summary, we have

PROPOSITION 2.3.1. *The equation* (2.3.7) *always has a solution in* AUG. *If* $f = ([BC]; a)$ *and* $\xi = (Y; \nu)$, *then* (2.3.11) *is one such solution*; *if* C *is positive, this solution is unique.*

In the case that C is positive and f is in SUR (i.e. f is a surrogate for a normal tree) we wish to establish that the unique solution to (2.3.7) is also in SUR. By Proposition 2.3.1, it is sufficient to show there is some ξ in SUR which satisfies (2.3.7). We will not give this argument, but rely on extrapolation from the particular case where $n = 1$, $p = 1$ and $f: 1 \to 2$ is the surrogate of the tree indicated in Fig. 2.3.1.

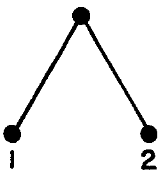


Figure 2.3.1

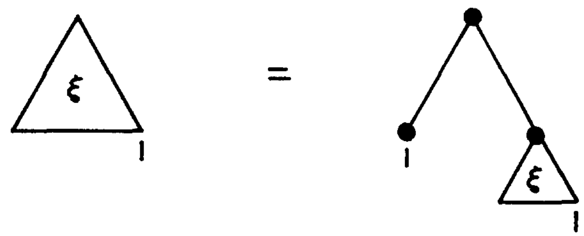


Figure 2.3.2

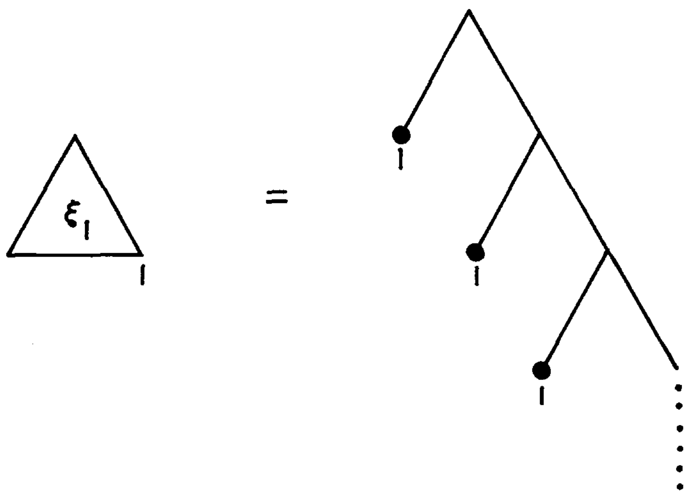


Figure 2.3.3

Then Eq. (2.3.7) is represented by Fig. 2.3.2, and Fig. 2.3.3 is a solution to (2.3.7). Thus, we have

THEOREM 2.3.2. *If* $f = ([BC]; a)$ *is in* SUR *and* C *is positive, then Eq.* (2.3.7) *has a unique solution in* SUR.

Permitting ourselves the extravagance of a different name, *Tr*, for the algebraic theory isomorphic to SUR whose elements ("morphisms") are normal trees, we have

COROLLARY 2.3.3. *If* $f: n \to p + n$ *is in Tr and* $i_n \cdot f$ *is not a root tree for each* $i \in [n]$, *then Eq.* (2.3.7) *has a unique solution in Tr.*

By virtue of Corollary 2.3.3, *Tr* is an "iterative algebraic theory" in the sense of [3] (cf. Remark 2.3.4).

We may describe the unique solution to Eq. (2.3.7) in more detail using the notion of "profile." If $T: n \to p$ is a normal tree and (i, v) $i \in [n]$, $v \in [\omega]^*$, is a vertex of T, let the *length* of (i, v) be the length of the word v. The *profile of* T *at length* d, $P_d(T)$, is the sequence of non-negative integers $w_1\rho$, $w_2\rho, \ldots, w_n\rho$, where $w_1, \ldots, w_n$, $n \geqslant 0$ is the sequence (from left to right) of vertices of T of length d.

Now if $\xi = f \cdot (1_p, \xi)$, where f satisfies the hypotheses of Corollary 2.3.3, it follows that for any $m \geqslant 0$,

$$\xi = f \cdot (1_p \oplus 0_n, f)^m \cdot (1_p, \xi)$$

where $1_p \oplus 0_n: p \to p + n$ is the p-rooted normal tree such that the ith root is a terminus labelled i, $i \in [p]$. Thus, as an unlabelled tree, ξ may be described by the fact that for every m the profile of ξ at length $d \leqslant m$ equals the profile of $f \cdot (1_p \oplus 0_n, f)^m$ at length d.

Remark 2.3.4. As was noted in [5], either Eq. (2.3.7) or the equation

$$\xi = f \cdot (\xi, 1_p) \tag{2.3.13}$$

where $f: n \to n + p$ is ideal, may be used to characterize iterative algebraic theories. Equations of the form (2.3.7) have unique solutions iff those of the form (2.5.13) do. The unique solution to (2.3.7) is denoted $f^\dagger$ and was called [5] the right iterate of f. The unique solution to (2.3.13) was called the left iterate of f.

In order to use some results of [3] without translation, we will rely on a temporary expedient. Namely we will adopt the (unsatisfactory) convention that if the source of a morphism f is $[n]$ and the target of f is *written* $[n + p]$ then $f^\dagger$ indicates the left iterate while if the target of f is written $[p + n]$ then $f^\dagger$ is the right iterate of f. This convention is clearly unsatisfactory as a permanent measure, e.g. ambiguity results when $n = p$. Despite this we believe that using this convention *here* will not cause any confusion.

For the sequel we require the following.

DEFINITION 2.3.5. If $f: n \to p$ is in Tr and $\mathbf{i}_n \cdot f$ is a root tree for each $i \in [n]$, then f is called *base.*

2.4. $\Gamma\, Tr$

By a *genus* Γ we mean a family of pairwise disjoint sets Γ_i, $i \in N$. Clearly there is a "canonical bijection" between genera and ranked sets, the choice between the two being mainly a matter of notational convenience. Thus, a genus Γ gives rise to a ranked set (A, ρ) where $A = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \cdots$ and $\rho: A \to N$ has the value i on the elements of Γ_i. Conversely, a ranked set (A, ρ) gives rise to the genus Γ, where $\Gamma_i = \rho^{-1}(i)$. Moreover, the compositions

$$\text{Genera} - \text{Ranked sets} \to \text{Genera}$$

and

$$\text{Ranked sets} - \text{Genera} \to \text{Ranked sets}$$

are respectively the identity Genera $\to$ Genera and the identity Ranked sets $\to$ Ranked sets.

Let Γ be a genus. By a Γ-tree $T: n \to p$ we mean a tree $T': n \to p$ in the sense of Section 2.2 together with a ("labelling") function λ: $V^- \to \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \mathscr{I} \ldots$, where V^- is the set of non-termini satisfying

$$\lambda(v) \in \Gamma_{\rho(v)} . \tag{2.4.1}$$

Thus specification of a Γ-tree $n \to p$ involves specifying a ranked set (V, ρ), roots $\epsilon_1, \ldots, \epsilon_n$, a successor function σ, terminus function τ and the function λ. Even if the tree is normal, so that the roots as well as ρ and σ need not be specified, one still has (V, τ, λ). The information, we shall see, can be compectly secured in an appropriate kind of augmented matrix which, as before, will serve as a convenient surrogate for "normal tree."

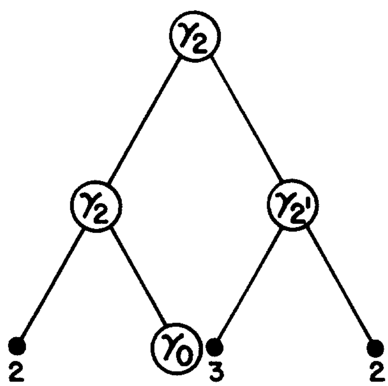


Figure 2.4.1

An example will facilitate the exposition. Consider Figure 2.4.1, where $\gamma_2, \gamma_2' \in \Gamma_2$ and $\gamma_0 \in \Gamma_0$, which is intended to represent a tree $1 \to 3$. The normal tree represented by Figure 2.4.1 is specified by the first four columns of the following table.

V	ρ	τ	λ	New names for vertices	Abbreviations for new names
Λ	2		γ_2	Λ	Λ
1	2		γ_2	$(\gamma_2, 1)$	$\gamma_2 1$
2	2		γ_2'	$(\gamma_2, 2)$	$\gamma_2 2$
11	0	2		$(\gamma_2, 1)(\gamma_2, 1)$	$\gamma_1 1 \gamma_2 1$
12	0		γ_0	$(\gamma_2, 1)(\gamma_2, 2)\gamma_0$	$\gamma_2 1 \gamma_2 2 \gamma_0$
21	0	3		$(\gamma_2, 2)(\gamma_2', 1)$	$\gamma_2 2 \gamma_2' 1$
22	0	2		$(\gamma_2, 2)(\gamma_2', 2)$	$\gamma_2 2 \gamma_2' 2$

If T: $1 \to 3$ is the Γ-tree specified by the above table, i.e. the Γ-tree depicted by Figure 2.4.1, we define $\bar{T}$ to be the 1×3 augmented matrix $(A; a)$ where $A_{11} = \varnothing$, $A_{12} = \{\gamma_2 1 \gamma_2 1, \gamma_2 2 \gamma_2' 2\}$, $A_{13} = \{\gamma_2 2 \gamma_2' 1\}$, $a_1 = \{\gamma_2 1 \gamma_2 2 \gamma_0, \gamma_2 1, \gamma_2 2, \Lambda\}$. In general, given any

Γ-tree $n \to p$, we define the augmented matrix $\bar{T}$ as in Section 2.2 but interpret the phrase "labels of paths" in the manner indicated by the above example (see the discussion preceding Proposition 2.2.1).

In the case of finite Γ-trees $n \to^T p$ a more efficient representation is possible (cf. Section 2.2) while preserving the injectiveness of the map $T \mapsto \bar{T}$. The more efficient representation applied to Fig. 2.4.1 yields $a_1 = \{\gamma_2 1 \gamma_2 2 \gamma_0\}$ while A_{1j} is unchanged, $j \in [3]$.

By a Γ-surmatrix we mean an augmented matrix of the form $\bar{T}$ where T is a Γ-tree.

Proposition 2.4.1. *Proposition* 2.2.1 *holds for Γ-trees T, U and Γ-surmatrices are closed under multiplication.*

Proposition 2.4.2. *Proposition* 2.2.1 *holds in the domain of finite Γ-trees even if $\bar{T}$ is interpreted as "the more efficient" representation of T.*

The discussion of Section 2.3 carries over to Γ-trees and their representations leading to

Theorem 2.4.3. Γ SUR *and* Γ Tr *are (isomorphic) iterative algebraic theories.*

In the special case that Γ_i is a one element set for each $i \in N$ each tree T: $n \to p$ in the sense of Section 2.2 may be made into a Γ-tree in eactly one way, i.e. there is exactly one function λ: $V^- \to \bigcup_{i=0}^{\infty} \Gamma_i$ satisfying (2.4.1). Thus the notion "Γ-tree $n \to p$" may be regarded as a generalization of the notion "tree $n \to p$". Thus

Proposition 2.4.4. *In the case that Γ is a family of singletons* Γ SUR $\approx$ SUR *and* Γ *Tr* $\approx$ *Tr.*

In the case that Γ is a family of singletons Fig. 2.4.1 may be "abbreviated" by the tree $1 \to 3$ represented by Fig. 2.4.2, in particular $\gamma_2 = \gamma_2'$.

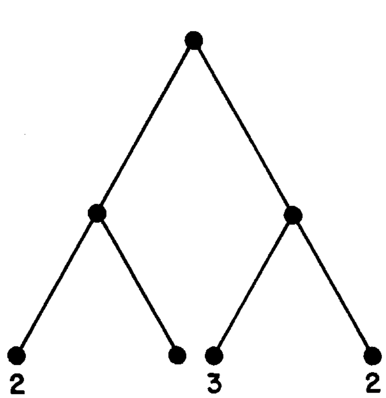


Figure 2.4.2

For use in Section 3, we point out a fundamental property of the theory Γ *Tr*. Call a Γ-tree T: $1 \to n$ *atomic* if T has $n + 1$ vertices: a root ϵ of rank n, and n immediate successors, all of which are termini; the ith successor is labelled i, $i \in [n]$; the value $\lambda(\epsilon) = \gamma$

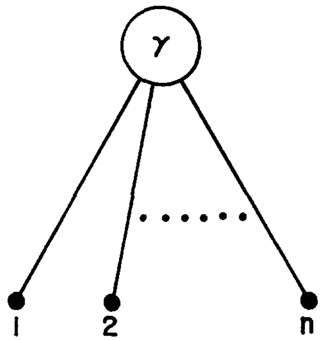


Figure 2.4.3

belongs to Γ_n. Such a tree may be represented by Fig. 2.4.3 (Clearly for each n there is a bijection between the set of normal atomic Γ-trees $1 \to n$ and the set Γ_n.)

The property of $\Gamma\ Tr$ we want to call attention to is the following:

PROPOSITION 2.4.5. *Let T_1, T_2: $1 \to n$ be* atomic *(normal) Γ-trees, and let U_1, U_2: $n \to p$ be arbitrary (normal) Γ-trees. If $T_1 \cdot U_1 = T_2 \cdot U_2$, then $T_1 = T_2$ and $U_1 = U_2$.*

We express this fact briefly by saying $\Gamma\ Tr$ has the *unique factorization property*.

We also require the notion "primitive tree." A tree T: $n \to p$ in $\Gamma\ Tr$ is *primitive* if for each $i \in [n]$,

$$\mathbf{i}_n \cdot T = \gamma_i \cdot f_i$$

for some atomic γ_i and base f_i.

2.5. *Trees of Finite Index and Iterates of Primitive Trees*

Call a Γ-tree T: $1 \to p$ *trivial* if T is isomorphic to the root tree $\mathbf{j}_p$, for some $j \in [p]$. Thus a tree T: $1 \to p$ with only one vertex ϵ (so $\epsilon\rho = 0$) is non-trivial iff ϵ is labelled with an element in Γ_0. By the *descendency index* of T (briefly, the *index* of T) we mean the number of distinct normal trees isomorphic to *non-trivial* descendency trees of T; i.e. the index of T is the cardinality of the set of non-trivial normal Γ-trees T': $1 \to p$ such that there is a vertex v with T' isomorphic to vD_T (see Proposition 2.1.4). Since the Γ-trees we are dealing with are locally finite, the index of T is either finite or denumerably infinite.

For example, the indices of the trees in Figs. 2.1.2–2.1.9 are respectively 1, 2, 2, 2, 3, 1, 1, 0, when regarded as trees $1 \to 1$ (where Γ is a family of singletons). The tree in Figure 2.1.11 has infinite index. When regarded as trees $1 \to 0$, the index of the trees in Figures 2.1.2–2.1.6, 2.1.8 and 2.1.9 are increased by one. The index of the tree in Fig. 2.1.7 remains one.

Now suppose T: $1 \to p$ has finite index $n > 0$ and let T_1, T_2,..., T_n be an enumeration without repetition of the non-trivial normal Γ-trees isomorphic to descendency trees of T, and suppose T_1 is isomorphic to T. We shall construct a primitive Γ-tree τ: $n \to p + n$

(i.e. a primitive morphism in $\Gamma\,\mathrm{Tr}$) such that $\mathbf{1}_n \cdot \tau^\dagger = T_1$. (Recall $\mathbf{1}_n$: $1 \to n$ is the root tree.)

For $i \in [n]$, let v_i be a vertex of T such that $v_i D_T$ is isomorphic to T_i. Suppose the label of v_i is $\gamma_i \in \Gamma_{v_i\rho}$. We define τ by the requirement that

$$\mathbf{i}_n \cdot \tau = \gamma_i \cdot f_i\,, \tag{2.5.1}$$

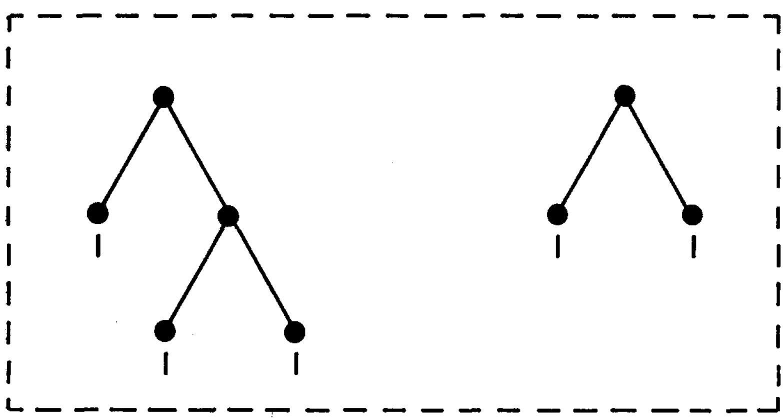
where γ_i is the atomic Γ-tree $1 \to v_i\rho$, and f_i (defined below) is base; i.e.

$$\mathbf{i}_n \cdot \tau\colon [1] \xrightarrow{\gamma_i} [v_i\rho] \xrightarrow{f_i} [p + n]. \tag{2.5.2}$$

Thus, in the case that $v_i\rho = 0, f_i = 0_{p+n}\colon [0] \to [p + n]$. Otherwise, let $k \in [v_i\rho]$. The kth successor $\sigma(v_i, k) = v'$ of v_i in T is either a terminus or not. If, in the former case, v' is labelled $j \in [p]$, we define $kf_i = j \in [p + n]$. Otherwise, if $v'D_T$ is isomorphic to T_l, $l \in [n]$, we define $kf_i = p + l \in [p + n]$.

$\tau\colon [2] \to [3]$

Figure 2.5.1



$\tau \cdot (1, \oplus 0_2, \tau)^d, d > 0$

Figure 2.5.2

For example, suppose that the tree T indicated in Fig. 2.1.4 is treated as a tree $1 \to 1$, and that all the vertices of rank 2 are labelled $\Delta \in \Gamma_2$. Then τ: $[2] \to [3]$ is indicated in Fig. 2.5.1, where again, the vertices of rank 2 are labelled Δ. Here $\mathbf{i}_2 \cdot \tau = \Delta \cdot f_i$, $i \in [2]$, where f_i: $[2] \to [3]$ and $1f_1 = 1$, $2f_1 = 3$; $1f_2 = 1$, $2f_2 = 1$.

$\tau \cdot (1_1 \oplus 0_2, \tau)^d$, for $d > 0$ is indicated in Fig. 2.5.2, where, as before, the vertices of rank 2 are labelled Δ.

As another example, if we treat the tree indicated in Fig. 2.1.8 as a tree T: $1 \to 0$, where each vertex of rank 2 is labelled $\Delta \in \Gamma_2$ and each vertex of rank 0 is labelled $\perp \in \Gamma_0$, then the index of T is 2, and τ: $[2] \to [2]$ is given by Fig. 2.5.3. τ^2, τ^3, τ^4 are indicated in Fig. 2.5.4. These examples illustrate the following theorem.

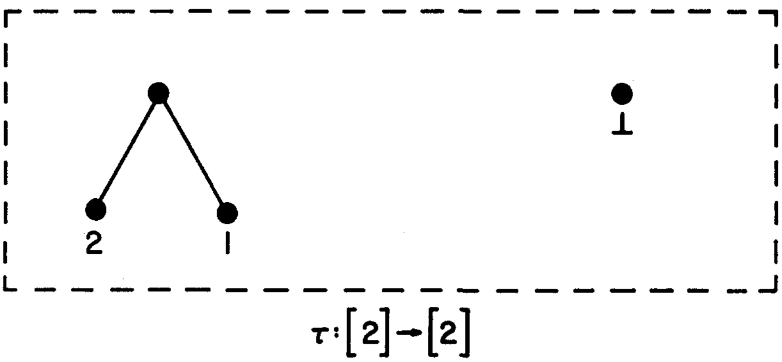


Figure 2.5.3

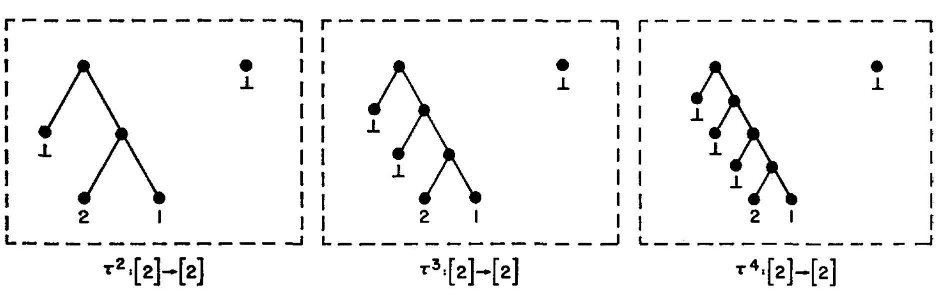


Figure 2.5.4

THEOREM 2.5.1. *If T: $1 \to p$ is a Γ-tree with finite index s, and τ: $[s] \to [p + s]$ is the primitive morphism in ΓTr described by* (2.5.1) *and* (2.5.2), *then $1_s \cdot \tau^\dagger$: $[1] \to [p]$ is isomorphic to T. In the case that T is a finite tree, we have, for all sufficiently large d,*

$$\tau \cdot (1_p \oplus 0_s, \tau)^d = \tau^\dagger \cdot (1_p \oplus 0_s).$$

The construction of the primitive tree τ described in the discussion preceding Theorem 2.5.1 "works" even when the Γ-tree T: $1 \to p$ does not have finite index. Indeed, suppose the index of T is ω. Define the primitive (infinitely rooted) normal tree $\tau\colon [\omega] \to [\omega]$ exactly as before. $\tau^\dagger\colon [\omega] \to [p]$ is to be taken as a "limit" of the trees

$$\tau \cdot (1_p \oplus 0_\omega, \tau)^d,$$

as $d \to \infty$. Then, as before $1_\omega \cdot \tau^\dagger$ is isomorphic to T. Thus, we have

THEOREM 2.5.2. *Theorem 2.5.1 holds even when the descendency index of T is ω.*

We call attention to the fact that the iterate of $\tau\colon [\omega] \to [\omega]$ is ambiguous until one specifies "the p", $0 \leqslant p < \omega$, which is to be the target of $\tau^\dagger$.

Remark 2.5.3. *It is important to note that "infinite vector iteration" is used to obtain Theorem 2.5.2 while only "finite vector iteration" is used in Theorem 2.5.1.*

Remark 2.5.4. The collection of trees of finite index is closed under composition, (finite) source-tupling and iteration. Indeed, if T has index n and U has index p, then $T \cdot U$ and (T, U) have (when defined) index at most $n + p$ and $T^\dagger$ has (when defined) index at most n.

3. INJECTIVITY

In this section, it is assumed that the reader is familiar with [3]. It would be helpful to the reader to have read [1] as well, but this is not essential. In [1] it was shown that for any genus Γ, there is an iterative theory $\Gamma\mathscr{I}$, freely generated by Γ; i.e. for any iterative theory J and any family h of functions mapping Γ_n into ideal morphisms $[1] \to [n]$ in J there is a unique ideal theory morphism $\Gamma\mathscr{I} \to J$ extending h. Furthermore, it was shown that $\Gamma\mathscr{I}$ contained $\Gamma\mathscr{T}$, the algebraic theory freely generated by Γ. The argument given in [1] showed that $\Gamma\mathscr{I}$ may be constructed as certain equivalence classes of "normal descriptions" (see below). A more concrete description of $\Gamma\mathscr{I}$ is obtained in this section (Corollary 3.2). Another description of $\Gamma\mathscr{I}$ is obtained in Section 4 (Corollary 4.1.2).

The objective of this section is to prove the following.

THEOREM 3.1. *The ideal theory morphism from $\Gamma\mathscr{I}$ into $\Gamma\, Tr$ induced by the map which takes the generator $\gamma \in \Gamma_n$ into the tree $\gamma\colon 1 \to n$ for each γ in Γ_n (and each $n \in N$) is injective.*

COROLLARY 3.2. *The iterative subtheory of $\Gamma\, Tr$ generated by Γ denoted $\Gamma\, tr$ is (a description of) the iterative theory freely generated by Γ.*

In [1] it was shown (cf. Theorem 4.1 end last paragraph of Section 6) that $\Gamma\mathscr{I}$ may be described as $\mathrm{ND}(\Gamma\mathscr{I})/\sim$ where, by definition, $D \sim D'$ iff for all ideal theory-morphisms $\phi \mapsto \bar{\phi}$ from $\Gamma\mathscr{T}$ into a arbitrary iterative theory J, we have $|\bar{D}|_J = |\bar{D}'|_J$. [If $D = (\beta; \tau)$ then $|\bar{D}|_J = \beta \cdot (\bar{\tau}^\dagger, 1_p)$.]

A morphism $n \xrightarrow{\phi} p$ in $\Gamma\mathscr{T}$ will be called *primitive* if for all $i \in [n]$, $i \cdot \phi$ has degree 1, i.e., $i \cdot \phi = \gamma_i g_i$, where γ_i is in Γ and g_i is a base morphism.[3] A normal description $D = (\beta; \tau)$ in ND($\Gamma\mathscr{T}$) will be called *primitive* if τ is primitive. By $\Gamma \cdot \mathscr{N}$ we mean the collection of all primitive morphisms and by ND($\Gamma \cdot \mathscr{N}$) we mean the collection of all primitive normal descriptions; $\Gamma \cdot \mathscr{N}$ is a "sort" in the sense of [3] and ND($\Gamma \cdot \mathscr{N}$) is the collection of all normal descriptions of sort $\Gamma \cdot \mathscr{N}$. Since the collection of all normal descriptions of a given sort is closed under composition, source tupling and iteration and contains for each n, p and for each function $[n] \to^b [p]$ the base normal description $(b; 0_p)$, we have ND$(\Gamma \cdot \mathscr{N})/\sim$ is a sub-iterative theory of ND$(\Gamma\mathscr{T})/\sim$ containing "a copy of Γ" and so ND$(\Gamma\mathscr{T})/\sim$ = ND$(\Gamma \cdot \mathscr{N})/\sim$. Thus ND$(\Gamma \cdot \mathscr{N})/\sim$ *is a description of the free iterative theory* $\Gamma\mathscr{I}$.

To prove Theorem 3.1 (and with it Corollary 3.2), we wish to show for primitive normal descriptions $[n] \to^{D_i}_{s_i} [p]$, $i \in [2]$: if $|D_1|_{\Gamma Tr} = |D_2|_{\Gamma Tr}$ then $D_1 \sim D_2$, i.e., for any iterative theory J and for any ideal theory-morphism $\phi \mapsto \bar{\phi}$ of $\Gamma\mathscr{T}$ into J, $|\bar{D}_1|_J = |\bar{D}_2|_J$. In fact, it is enough to prove this for $n = 1$. It is then sufficient to prove:

$$\text{if} \quad [2] \xrightarrow[s]{D} [p] \quad \text{and} \quad 1 \cdot |D|_{\Gamma Tr} = 2 \cdot |D|_{\Gamma Tr} \quad \text{then} \quad 1 \cdot |\bar{D}|_J = 2 \cdot |\bar{D}|_J$$

for this reduces to the former assertion by taking $D = (D_1, D_2)\colon [2] \to [p]$. Now if $D = (\beta; \tau)$ and $1 \mapsto^\beta i$, $2 \mapsto^\beta j$, then the latter assertion reduces to:

$$i \cdot \tau^\dagger = j \cdot \tau^\dagger \text{ in } \Gamma\, Tr \Rightarrow i \cdot \bar{\tau}^\dagger = j \cdot \bar{\tau}^\dagger \text{ in } J, \text{ which is Proposition 3.4 (3.8).}$$

We pause to make the following observation.

PROPOSITION 3.3. (a) *Let* $\tau\colon [s] \to [s + p]$ *be an ideal morphism in an iterative theory. Define the base morphism* $[s] \to^\alpha [s]$ *by the requirement* $i \cdot \alpha = \inf\{k \in [s] \mid k \cdot \tau^\dagger = i \cdot \tau^\dagger\}$. *Then*

$$\alpha \cdot \tau^\dagger = \tau^\dagger \tag{3.1}$$

$$i \cdot \tau^\dagger = j \cdot \tau^\dagger \Leftrightarrow i \cdot \alpha = j \cdot \alpha. \tag{3.2}$$

(b) *The conjunction of* (3.1) *and* (3.2) *is equivalent to* $\alpha\colon [s] \to^\nu [s]/\equiv\, \to^c [s]$, (*i.e.* $\alpha = \nu \cdot c$) *for some* c *where* $i \equiv j \Leftrightarrow i \cdot \tau^\dagger = j \cdot \tau^\dagger$, $[s]/\equiv$ *is the partition induced by the equivalence relation* $\equiv$ *on* $[s]$, ν *takes* $i \in [s]$ *into its equivalence class* $i/\equiv$ *and* c *is a choice function, i.e.* $c(E) \in E$ *where* E *is an* $\equiv$*-equivalence class.*

THEOREM 3.4. (a) *In an iterative theory let* $[s] \to^\tau [s + p]$ *be an ideal morphism and* $[s] \to^\alpha [s]$ *the base morphism of Proposition* 3.3(a). *Define* ψ *by*

$$\psi\colon [s] \xrightarrow{\tau} [s + p] \xrightarrow{\alpha \oplus 1_p} [s + p], \qquad \text{i.e.} \quad \psi = \tau \cdot [\alpha \oplus 1_p].$$

[3] In this section we write $i \cdot \phi$ in place of $\mathbf{i}_n \cdot \phi$ since the source of the morphism ϕ will be clear from context.

Then

$$\psi^\dagger = \tau^\dagger. \tag{3.3}$$

(b) *Suppose the iterative theory is Γ Tr and the ideal morphism τ is primitive. Then ψ is* standard, i.e.

$$i \cdot \psi^\dagger = j \cdot \psi^\dagger \Rightarrow i \cdot \psi = j \cdot \psi: \tag{3.4}$$

Furthermore

$$\alpha \cdot \psi = \psi, \tag{3.5}$$

$$\psi \cdot (\psi, 0_s \oplus 1_p) = \tau \cdot (\psi, 0_s \oplus 1_p), \tag{3.6}$$

$$\bar{\psi}^\dagger = \bar{\tau}^\dagger \text{ in } J, \tag{3.7}$$

$$i \cdot \tau^\dagger = j \cdot \tau^\dagger \text{ in } \Gamma\, Tr \Rightarrow i \cdot \bar{\tau}^\dagger = j \cdot \bar{\tau}^\dagger \text{ in } J. \tag{3.8}$$

Proof. (a)
$$\begin{aligned} \psi \cdot (\tau^\dagger, 1_p) &= \tau \cdot [\alpha \oplus 1_p] \cdot (\tau^\dagger, 1_p) && \text{by definition} \\ &= \tau \cdot (\alpha \cdot \tau^\dagger, 1_p) && \text{by [3, (2.5.16)]} \\ &= \tau \cdot (\tau^\dagger, 1_p) && \text{by (3.1)} \\ &= \tau^\dagger. \end{aligned}$$

Thus (3.3) follows by the unique solution property i.e. by [3, (4.1.3)].

(b) Assume $i \cdot \psi^\dagger = j \cdot \psi^\dagger$ and suppose $i \cdot \tau = \gamma_i \cdot f_i$ for each $i \in [s]$ where f_i is base. By the assumption, (3.3) and definition of ψ we have

$$\gamma_i \cdot f_i \cdot [\alpha \oplus 1_p] \cdot (\tau^\dagger, 1_p) = \gamma_j \cdot f_j \cdot [\alpha \oplus 1_p] \cdot (\tau^\dagger, 1_p).$$

From the unique factorization property in Γ *Tr*, we have $\gamma_i = \gamma_j$ and $f_i \cdot (\alpha \cdot \tau^\dagger, 1_p) = f_i \cdot [\alpha \oplus 1_p] \cdot (\tau^\dagger, 1_p) = f_j \cdot [\alpha \oplus 1_p] \cdot (\tau^\dagger, 1_p) = f_j \cdot (\alpha \cdot \tau^\dagger, 1_p)$. Using (3.1), we obtain

$$f_i \cdot (\tau^\dagger, 1_p) = f_j \cdot (\tau^\dagger, 1_p). \tag{3.9}$$

Let $k \in [s]$ and suppose $k \overset{f_i}{\mapsto} l$, $k \overset{f_j}{\mapsto} l'$. If $l \in [s]$, then so is l' and $l \cdot \tau^\dagger = l' \cdot \tau^\dagger$ from the last equlaty. Using (3.2), we conclude that $l \cdot \alpha = l' \cdot \alpha$. If $l \in s + [p]$, we obtain $l = l'$ from (3.9). It follows then for $l \in [s + p]$, $l \cdot [\alpha \oplus 1_p] = l' \cdot [\alpha \oplus 1_p]$ so that $f_i \cdot [\alpha \oplus 1_p] = f_j \cdot [\alpha \oplus 1_p]$, $\gamma_i \cdot f_i \cdot [\alpha \oplus 1_p] = \gamma_j \cdot f_j \cdot [\alpha \oplus 1_p]$, $i \cdot \tau \cdot [\alpha \oplus 1_p] = j \cdot \tau \cdot [\alpha \oplus 1_p]$ and $i \cdot \psi = j \cdot \psi$. Thus (3.4) is proved.

Now (3.5) readily follows from (3.1), (3.3) and (3.4).

To obtain (3.6) we calculate:

$$\begin{aligned} \psi \cdot (\psi, 0_s \oplus 1_p) &= \tau \cdot [\alpha \oplus 1_p] \cdot (\psi, 0_s \oplus 1_p) \\ &= \tau \cdot (\alpha \cdot \psi, 0_s \oplus 1_p) \\ &= \tau \cdot (\psi, 0_s \oplus 1_p) \qquad \text{by (3.5).} \end{aligned}$$

It remains to prove (3.7) and (3.8). Note first that if τ and ψ are any ideal morphisms satisfying (3.6) (in *any* iterative theory) then

$$\begin{aligned}\psi^\dagger &= \psi \cdot (\psi, 0_s \oplus 1_p) \cdot (\psi^\dagger, 1_p)\\ &= \tau \cdot (\psi, 0_s \oplus 1_p) \cdot (\psi^\dagger, 1_p) = \tau \cdot (\psi^\dagger, 1_p)\end{aligned}$$

so that $\psi^\dagger = \tau^\dagger$. From (3.6), $\bar{\psi} \cdot (\bar{\psi}, 0_s \oplus 1_p) = \bar{\tau} \cdot (\bar{\psi}, 0_s \oplus 1\theta)$ and so $\bar{\psi}^\dagger = \bar{\tau}^\dagger$ by the argument immediately above, which proves (3.7). For the final assertion (3.8) assume $i \cdot \tau^\dagger = j \cdot \tau^\dagger$. By (3.3) and (3.4), $i \cdot \psi = j \cdot \psi$ so that $i \cdot \bar{\psi} = j \cdot \bar{\psi}$ and $i \cdot \bar{\psi}^\dagger = j \cdot \bar{\psi}^\dagger$ from the defining equation for $\bar{\psi}^\dagger$. Since $\bar{\psi}^\dagger = \bar{\tau}^\dagger$, we have $i \cdot \bar{\tau}^\dagger = j \cdot \bar{\tau}^\dagger$. ▌

4

4.1. *Characterizing $\Gamma\,tr$*

In Section 3, it was proved that the morphism $\Gamma\mathscr{I} \to \Gamma\,Tr$ from the iterative theory $\Gamma\mathscr{I}$, freely generated by the genus Γ into $\Gamma\,Tr$, which takes $\gamma \in \Gamma_p$ to the primitive tree $\gamma\colon 1 \to p$, is injective. This yielded one "concrete" description of Γ, namely that given in Corollary 3.2. In this section, we note that the elementary observation contained in Theorem 2.5.1 gives an even simpler description of $\Gamma\mathscr{I}$. Theorem 4.1.1 together with its Corollary constitute the main result of this paper.

Recall that $\Gamma\,tr$ is the iterative subtheory of $\Gamma\,Tr$ generated by the atomic Γ-trees.

THEOREM 4.1.1. *The morphisms in $\Gamma\,tr$ consist precisely of those normal Γ-trees of finite index.*

Proof. Each primitive Γ-tree $1 \to p$ has finite index, and this property is preserved by the operations of composition, source-tupling and iteration (cf. 2.5.4). Thus every morphism in $\Gamma\,tr$ has finite index.

Conversely, suppose $T\colon 1 \to p$ is a normal Γ-tree with finite index s. If $s = 0$, T is $\mathbf{j}_p$,for some $j \in [p]$. Otherwise, by Theorem 2.5.1, T is $1_s \cdot \tau^\dagger$, where $\tau\colon s \to p + s$ is a primitive Γ-tree. Thus, in either case, T is a morphism in $\Gamma\,tr$, completing the proof.

COROLLARY 4.1.2. *The iterative theory, $\Gamma\mathscr{I}$ is isomorphic to $\Gamma\,tr$.*

Proof. By Theorem 4.1.1 and Corollary 3.2.

In section 2.5 it was noted that every Γ-tree of finite index is a component of a finite vector iterate of a primitive Γ-tree while every Γ-tree of infinite index is a component of an infinite vector iterate of a primitive Γ-tree. In order to provide an algebraic theory setting for the discussion of infinite vector iteration, we "replace" (in the next section) the base category consisting of the skeletal category of finite sets by the category of all sets (of all cardinalities). We do not employ a skeletal category here in order to avoid getting involved with cardinal or ordinal arithmetic. Indeed in Section 2.5, in the case that the index of the tree T was ω, we were tempted to describe the primitive tree τ by

$\tau: [\omega] \to [p + \omega]$, even though $p + \omega = \omega$. Nevertheless this notation may be useful. The reader may recall from ordinal arithmetic that $\omega + p \neq \omega$. Thus ordinal arithmetic here suggests a spurious distinction between "right" and left" infinite vector iteration.

4.2. *Algebraic Theories with Base $\mathscr{S}$; Completely Iterative Theories*

The algebraic theories used in Sections 2, 3, and 4.1 (see also [1, 3, 9]) might be more precisely described as "algebraic theories with base $\mathscr{N}$," where $\mathscr{N}$ is the category whose morphisms are functions $[n] \to [p]$. Indeed, in [3], the functions $[n] \to [p]$ were called the "base morphisms" in any algebraic theory. In this section we define the notion of an algebraic theory with base $\mathscr{S}$, where $\mathscr{S}$ is the category of sets. The category $\mathscr{N}$ may be described as the full subcategory of $\mathscr{S}$ determined by restricting the objects of $\mathscr{S}$ to be $[n]$, for $n = 0, 1, 2,\ldots$. We will also indicate how the definitions and most of the results of [3] extend to theories with base $\mathscr{S}$. Theorem 3.1 and its corollary have an interesting generalization in this setting.

DEFINITION 4.2.1. An algebraic theory T with base $\mathscr{S}$ (briefly "$\mathscr{S}$-theory") is a category having the class of all sets as its class of objects. Furthermore, for each set A and each $a \in A$, there is a *distinguished* morphism

$$\mathbf{a}: [1] \to A$$

satisfying

for any family $\phi_a: [1] \to B$ of morphisms indexed by $a \in A$, there is a *unique* morphism (4.2.1)

$$\phi: A \to B$$

such that for each $a \in A$

$$\phi_a: [1] \xrightarrow{\mathbf{a}} A \xrightarrow{\phi} B.$$

ϕ is called the source-tupling of the family $(\phi_a: a \in A)$.

The $\mathscr{S}$-theory T is *nondegenerate* if for $a \neq a'$ in $A = [2]$, the distinguished morphisms $\mathbf{a}, \mathbf{a}': [1] \to A$ are distinct. It follows that if T is nondegenerate and a, a' are distinct members of any set A, then the distinguished morphisms $\mathbf{a}, \mathbf{a}': [1] \to A$ are also distinct.

If T is nondegenerate, a function $f: A \to B$ may be identified with the source-tupling of the morphisms $(\phi_a: [1] \to B \mid a \in A)$, where for each $a \in A$, if $af = b$ then ϕ_a is the distinguished morphism $\mathbf{b}: [1] \to B$. In this way, $\mathscr{S}$ is (isomorphic to) a subtheory of any nondegenerate $\mathscr{S}$-theory.

Henceforth, all $\mathscr{S}$-theories are assumed nondegenerate.

The (isomorphic images of) functions $f: A \to B$ in $\mathscr{S}$ are called the *base morphisms* in T. The distinguished $\mathbf{a}: [1] \to A$ is of course base, being identified with the function $1 \mapsto a$.

A morphism $\phi: [1] \to A$ in an $\mathscr{S}$-theory T is *ideal* if for any morphism $\psi: A \to B$, the composition $\phi \cdot \psi: [1] \to B$ is not base. A morphism $\phi: A \to B$ is *ideal* if for each $a \in A$, $\mathbf{a} \cdot \phi$ is ideal. The algebraic theory T itself is ideal if every nondistinguished morphism $[1] \to A$ is ideal, for every set A.

Assume a *choice* for forming the "disjoint union" $C_1 + C_2$ of the sets C_1, C_2 has been made, along with the corresponding base injections $\iota_i: C_i \to C_1 + C_2$, $i = 1, 2$. It may be shown that the following "universal property" holds in any $\mathscr{S}$-theory.

For any morphisms $\phi_i: C_i \to D$, $i = 1, 2$ (with a common target) there is a unique morphism (4.2.2)

$$\theta: C_1 + C_2 \to D$$

such that

$$\phi_i: C_i \xrightarrow{\iota_i} C_1 + C_2 \xrightarrow{\theta} D, \qquad i = 1, 2, \qquad \text{i.e. } \phi_i = \iota_i \cdot \theta.$$

The morphism θ is called the *source pairing* of ϕ_1, ϕ_2 and is denoted (ϕ_1, ϕ_2).

DEFINITION 4.2.2. An ideal $\mathscr{S}$-theory T is *completely iterative* if for each morphism $\phi: A \to B + A$ there is a unique morphism $\phi^\dagger: A \to B$ satisfying

$$\phi^\dagger = \phi \cdot (1_B, \phi^\dagger). \tag{4.2.3}$$

The morphism $\phi^\dagger$ is the *infinite vector iterate* of ϕ if the cardinality of A is infinite.

Note that Eq. (4.2.3) is the analogue of Eq. (2.3.7) (see Remark 2.3.4). The morphism $(1_B, \phi^\dagger)$ is the source pairing of the identity morphism (function) $1_B: B \to B$ and $\phi^\dagger: A \to B$. It may be shown that the property of being "completely iterative" does not depend on the "choice" made above (preceding (4.2.2)).

We now indicate briefly how the main results of [1, 3] extend to ideal and completely iterative $\mathscr{S}$-theories.

Let T be an ideal $\mathscr{S}$-theory. An *$\mathscr{S}$-normal description $D = (\beta; \tau): A \to_S B$ over T of weight S* consists of a morphism

$$\beta: A \to B + S$$

and an ideal morphism

$$\tau: S \to B + S$$

(where A, B, S are sets; i.e., objects of T). If T is completely iterative, the *behavior* of D, denoted $| D |$, is the morphism

$$| D |: A \xrightarrow{\beta} B + S \xrightarrow{(1_B, \tau^\dagger)} B$$

A *sort* Σ in an ideal $\mathscr{S}$-theory T is a collection $\{\Sigma_A: A \text{ an object in } T\}$, where for each set A, Σ_A is a set of ideal morphisms $[1] \to A$, such that if $[1] \to^\sigma A$ is in Σ_A and $A \to B$

is base, then $[1] \xrightarrow{\sigma} A \xrightarrow{f} B$ is in Σ_B. If Σ is a sort, we let Σ^0 be the collection of all morphisms $\sigma: A \to B$ such that, for each distinguished morphism $\mathbf{a}: [1] \to A$,

$$\mathbf{a} \cdot \sigma \in \Sigma_B .$$

THEOREM 4.2.3. (An analogue for completely iterative theories of part of the Main Theorem in [3]). *Let Σ be a sort in the completely iterative theory T. The least completely iterative subtheory of T containing Σ consists precisely of the behaviors of all normal descriptions $D = (\beta; \tau)$ over T such that $\tau \in \Sigma^0$.*

The proof of Theorem 4.2.3 may be obtained by essentially notational changes in the proof given in [3].

The part of the Main Theorem of [3] that does not generalize to completely iterative theories concerns the relation between "scalar" and "vector" iteration because the generalization admits "infinite vector" iteration. (See Remark 4.2.7). For iterative theories (with base $\mathscr{N}$), scalar iteration is as powerful as vector iteration (see [2]). For completely iterative theories, scalar iteration is weaker than vector iteration, as we will explain below.

By making use of the constructions involved in the proof indicated above of Theorem 4.2.3, and with only minor changes in the argument in [1, Sections 5, 6], one can prove

For any genus $\Gamma = (\Gamma_n: n \in N)$, there is a completely iterative theory freely generated by Γ. (4.2.4)

In fact, by generalizing the notion of genus (or, equivalently, ranked set) a stronger theorem may be proved with no additional labor. An *$\mathscr{S}$-ranked set* consists of a set Γ and a function

$$\rho: \Gamma \to \mathscr{S}$$

where $\mathscr{S}$, here, is merely the class of all sets. Thus an $\mathscr{S}$-ranked set is equivalent to an "$\mathscr{S}$-genus": a collection $\{\Gamma_i: i \in \mathscr{S}\}$ of pairwise disjoint set indexed by $\mathscr{S}$.

THEOREM 4.2.4. (Analogue of [1]). *For any $\mathscr{S}$-genus $\{\Gamma_i: i \in \mathscr{S}\}$, there is a completely iterative theory $\Gamma\mathscr{C}(\mathscr{S})$, freely generated by Γ; i.e., for any completely iterative theory J and any function F taking $\gamma \in \Gamma_i$, $i \in \mathscr{S}$ to an ideal morphism $\gamma F: [1] \to i$ in J, there is a unique $\mathscr{S}$-ideal theory morphism $\bar{F}: \Gamma\mathscr{C}(\mathscr{S}) \to J$ extending F.*

Arguing as in [1], one first shows there is an $\mathscr{S}$-theory, $\Gamma\mathscr{T}(\mathscr{S})$, freely generated by Γ. Then the elements of $\Gamma\mathscr{C}(\mathscr{S})$ can be described as certain equivalence classes of those (primitive) normal descriptions $D = (\beta; \tau): A \to_S B$, over $\Gamma\mathscr{T}(\mathscr{S})$, where for each distinguished $\mathbf{s}: [1] \to S$, $s \in S$, $\mathbf{s} \cdot \tau: [1] \to B + S$ factors uniquely as

$$\mathbf{s} \cdot \tau: [1] \xrightarrow{\gamma} i \xrightarrow{f} B + S$$

for some $\gamma \in \Gamma_i$, $i \in \mathscr{S}$, and some base morphism f.

Again using trees, a "concrete" description of $\Gamma\mathscr{T}(\mathscr{S})$ and $\Gamma\mathscr{C}(\mathscr{S})$ may be given.

DEFINITION 4.2.5. For any set A, an *A-rooted $\mathscr{S}$-tree* consists of:

an $\mathscr{S}$-ranked set $\rho\colon V \to \mathscr{S}$; (4.2.4)

a ("root") function $r\colon A \to V$ (4.2.5)

a ("successor") function $\sigma\colon E \to V$, where $E = \{(v, i) \mid v \in V, i \in v\rho\}$ satisfying the following requirements: (4.2.6)

σ and r are injective functions; (4.2.7)

no element ar, $a \in A$, is in the image of σ; (4.2.8)

any subset V' of V, containing each element ar, $a \in A$ and closed under σ, coincides with V. (4.2.9)

Clearly Definition 4.2.5 is a generalization of "n-rooted tree." Isomorphism of A-rooted $\mathscr{S}$-trees is defined analogously to the numeric case so that if two A-rooted $\mathscr{S}$-trees are isomorphic, they are uniquely isomorphic. In the obvious way now, we may define the notion of an $\mathscr{S}$-tree $T\colon A \to B$, and a ***normal*** $\mathscr{S}$-tree $T\colon A \to B$. (The vertices of a normal $\mathscr{S}$-tree are elements of I^*, the set of finite sequences of elements of I, where $I = \bigcup_{v \in V} \rho(v)$). Note that in general, $\mathscr{S}$-trees are neither locally finite nor locally ordered but they are locally indexed and this indexing, to a great extent, serves as a substitute for the order. If $\rho_\Gamma\colon \Gamma \to \mathscr{S}$ is an $\mathscr{S}$-ranked set, then a (normal) *$\Gamma\mathscr{S}$-tree* $T\colon A \to B$ consists of a (normal) $\mathscr{S}$-tree $T'\colon A \to B$ together with a labelling function $\lambda\colon V^- \to \Gamma$ (where V^- is the set of non-termini) such that the following diagram commutes.

$$\begin{tikzcd}
V^- \arrow[r, "\rho"] \arrow[d, "\lambda"'] & \mathscr{S} \\
\Gamma \arrow[ur, "\rho_\Gamma"'] &
\end{tikzcd}$$

The atomic $\Gamma\mathscr{S}$-tree $\gamma\colon [1] \to i$ corresponding to $\gamma \in \Gamma_i$, $i \in \mathscr{S}$ is indicated in Fig. 4.2.1.[4] The set of vertices of the normal $\Gamma\mathscr{S}$-tree indicated in the figure consists of the empty sequence Λ, and all words a in i^* of length one such that $a \in i$; the termini function takes $a \in i$ to a.

[4] In accordance with the discussion preceding Theorem 4.2.4, Γ_i is the set of elements of Γ whose rank is i; i.e. $x \in \Gamma_i$ if $x\rho_\Gamma = i$.

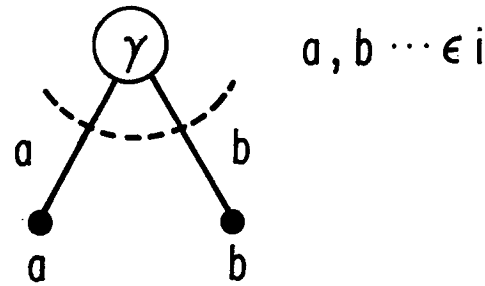


Figure 4.2.1

With the by now familiar definitions of composition and the distinguished ("root") trees $\mathbf{a}$: $[1] \to A$, it may be shown in a straightforward manner that the collection of normal $\Gamma\mathscr{S}$-trees T: $A \to B$ forms a completely iterative theory; we denote this theory $\Gamma\, Tr(\mathscr{S})$.

Clearly, the least subtheory (note: *not* completely iterative subtheory) of $\Gamma\, Tr(\mathscr{S})$ containing the normal primitive $\Gamma\mathscr{S}$-trees γ: $[1] \to i$, for $\gamma \in \Gamma_i$, is (isomorphic to) the $\mathscr{S}$-theory $\Gamma\mathscr{T}(\mathscr{S})$, freely generated by Γ. The morphisms in this copy of $\Gamma\mathscr{T}(\mathscr{S})$ consist of those normal $\Gamma\mathscr{S}$-trees having no infinite paths.

The argument of Section 3 carries over to prove

THEOREM 4.2.6. *The unique ideal $\mathscr{S}$-theory morphism F: $\Gamma\mathscr{C}(\mathscr{S}) \to \Gamma\, Tr(\mathscr{S})$, taking $\gamma \in \Gamma_i$, to the primitive normal $\Gamma\mathscr{S}$-tree γ: $[1] \to i$, for all $i \in \mathscr{S}$, in $\Gamma\, Tr(\mathscr{S})$, is an injection.*

The morphism F takes the equivalence class of the primitive normal description $D = (\beta; \tau)$: $A \to_S B$ to the morphism $\beta \cdot (1_B, \tau^\dagger)$ in $\Gamma\, Tr(\mathscr{S})$.

The idea used to prove Theorem 4.1.1 (and Theorem 2.5.1) can be used to show that F is not only injective, but surjective as well.

Indeed, let ϕ: $[1] \to A$ be any normal $\Gamma\mathscr{S}$-tree T. Let S be the set of all non-trivial normal trees isomorphic to descendancy trees of T. Define the primitive morphism τ: $S \to A + S$ by the requirement that for each $s \in S$

$$\mathbf{s} \cdot \tau\colon [1] \xrightarrow{\gamma_s} i \xrightarrow{f_s} A + S$$

where $\gamma_s \in \Gamma_i$ is the label of the vertex v, where $vD_T \approx s$, and f_s is the base function $i \to A + S$ defined by:

For $b \in i$ if edge (v, b) points to v' in T then

(a) if v' is a non-terminus and $v'D_T \approx s'$ define $bf_s = s'$

(b) if v' is a terminus of T labelled $a \in A$ then $bf_s = a$.

Let β: $[1] \to A + S$ be the base function taking 1 to s_0 where s_0 is the normal tree isomorphic to T.

It may be verified that $\beta \cdot (1_A, \tau^\dagger) = \phi$ in $\Gamma\, Tr(\mathscr{S})$. Thus for $\mathscr{S}$-theories we have

THEOREM 4.2.7. *$\Gamma\mathscr{C}(\mathscr{S})$ is isomorphic to $\Gamma\, Tr(\mathscr{S})$.*

Remark 4.2.8. Even when $\rho: \Gamma \to \mathscr{S}$ is an $\mathscr{S}$-ranked set, such that for each $\gamma \in \Gamma$, $\gamma\rho = [n]$ for some n, we can use the completely iterative theory $\Gamma\, Tr(\mathscr{S})$ to show that "scalar" and infinite vector iteration are not equivalent. Indeed, suppose $\gamma\rho = [1]$ for all γ in the countably infinite set $\Gamma = \{\gamma_1, \gamma_2, ...\}$. Let τ: $[1] \to \phi$ be the normal $\Gamma\mathscr{S}$-tree indicated by Fig. 4.2.2. τ does not belong to the least subtheory T of $\Gamma\, Tr(\mathscr{S})$ containing the atomic trees γ closed under *scalar iteration* since τ has infinitely many non-isomorphic subtrees. These matters are more fully explained below.

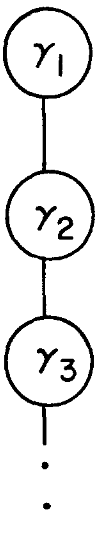


Figure 4.2.2

An ideal $\mathscr{S}$-theory is *finitely iterative* (respectively *scalar iterative*) if for any finite set A (respectively, any singleton set A) and any ideal morphism ϕ: $A \to B + A$ there is a unique morphism $\phi^\dagger$: $A \to B$ such that $\phi^\dagger = \phi \cdot (1_B, \phi^\dagger)$. The morphism $\phi^\dagger$ is called the "finite vector iterate of ϕ" (respectively, the "scalar iterate of ϕ").

Call an ideal morphism ϕ "numerical" if ϕ: $[s] \to B + [s]$ for some set B, some number $s \geqslant 0$; it may be easily shown that an ideal $\mathscr{S}$-theory T is finitely iterative iff every numerical ideal morphism has a finite vector iterate; also T is scalar iterative iff every numerical ideal morphism with source [1] has a scalar iterate. Using these facts and the argument of [2], one obtains

PROPOSITION 4.2.9. *Every scalar iterative $\mathscr{S}$-theory is finitely iterative.*

If the rank of each γ in the ranked set Γ is a finite set, Γ is called *finitary*.

THEOREM 4.2.10. *Suppose that Γ is a finitary ranked set. Let T be the least subtheory of $\Gamma\, Tr(\mathscr{S})$ closed under scalar iteration (or equivalently, by 4.2.9, closed under finite iteration). Then*

(a) *Every function taking* $\gamma\colon [1] \to A$, $\gamma \in \Gamma$ *to* $\bar{\gamma}\colon [1] \to A$ *in a scalar iterative theory* J *extends uniquely to a* $\mathscr{S}$*-theory morphism* $\mathbf{F}\colon T \to J$ *(briefly,* T *is the scalar iterative* $\mathscr{S}$*-theory freely generated by* Γ*).*

(b) *The morphisms* $A \to B$ *in* T*, where* A *is a singleton set, consist precisely of those normal* $\Gamma\mathscr{S}$*-trees in* $\Gamma\ Tr$ *with finite descendency index.*

The proof of this theorem makes use of the observation that any Γ tree $T\colon [1] \to A$ of finite index can be written as a composition

$$T\colon [1] \xrightarrow{T'} [n] \xrightarrow{f} A$$

where f is an injective function.

Problem. If the ranked set Γ is not finitary and if T is the scalar iterative subtheory of $\Gamma\ Tr(\mathscr{S})$ generated by Γ, is T freely generated by Γ?

We suspect the answer to the problem is "yes."

APPENDIX I: Comparison of Definitions of Rooted Trees

A popular definition of "tree" (cf. e.g. [10]) has it that a tree is a connected acyclic (undirected) graph. In this appendix, the connection between this popular definition and the definition of "rooted tree" given in Section 2.1 is discussed.

In [10] a graph G consists of a finite set V (of vertices or nodes) and a set E of (edges or lines) i.e., doubletons $\{v, v'\}$, $v, v' \in V$, $v \neq v'$. We immediately delete the requirement that V be finite (since our trees are permitted to be infinite) but otherwise embrace this definition. A *path* from v to v' in $G = (V, E)$ is a word $p = v_0 v_1 \cdots v_n$ in V^+ (the set of all finite sequences of elements of V of positive length) such that $v = v_0$, $v' = v_n$, $\{v_{i-1}, v_i\} \in E$ for all $i \in [n]$ and whenever $i \neq j$, $v_i \neq v_j$. (*Remark*: if p is a path in G from v to v then $n = 0$; thus a path from v to v is unique.) The *edge count* of p is n; its *node count* is $1 + n$. A *cycle* in a graph G with vertex set V is a word of edge count $\geqslant 3$ of the form vwv, where $v \in V$, $w \in V^+$ and both vw and wv are paths in G; a graph G is *acyclic* if there are no cycles in G. The graph G is *connected* if for any $v, v' \in V$, there is a path from v to v'.

We formally record the "popular" definition of "tree."

Definition A. A graph $G = (V, E)$, is a *tree* if G is connected and acyclic.

Proposition B. *For a graph* $G = (V, E)$ *the following conditions are equivalent.*

(B1) *G is connected and acyclic; i.e., G is a tree.*

(B2) *For any vertices $a, b \in V$, there is a unique path from a to b.*

Proof. Since the implication (B2) $\Rightarrow$ (B1) is easily established we prove only that if p_1 and p_2 are paths in G from a to b then $p_1 = p_2$. Suppose the edge counts of p_1

and p_2 are e_1 and e_2. The proof proceeds by induction on $e_1 + e_2$. The case $e_1 + e_2 \leqslant 2$ is easily disposed of using the parenthetical remark above.

Suppose now $e_1 + e_2 \geqslant 3$ so that $a \neq b$. Suppose too, $p_1 = u_1 b$, $p_2 = u_2 b$, where $u_1, u_2 \in V^*$. Then the word $u_1 b u_2^{\smile}$ (where $u_2^{\smile}$ is the word "u_2 in reverse order") which has edge count $e_1 + e_2$ would be a cycle if no vertex, other than a, occurred more than once in that word. But G is acyclic. Hence there is a vertex $v \neq a$ which occurs in $u_1 b u_2^{\smile}$ more than once. Now $v \neq b$ since no vertex occurs more than once in p_i for each $i \in [2]$. Making use of the "distinctness" property of p_i again, we conclude there is an occurrence of v "to the left of b" and one "to the right of b" in $u_1 b u_2^{\smile}$. Thus we have

$$p_1 = aw_1 v w'_1 b$$

$$p_2 = aw_2 v w'_2 b.$$

By inductive assumption, $aw_1 v = aw_2 v$ and $vw'_1 b = vw'_2 b$ so that $p_1 = p_2$. ▮

Let $G = (V, E)$ be a graph and suppose $\epsilon \in V$. We call (G, ϵ) a *rooted* graph; ϵ is the *root* of (G, ϵ). Suppose G is *connected* and *acyclic*. We define the *immediate successor relation* $s \subseteq V \times V$ in G *as follows*: $(v_1, v_2) \in s$ iff the unique path (cf. Proposition B) from ϵ to v_2 is of the form $\cdots v_1 v_2$. Now suppose (G, ϵ) is *locally finite* i.e., for each $v \in V$, the set of $v' \in V$ such that $(v, v') \in s$, is finite. Let $\rho(v)$ be the number of immediate successor of v. (Actually "local finiteness" is independent of ϵ; it may be stated: for each $v \in V$, the number of doubletons $e \in E$ such that $v \in e$, is finite.) Suppose further that for each $v \in V$, the (finite) set of immediate successors of v is *ordered*. We then define $\sigma(v, i) = v'$ if v' is the ith successor of v, $i \in [\rho(v)]$. We take it as generally known that the data $(V, \rho, \sigma, \epsilon)$ satisfies Definition 2.1.2 in the case of singly rooted trees and focus attention on the reverse direction.

Now suppose $T = (V, \rho, \sigma, \epsilon)$ satisfies Definition 2.1.2 in the case $n = 1$. We define the graph $G = (V, E)$ by taking $E = \{\{v, v'\} \mid \sigma(v, i) = v' \text{ for some } i \in [\rho(v)]\}$ and wish to show that G is connected and acyclic.

We first observe that by the principle of tree induction (cf. 2.1.13), there exists at most one function $l\colon V \to N$ satisfying: $l(\epsilon) = 0$, $l(v) = n \Rightarrow l(\sigma(v, i)) = 1 + n$ for each $i \in [\rho(v)]$. The proof that there exists at least one function satisfying these conditions makes use of (2.1.1), (2.1.2) and (2.1.3). (This function is sometimes called, level or length or depth or $\cdots$; if T is normal $l(v)$ is the length of v.) To show that G is connected, assume $v, v' \in V$ and suppose $l(v) \leqslant l(v')$. We construct the sequence (of *ancestors* of v')

$$(v_0, v_1, v_2, \ldots, v_n)$$

with $v_0 = v'$ and v_i an immediate successor of v_{i+1}, $0 \leqslant i < n$, $v_n = \epsilon$, (so that $n = l(v')$). Let i, $0 \leqslant i \leqslant n$, be the smallest index such that v_i is an ancestor of v (such an i exists since ϵ is a common ancestor of v and v'). Then the sequence

$$(v_0, v_1, v_2, \ldots, v_i = w_i, w_{i+1}, w_{i+2}, \ldots, w_{i+p}), \qquad p \geqslant 0,$$

where w_{i+j} is an immediate successor of w_{i+j-1} for $j \in [p]$ and $w_{i+p} = v$, is a path in G.

It remains to show that G is acyclic. Suppose the contrary so that $(v_1, v_2, \ldots, v_n)$, $v_1 = v_n$, $n \geqslant 4$, is a cycle. Since σ is injective, either all the (ordered) pairs (v_1, v_2), $(v_2, v_3), \ldots, (v_n, v_1)$ are in the immediate successor relation s or all the reversed pairs are in s. Suppose the former. Then $l(v_1) < l(v_2) < \cdots < l(v_n) < l(v_1)$. The contradiction compels the conclusion that G is acyclic.

APPENDIX II: Replacing Finite Unsuccessful Paths by Infinite Unsuccessful Paths

In Subsections 2.2, 2.3, 2.4 certain augmented matrices, the "surmatrices," were used to (faithfully) represent rooted trees and rooted Γ-trees. When $T: 1 \to 1$ is a tree, T is represented by the surmatrix $(A; a)$ where A is the set of all labels of paths from the root of T to a terminus; a is the set of all labels of paths from the root to a non-terminus. In 2.2 and 2.3, both A and a are subsets of Σ^* where $\Sigma = [\omega]$; in 2.4, A and a are subsets of Σ^* where $\Sigma = \Gamma_0 \cup \bigcup_{n \geqslant 1} (\Gamma_n \times [n])$.

If we call a path in a tree from a root to a leaf *successful* then the elements of A are labels of successful paths. The elements of a may be partitioned into the set a_1 of labels of successful paths (which end with a non-terminus) and the set a_2 of labels of unsuccessful paths (which begin with the root). Thus $a_1 \cup a_2 = a$, $a_1 \cap a_2 = \varnothing$.

The two cases mentioned above may be (essentially) subsumed under a single case by passing to $\mathscr{S}$-ranked sets Γ. The case $\Sigma = [\omega]$ is then replaced by the case $\Gamma = \{\gamma, \gamma_0\}$ with $\rho(\gamma) = [\omega]$, $\rho(\gamma_0) = [0]$. Then i gets replaced by the "letter" (γ, i) so that the "new" $A \cup a_1 \subseteq (\{\gamma\} \times [\omega])^*$ while $u \in a_2$ gets replaced by $u\gamma_0$ so that the "new" $a_2 \subseteq (\{\gamma\} \times [\omega])^* \{\gamma_0\}$.

The main objective of this Appendix is to show that the tree $T: 1 \to 1$ may equally well be represented by $(A; a_1 \cup b)$ where b is the set of labels of infinite paths in T which begin with the root.

Let Σ be any set. As usual Σ^* devotes the set of all finite sequences (words) of elements of Σ while Σ^∞ denotes the set of all infinite sequences (functions) $f: [\omega] \to \Sigma$, where $[\omega] = \{1, 2, 3, \ldots\}$ is the set of positive integers. Thus an element of Σ^∞ may be called an "infinite word on Σ." There is a partial ordering $\leqslant$ on $\Sigma^* \cup \Sigma^\infty$: $u \leqslant v$ iff u is a prefix of v (i.e., u is an initial segment of v). With respect to this ordering all elements of Σ^∞ are maximal and two distinct infinite words are incomparable. We write $u < v$ if u is a *proper prefix* of v, i.e., $u \leqslant v$ and $u \neq v$. If X is a set, we write $X^\wedge$ for the set of all subsets of X.

We define the function

$$\mathit{pref}: (\Sigma^* \cup \Sigma^\infty)^\wedge \to \Sigma^{*\wedge}$$

by

$$\mathit{pref}(M) = \{u \in \Sigma^* \mid u \leqslant v, \text{ for some } v \in M\}, \qquad M \subseteq \Sigma^* \cup \Sigma^\infty.$$

Thus, $\mathit{pref}(M)$ is the set of all finite prefixes of words in M. If M is a singleton consisting of v alone, we write "$\mathit{pref}(v)$" for $\mathit{pref}(M)$. Clearly,

PROPOSITION A. $M_1 \subseteq M_2 \subseteq \Sigma^* \cup \Sigma^\infty \Rightarrow \mathit{pref}\, M_1 \subseteq \mathit{pref}\, M_2 \subseteq \Sigma^*$.

A subset F of Σ^* is *prefix closed* if $F = \mathit{pref}\, F$. Let $\mathit{Pref}(\Sigma^*)$ be the set of all prefix closed subsets of Σ^*.

We define the function

$$\mathit{lim}\colon \Sigma^{*\wedge} \to \Sigma^{\infty\wedge}$$

by the following requirement: for $v \in \Sigma^\infty$, $F \subseteq \Sigma^*$

$$v \in \mathit{lim}\, F \qquad \text{iff} \quad \mathit{pref}\, v \subseteq F.$$

The following two propositions are obvious.

PROPOSITION B. $F_1 \subseteq F_2 \subseteq \Sigma^* \Rightarrow \mathit{lim}\, F_1 \subseteq \mathit{lim}\, F_2 \subseteq \Sigma^\infty$.

PROPOSITION C. *For* $I \subseteq \Sigma^\infty$, $I \subseteq \mathit{lim}\, (\mathit{pref}\, I)$.

Before stating the theorem which leads to our main objective, we require two more functions. The function

$$\mathit{max}\colon (\Sigma^* \cup \Sigma^\infty)^\wedge \to \Sigma^{*\wedge}$$

is defined as follows: for $M \subseteq \Sigma^* \cup \Sigma^\infty$

$$\mathit{max}(M) = \{u \in M \cap \Sigma^* \mid u < v \text{ for no } v \in M\}$$

i.e., $\mathit{max}(M)$ is the set of finite words in M which are maximal in M. Clearly,

PROPOSITION D. $F \in \mathit{Pref}\, \Sigma^* \Rightarrow (F - \max(F)) \in \mathit{Pref}\, \Sigma^*$; $\lim F = \mathit{lim}(F - \max(F))$ *for* $F \in \mathrm{Pref}\, \Sigma^*$.

The function

$$\mu\colon \mathit{Pref}\, \Sigma^* \to (\Sigma^* \cup \Sigma^\infty)^\wedge$$

is defined by the following, where $F \subseteq \Sigma^*$, $F = \mathit{pref}\, F$:

$$\mu(F) = \mathit{max}(F) \cup \mathit{lim}(F).$$

THEOREM E. *The function* μ *is injective i.e. for* $F_1, F_2 \in \mathit{Pref}\, \Sigma^*$, $\mu(F_1) = \mu(F_2) \Rightarrow F_1 = F_2$.

With A, a, a_1, a_2, b as in the beginning of this Appendix, we have $A \cup a$ is prefix closed, $\max(A \cup a) = A \cup a_1$, $a_2 = (A \cup a) - (A \cup a_1)$ is prefix closed (by Proposition D) and $b = \mathit{lim}\, a_2$.

COROLLARY F. *With A, a_1, a_2, b as above, the function which takes $(A; a_1 \cup a_2)$ into $(A; a_1 \cup b)$ is injective.*

Proof of Theorem E. Suppose $F_1, F_2 \in Pref\,\Sigma^*$ and $\mu(F_1) = \mu(F_2)$. Then $max\,F_1 = max\,F_2$ and $lim\,F_1 = lim\,F_2$. Suppose $u \in F_1$. Either $u \leqslant v$ for some $v \in \max F_1$ or else there is an infinite chain: $u < u_1 < u_2 < \cdots$, $u_i \in F_1$, so that the unique $v \in F_1{}^\infty$ satisfying $u_i < v$ for all i, is in $lim\,F_1$. In the former case, $v \in max\,F_2 \subseteq F_2$ and so, by prefix closure, $u \in F_2$. In the latter case, $v \in lim\,F_2$ and $u \in pref(v) \subseteq F_2$. Thus, in either case, $u \in F_1 \Rightarrow u \in F_2$, i.e., $F_1 \subseteq F_2$. By symmetry we obtain $F_2 \subseteq F_1$ which concludes the proof.

We now ask: Which subsets of $\Sigma^* \cup \Sigma^\infty$ are in the image of μ?

THEOREM G. (a) *Suppose $F \cup f \in Pref\,\Sigma^*$ and $F = \max(F \cup f)$. Let $g = lim(F \cup f)$ so that $\mu(F \cup f) = F \cup g$. Then $F \subseteq max(F \cup g)$ and $lim\,pref(F \cup g) \subseteq g$.*

(b) *Suppose $F \subseteq \Sigma^*$, $g \subseteq \Sigma^\infty$, $F \subseteq max(F \cup g)$, $lim\,pref(F \cup g) \subseteq g$. Then $\mu(pref(F \cup g)) = F \cup g$.*

Proof. (a) Let $u \in F$. If $u \notin max(F \cup g)$ then $u < v \in g$ for some v and so $u < w \in pref\,v \subseteq F \cup f$. We conclude $u \notin max(F \cup f)$ which contradicts the supposition $F = max(F \cup f)$. Thus: $F \subseteq max(F \cup g)$. Now:

$$\begin{aligned} pref(F \cup g) &\subseteq pref\,F \cup pref\,g \\ &\subseteq pref\,F \cup pref(F \cup f) \\ &\subseteq F \cup f. \end{aligned}$$

Thus: $lim\,pref(F \cup g) \subseteq lim(F \cup f) = g$.

(b) From the supposition $F \subseteq \max(F \cup g)$, we conclude $F \subseteq max\,pref(F \cup g)$. Now:

$$\begin{aligned} max\,pref(F \cup g) \subseteq \max(pref\,F \cup pref\,g) &\subseteq max\,pref\,F \cup max\,pref\,g \\ &\subseteq F \cup \varnothing \subseteq F. \end{aligned}$$

Thus: $F = max\,pref(F \cup g)$.

Now, $g \subseteq lim\,pref\,g \subseteq lim\,pref(F \cup g)$ while by supposition the opposite inclusion holds. Thus $g = lim\,pref(F \cup g)$ which concludes the proof.

Observation. As a point of independent interest suggested by the condition $lim\,pref(F \cup g) \subseteq g$, $F \subseteq \Sigma^*$, $g \subseteq \Sigma^\infty$, we observe that the condition $lim\,pref\,g \subseteq g$, (which is implied by the previous condition), i.e., $g = lim\,pref\,g$, is equivalent to the condition that g is topologically closed if Σ is given the discrete topology and Σ^∞ the induced product topology. Explicitly, $lim\,pref\,g \subseteq g$ iff g is the complement of an arbitrary union of sets of the form

$$A_1 \times A_2 \times \cdots \times A_n \times \Sigma^\infty = A_1 \times A_2 \times \cdots \times A_n \times \Sigma \times \Sigma \times \cdots$$

where, for each $i \in [n]$, $A_i \subseteq \Sigma$.

If $F, G \subseteq \Sigma^*$ and $f, g \subseteq \Sigma^* \cup \Sigma^\infty$, we define $(F; f) \cdot (G; g) = (FG; f \cup Fg)$. It is straightforward but tedious to verify directly that

PROPOSITION H. *The function that takes* $(A; a_1 \cup a_2)$ *into* $(A; a_1 \cup b)$, *where* $b = \mathit{lim}\ a_2 = \lim(A \cup a_1 \cup a_2)$, *preserves composition (thus giving rise to an isomorphism).*

The above considerations extend without difficulty to the case $T: n \to p$ and $n \times p$ augmented matrices.

Certain trees do not require *augmented* matrices for their representation. The fact that finite trees have a "more efficient" representation was mentioned in Section 2.4. There are other trees as well, however, which can be faithfully represented by labels of successful paths only.

DEFINITION. Call a tree $T: n \to p$ *biaccessible* if every vertex of the tree lies on a successful path. (Thus finite trees are biaccessible.) The corresponding notion for 1×1 surmatrices $(A; a_1 \cup a_2)$ where $A \cup a_1 \cup a_2)$ is prefix closed and $A \cup a_1 = \max(A \cup a_1 \cup a_2)$ is given by the following.

PROPOSITION I. *A tree* $T: 1 \to 1$ *is biaccessible iff its surrogate* $(A; a_1 \cup a_2)$ *satisfies* $a_2 \subseteq \mathit{pref}(A \cup a_1)$ *or, equivalently,* $A \cup a_1 \cup a_2 = \mathit{pref}(A \cup a_1)$.

PROPOSITION J. $\mathit{lim}\ a_2 \subseteq \mathit{lim}\ \mathit{pref}(A \cup a_1) \Leftrightarrow a_2 \subseteq \mathit{pref}(A \cup a_1)$.

Proof $\Rightarrow$. $\mathit{lim}\ a_2 \subseteq \mathit{lim}\ \mathit{pref}(A \cup a_1) \subseteq \mathit{lim}\ \mathit{pref}(A \cup a_1 \cup a_2) \subseteq \mathit{lim}(A \cup a_1 \cup a_2) \subseteq \mathit{lim}\ a_2$ so that $\mathit{lim}\ \mathit{pref}(A \cup a_1) = \mathit{lim}\ a_2$. Notice that $A \cup a_1 = \mathit{max}\ \mathit{pref}(A \cup a_1)$ so that $\mu(\mathit{pref}(A \cup a_1)) = A \cup a_1 \cup \lim a_2 = \mu(A \cup a_1 \cup a_2)$ and by the injectiveness (Theorem E) of μ, we have $(A \cup a^0 \cup a_2) = \mathit{pref}(A \cup a^0)$ which proves $\Rightarrow$. The opposite implication is obvious (Proposition B).

ACKNOWLEDGMENT

We are indebted to Martha Mierce for her painstaking attention to the manuscript (including locating misprints) resulting in the production of a document pleasing to the eye.

REFERENCES

1. S. L. BLOOM AND C. C. ELGOT. The Existence and Construction of Free iterative Theories, *J. Comput. System Sci.* **12** (1976), 305–318.
2. S. L. BLOOM, S. GINALI, AND J. D. RUTLEDGE, Scalar and Vector Iteration, *J. Comput. System Sci.* **14** (1977), 251–256.
3. C. C. ELGOT, Monadic computation and iterative algebraic theories, *in* "Logic Colloquium 1973," Vol. 80, "Studies in Logic" (H. E. Rose and J. C. Shepherdson, Eds.), North-Holland, Amsterdam, 1975.
4. C. C. ELGOT, Matricial theories, *J. Algebra* **42** (1976), 391–421.

5. C. C. ELGOT, Structured programming with and without GO TO statements, *IEEE Trans. Software Eng. SE-2*, No. 1 (March 1976); Erratum and Corrigendum (September 1976).
6. D. E. KNUTH, "The Art of Computer Programming," Vol. I, Addison–Wesley, Reading, Mass. 1969.
7. E. ENGELER, Structure and meaning of elementary programs, *in* "Symposium on Semantics of Algorithmic Languages Proceedings," pp. 89–101, Springer–Verlag, New York/Berlin, 1971.
8. E. G. MANES, "Algebraic Theories," Academic Press, New York, 1976.
9. S. EILENBERG AND J. B. WRIGHT, Automata in general algebras, *Inform. Contr.* 11 (1967), 452–470.
10. F. HARARY, "Graph Theory," Addison–Wesley, Reading, Mass. 1969.
11. J. GOGUEN, J. THATCHER, E. WAGNER, AND J. WRIGHT, Initial algebra semantics and continuous algebras, *J. Assoc. Comput. Mach.*, to appear.
12. M. WAND, "Mathematical Foundations of Language Theory," Dissertation. Project MAC, MIT, 1973.
13. S. GINALI, "Iterative Algebraic Theories, Infinite Trees, and Program Schemata," Dissertation, University of Chicago, June 1976.
14. W. LAWVERE, Functorial semantics of algebraic theories, *Proc. Nat. Acad. Sci. U.S.A.* **50** (1963), 869–872.