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Singular stochastic equations on Hilbert spaces: Harnack inequalities for their transition semigroups ☆

Giuseppe Da Prato [b], Michael Röckner [c,d], Feng-Yu Wang [a,e,*]

[a] *School of Math. Sci. and Lab. Math Com. Sys., Beijing Normal University, 100875, China*
[b] *Scuola Normale Superiore di Pisa, Italy*
[c] *Faculty of Mathematics, University of Bielefeld, Germany*
[d] *Department of Mathematics and Statistics, Purdue University, W. Lafayette, 47906 IN, USA*
[e] *Department of Mathematics, Swansea University, Singleton Park, SA2 8PP, Swansea, UK*

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Abstract

We consider stochastic equations in Hilbert spaces with singular drift in the framework of [G. Da Prato, M. Röckner, Singular dissipative stochastic equations in Hilbert spaces, Probab. Theory Related Fields 124 (2) (2002) 261–303]. We prove a Harnack inequality (in the sense of [F.-Y. Wang, Logarithmic Sobolev inequalities on noncompact Riemannian manifolds, Probab. Theory Related Fields 109 (1997) 417–424]) for its transition semigroup and exploit its consequences. In particular, we prove regularizing and ultra-boundedness properties of the transition semigroup as well as that the corresponding Kolmogorov operator has at most one infinitesimally invariant measure μ (satisfying some mild integrability conditions). Finally, we prove existence of such a measure μ for noncontinuous drifts.

Keywords: Stochastic differential equations; Harnack inequality; Monotone coefficients; Yosida approximation; Kolmogorov operators

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* Corresponding author at: School of Math. Sci. and Lab. Math Com. Sys. Beijing Normal University, 100875, China.
E-mail address: wangfy@bnu.edu.cn (F.-Y. Wang).

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1. Introduction, framework and main results

In this paper we continue our study of stochastic equations in Hilbert spaces with singular drift through its associated Kolmogorov equations started in [6]. The main aim is to prove a Harnack inequality for its transition semigroup in the sense of [16] (see also [1,14,17] for further development) and exploit its consequences. See also [12] for an improvement of the main results in [14] concerning generalized Mehler semigroups. To describe our results more precisely, let us first recall the framework from [6].

Consider the stochastic equation

$$\begin{cases} dX(t) = \big(AX(t) + F\big(X(t)\big)\big)\,dt + \sigma\,dW(t), \\ X(0) = x \in H. \end{cases} \tag{1.1}$$

Here H is a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$, $W = W(t)$, $t \geqslant 0$, is a cylindrical Brownian motion on H defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ and the coefficients satisfy the following hypotheses:

(H1) $(A, D(A))$ is the generator of a C_0-semigroup, $T_t = e^{tA}$, $t \geqslant 0$, on H and for some $\omega \in \mathbb{R}$

$$\langle Ax, x\rangle \leqslant \omega|x|^2, \quad \forall x \in D(A). \tag{1.2}$$

(H2) $\sigma \in L(H)$ (the space of all bounded linear operators on H) such that σ is positive definite, self-adjoint and
 (i) $\int_0^\infty (1 + t^{-\alpha})\|T_t\sigma\|_{HS}^2\,dt < \infty$ for some $\alpha > 0$, where $\|\cdot\|_{HS}$ denotes the norm on the space of all Hilbert–Schmidt operators on H.
 (ii) $\sigma^{-1} \in L(H)$.

(H3) $F : D(F) \subset H \to 2^H$ is an m-dissipative map, i.e.,

$$\langle u - v, x - y\rangle \leqslant 0, \quad \forall x, y \in D(F),\ u \in F(x),\ v \in F(y),$$

("dissipativity") and

$$\mathrm{Range}(I - F) := \bigcup_{x\in D(F)} \big(x - F(x)\big) = H.$$

Furthermore, $F_0(x) \in F(x)$, $x \in D(F)$, is such that

$$\big|F_0(x)\big| = \min_{y\in F(x)} |y|.$$

Here we recall that for F as in (H3) we have that $F(x)$ is closed, nonempty and convex.

The corresponding Kolmogorov operator is then given as follows: Let $\mathcal{E}_A(H)$ denote the linear span of all real parts of functions of the form $\varphi = e^{i\langle h,\cdot\rangle}$, $h \in D(A^*)$, where A^* denotes the adjoint operator of A, and define for any $x \in D(F)$,

$$L_0\varphi(x) = \frac{1}{2}\,\mathrm{Tr}\big(\sigma^2 D^2\varphi(x)\big) + \big\langle x, A^* D\varphi(x)\big\rangle + \big\langle F_0(x), D\varphi(x)\big\rangle, \quad \varphi \in \mathcal{E}_A(H). \tag{1.3}$$

Additionally, we assume:

(H4) There exists a probability measure μ on H (equipped with its Borel σ-algebra $\mathcal{B}(H)$) such that
  (i) $\mu(D(F)) = 1$,
  (ii) $\int_H (1 + |x|^2)(1 + |F_0(x)|)\mu(dx) < \infty$,
  (iii) $\int_H L_0\varphi d\mu = 0$ for all $\varphi \in \mathcal{E}_A(H)$.

Remark 1.1. (i) A measure for which the last equality in (H4) (makes sense and) holds is called *infinitesimally invariant* for $(L_0, \mathcal{E}_A(H))$.

(ii) Since ω in (1.2) is an arbitrary real number we can relax (H3) by allowing that for some $c \in (0,\infty)$

$$\langle u - v, x - y\rangle \leqslant c|x - y|^2, \quad \forall x, y \in D(F),\ u \in F(x),\ v \in F(y).$$

We simply replace F by $F - c$ and A by $A + c$ to reduce this case to (H3).

(iii) At this point we would like to stress that under the above assumptions (H1)–(H4) (and (H5) below) because F_0 is merely measurable and σ is not Hilbert–Schmidt, it is unknown whether (1.1) has a strong solution.

(iv) Similarly as in [6] (see [6, Remark 4.4] in particular) we expect that (H2)(ii) can be relaxed to the condition that $\sigma = (-A)^{-\gamma}$ for some $\gamma \in [0, 1/2]$. However, some of the approximation arguments below become more involved. So, for simplicity we assume (H2)(ii).

The following are the main results of [6] which we shall use below.

Theorem 1.2. *(Cf. [5, Theorem 2.3 and Corollary 2.5].) Assume* (H1)*,* (H2)(i)*,* (H3) *and* (H4)*. Then for any measure μ as in $(H4)$ the operator $(L_0, \mathcal{E}_A(H))$ is dissipative on $L^1(H,\mu)$, hence closable. Its closure $(L_\mu, D(L_\mu))$ generates a C_0-semigroup P_t^μ, $t \geqslant 0$, on $L^1(H,\mu)$ which is Markovian, i.e., $P_t^\mu 1 = 1$ and $P_t^\mu f \geqslant 0$ for all nonnegative $f \in L^1(H,\mu)$ and all $t > 0$. Furthermore, μ is P_t^μ-invariant, i.e.,*

$$\int_H P_t^\mu f\, d\mu = \int_H f\, d\mu, \quad \forall f \in L^1(H,\mu).$$

Below $B_b(H)$, $C_b(H)$ denote the bounded Borel-measurable, continuous functions respectively from H into $\mathbb{R}$ and $\|\cdot\|$ denotes the usual norm on $L(H)$.

Theorem 1.3. *(Cf. [5, Proposition 5.7].) Assume* (H1)–(H4) *hold. Then for any measure μ as in* (H4) *and $H_0 :=$ supp μ (:= largest closed set of H whose complement is a μ-zero set) there exists a semigroup $p_t^\mu(x, dy)$, $x \in H_0$, $t > 0$, of kernels such that $p_t^\mu f$ is a μ-version of $P_t^\mu f$ for all $f \in B_b(H)$, $t > 0$, where as usual*

$$p_t^\mu f(x) = \int_H f(y) p_t^\mu(x, dy), \quad x \in H_0.$$

Furthermore, for all $f \in B_b(H)$, $t > 0$, $x, y \in H_0$,

$$\left|p_t^\mu f(x) - p_t^\mu f(y)\right| \leqslant \frac{e^{|\omega| t}}{\sqrt{t \wedge 1}} \|f\|_0 \left\|\sigma^{-1}\right\| |x - y| \tag{1.4}$$

and for all $f \in \mathrm{Lip}_b(H)$ (:= *all bounded Lipschitz functions on* H)

$$\left|p_t^\mu f(x) - p_t^\mu f(y)\right| \leqslant e^{|\omega|t}\|f\|_{\mathrm{Lip}}|x-y|, \quad \forall t>0,\ x,y \in H_0, \tag{1.5}$$

and

$$\lim_{t\to 0} p_t^\mu f(x) = f(x), \quad \forall x \in H_0. \tag{1.6}$$

(*Here* $\|f\|_0$, $\|f\|_{\mathrm{Lip}}$ *denote the supremum, Lipschitz norm of* f *respectively.*) *Finally,* μ *is* p_t^μ-*invariant.*

Remark 1.4. (i) Both results above have been proved in [6] on $L^2(H,\mu)$ rather than on $L^1(H,\mu)$, but the proofs for $L^1(H,\mu)$ are entirely analogous.

(ii) In [6] we assume ω in (H1) to be negative, getting a stronger estimate than (1.4) (cf. [6, (5.11)]). But the same proof as in [6] leads to (1.4) for arbitrary $\omega \in \mathbb{R}$ (cf. the proof of [6, Proposition 4.3] for $t \in [0,1]$). Then by virtue of the semigroup property and since p_t^μ is Markov we get (1.4) for all $t>0$.

(iii) Theorem 1.3 holds in more general situations since (H2)(ii) can be relaxed (cf. [6, Remark 4.4] and [4, Proposition 8.3.3]).

(iv) (1.4) above implies that p_t^μ, $t>0$, is strongly Feller, i.e., $p_t^\mu(B_b(H)) \subset C(H_0)$ (= all continuous functions on H_0). We shall prove below that under the additional condition (H5) we even have $p_t^\mu(L^p(H,\mu)) \subset C(H_0)$ for all $p>1$ and that μ in (H4) is unique. However, so far we have not been able to prove that for this unique μ we have $\operatorname{supp}\mu = H$, though we conjecture that this is true.

For the results on Harnack inequalities, in this paper we need one more condition.

(H5) (i) $(1+\omega-A, D(A))$ satisfies the weak sector condition (cf. e.g. [10]), i.e., there exists a constant $K>0$ such that

$$\begin{aligned}&\left\langle(1+\omega-A)x, y\right\rangle \leqslant K\left\langle(1+\omega-A)x,x\right\rangle^{1/2}\left\langle(1+\omega-A)y,y\right\rangle^{1/2},\\ &\forall x,y \in D(A).\end{aligned} \tag{1.7}$$

(ii) There exists a sequence of A-invariant finite dimensional subspaces $H_n \subset D(A)$ such that $\bigcup_{n=1}^\infty H_n$ is dense in H.

We note that if A is self-adjoint, then (H2) implies that A has a discrete spectrum which in turn implies that (H5)(ii) holds.

Remark 1.5. Let $(A, D(A))$ satisfy (H1). Then the following is well known:

(i) (H5)(i) is equivalent to the fact that the semigroup generated by $(1+\omega-A, D(A))$ on the complexification $H_{\mathbb{C}}$ of H is a holomorphic contraction semigroup on $H_{\mathbb{C}}$ (cf. e.g. [10, Chapter I, Corollary 2.21]).

(ii) (H5)(i) is equivalent to $(1+\omega-A, D(A))$ being variational. Indeed, let $(\mathcal{E}, D(\mathcal{E}))$ be the coercive closed form generated by $(1+\omega-A, D(A))$ (cf. [10, Chapter I, Section 2]) and $(\widetilde{\mathcal{E}}, D(\mathcal{E}))$ be its symmetric part. Then define

$$V := D(\mathcal{E}) \quad \text{with inner product } \widetilde{\mathcal{E}} \text{ and } V^* \text{ to be its dual.} \tag{1.8}$$

Then

$$V \subset H \subset V^* \tag{1.9}$$

and $1 + \omega - A : D(A) \to H$ has a natural unique continuous extension from V to V^* satisfying all the required properties (cf. [10, Chapter I, Section 2, in particular Remark 2.5]).

Now we can formulate the main result of this paper, namely the Harnack inequality for p_t^μ, $t > 0$.

Theorem 1.6. *Suppose* (H1)–(H5) *hold and let μ be any measure as in* (H4) *and $p_t^\mu(x, dy)$ as in Theorem* 1.3 *above. Let $p \in (1, \infty)$. Then for all $f \in B_b(H)$, $f \geqslant 0$,*

$$\big(p_t^\mu f(x)\big)^p \leqslant p_t^\mu f^p(y) \exp\left[\|\sigma^{-1}\|^2 \frac{p\omega|x-y|^2}{(p-1)(1-e^{-2\omega t})}\right], \quad t > 0,\ x, y \in H_0. \tag{1.10}$$

As consequences in the situation of Theorem 1.6 (i.e. assuming (H1)–(H5)) we obtain:

Corollary 1.7. *For all $t > 0$ and $p \in (1, \infty)$*

$$p_t^\mu\big(L^p(H, \mu)\big) \subset C(H_0).$$

Corollary 1.8. *μ in* (H4) *is unique.*

Because of this result below we write $p_t(x, dy)$ instead of $p_t^\mu(x, dy)$.
Finally, we have

Corollary 1.9.

(i) *For every $x \in H_0$, $p_t(x, dy)$ has a density $\rho_t(x, y)$ with respect to μ and*

$$\big\|\rho_t(x, \cdot)\big\|_p^{p/(p-1)} \leqslant \frac{1}{\int_H \exp\left[-\|\sigma^{-1}\|^2 \frac{p\omega|x-y|^2}{(1-e^{-2\omega t})}\right]\mu(dy)}, \quad x \in H_0,\ p \in (1, \infty). \tag{1.11}$$

(ii) *If $\mu(e^{\lambda|\cdot|^2}) < \infty$ for some $\lambda > 2(\omega \wedge 0)^2\|\sigma^{-1}\|^2$, then p_t is hyperbounded, i.e. $\|p_t\|_{L^2(H,\mu)\to L^4(H,\mu)} < \infty$ for some $t > 0$.*

Corollary 1.10. *For simplicity, let $\sigma = I$ and instead of* (H1) *assume that more strongly $(A, D(A))$ is self-adjoint satisfying* (1.2)*. We furthermore assume that $|F_0| \in L^2(H, \mu)$.*

(i) *There exists $M \in \mathcal{B}(H_0)$, $M \subset D(F)$, $\mu(M) = 1$ such that for every $x \in M$ Eq.* (1.1) *has a pointwise unique continuous strong solution* (*in the mild sense see* (4.11) *below*)*, such that $X(t) \in M$ for all $t \geqslant 0$ $\mathbb{P}$-a.s.*

(ii) *Suppose there exists* $\Phi \in C([0,\infty))$ *positive and strictly increasing such that* $\lim_{s\to\infty} s^{-1}\Phi(s) = \infty$ *and*

$$\Psi(s) := \int_s^\infty \frac{dr}{\Phi(r)} < \infty, \quad \forall s > 0. \tag{1.12}$$

If there exists a constant $c > 0$ *such that*

$$\langle F_0(x) - F_0(y), x - y\rangle \leqslant c - \Phi\big(|x-y|^2\big), \quad \forall x, y \in D(F), \tag{1.13}$$

then p_t *is ultrabounded with*

$$\|p_t\|_{L^2(H,\mu)\to L^\infty(H,\mu)} \leqslant \exp\left[\frac{\lambda(1+\Psi^{-1}(t/4))}{(1-\varepsilon^{-\omega t/2})^2}\right], \quad t > 0,$$

holding for some constant $\lambda > 0$.

Remark 1.11. We emphasize that since the nonlinear part F_0 of our Kolmogorov operator is in general not continuous, it was quite surprising for us that in this infinite dimensional case nevertheless the generated semigroup P_t maps L^1-functions to continuous ones as stated in Corollary 1.7.

The proof that Corollary 1.9 follows from Theorem 1.6 is completely standard. So, we will omit the proofs and instead refer to [14,17].

Corollary 1.7 is new and follows whenever a semigroup p_t satisfies the Harnack inequality (see Proposition 4.1 below).

Corollary 1.8 is new. Since (1.10) implies irreducibility of p_t^μ and Corollary 1.7 implies that it is strongly Feller, a well known theorem due to Doob immediately implies that μ is the unique invariant measure for p_t^μ, $t > 0$. p_t^μ, however, depends on μ, so Corollary 1.8 is a stronger statement. Corollary 1.10 is also new.

Theorem 1.6 as well as Corollaries 1.7, 1.8 and 1.10 will be proved in Section 4. In Section 3 we first prove Theorem 1.6 in case F_0 is Lipschitz, and in Section 2 we prepare the tools that allow us to reduce the general case to the Lipschitz case. In Section 5 we prove two results (see Theorems 5.2 and 5.4) on the existence of a measure satisfying (H4) under some additional conditions and present an application to an example where F_0 is not continuous. For a discussion of a number of other explicit examples satisfying our conditions see [6, Section 9].

2. Reduction to regular F_0

Let F be as in (H3). As in [6] we may consider the Yosida approximation of F, i.e., for any $\alpha > 0$ we set

$$F_\alpha(x) := \frac{1}{\alpha}\big(J_\alpha(x) - x\big), \quad x \in H, \tag{2.1}$$

where for $x \in H$

$$J_\alpha(x) := (I - \alpha F)^{-1}(x), \quad \alpha > 0,$$

and $I(x) := x$. Then each F_α is single valued, dissipative and it is well known that

$$\lim_{\alpha\to 0} F_\alpha(x) = F_0(x), \quad \forall x \in D(F), \tag{2.2}$$

$$\left|F_\alpha(x)\right| \leqslant \left|F_0(x)\right|, \quad \forall x \in D(F). \tag{2.3}$$

Moreover, F_α is Lipschitz continuous, so F_0 is $\mathcal{B}(H)$-measurable. Since F_α is not differentiable in general, as in [6] we introduce a further regularization by setting

$$F_{\alpha,\beta}(x) := \int_H e^{\beta B} F_\alpha\big(e^{\beta B}x + y\big) N_{\frac{1}{2}B^{-1}(e^{2\beta B}-1)}(dy), \quad \alpha, \beta > 0, \tag{2.4}$$

where $B : D(B) \subset H \to H$ is a self-adjoint, negative definite linear operator such that B^{-1} is of trace class and as usual for a trace class operator Q the measure N_Q is just the standard centered Gaussian measure with covariance given by Q.

$F_{\alpha,\beta}$ is dissipative, of class C^∞, has bounded derivatives of all the orders and $F_{\alpha,\beta} \to F_\alpha$ pointwise as $\beta \to 0$.

Furthermore, for $\alpha > 0$

$$c_\alpha := \sup\left\{\frac{|F_{\alpha,\beta}(x)|}{1+|x|} \colon x \in H,\ \beta \in (0,1]\right\} < \infty. \tag{2.5}$$

We refer to [8, Theorem 9.19] for details.

Now we consider the following regularized stochastic equation

$$\begin{cases} dX_{\alpha,\beta}(t) = \big(AX_{\alpha,\beta}(t) + F_{\alpha,\beta}\big(X_{\alpha,\beta}(t)\big)\big)\,dt + \sigma\,dW(t), \\ X_{\alpha,\beta}(0) = x \in H. \end{cases} \tag{2.6}$$

It is well known that (2.6) has a unique mild solution $X_{\alpha,\beta}(t,x)$, $t \geqslant 0$. Its associated transition semigroup is given by

$$P_t^{\alpha,\beta} f(x) = \mathbb{E}\big[f\big(X_{\alpha,\beta}(t,x)\big)\big], \quad t > 0,\ x \in H,$$

for any $f \in B_b(H)$. Here $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$.

Proposition 2.1. *Assume* (H1)–(H4). *Then there exists a K_σ-set $K \subset H$ such that $\mu(K) = 1$ and for all $f \in B_b(H)$, $T > 0$ there exist subsequences* $(\alpha_n), (\beta_n) \to 0$ *such that for all $x \in K$*

$$\lim_{n\to\infty}\lim_{m\to\infty} P_\bullet^{\alpha_n,\beta_m} f(x) = p_\bullet^\mu f(x) \quad \textit{weakly in } L^2(0,T;dt). \tag{2.7}$$

Proof. This follows immediately from the proof of [6, Proposition 5.7]. (A closer look at the proof even shows that (2.7) holds for all $x \in H_0 = \operatorname{supp} \mu$.) □

As we shall see in Section 4, the proof of Theorem 1.6 follows from Proposition 2.1 if we can prove the corresponding Harnack inequality for each $P_t^{\alpha,\beta}$. Hence in the next section we confine ourselves to the case when F_0 is dissipative and Lipschitz.

3. The Lipschitz case

In this section we assume that (H1)–(H3) and (H5) hold and that F_0 in (H3) is in addition Lipschitz continuous. The aim of this section is to prove Theorem 1.6 for such special F_0 (see Proposition 3.1 below). We shall do this by finite dimensional (Galerkin) approximations, since for the approximating finite dimensional processes we can apply the usual coupling argument.

We first note that since F_0 is Lipschitz (1.1) has a unique mild solution $X(t,x)$, $t \geqslant 0$, for every initial condition $x \in H$ (cf.[8]) and we denote the corresponding transition semigroup by P_t, $t > 0$, i.e.

$$P_t f(x) := \mathbb{E}\big[f\big(X(t,x)\big)\big], \quad t > 0,\ x \in X,$$

where $f \in B_b(H)$.

Now we need to consider an appropriate Galerkin approximation. To this end let $e_k \in D(A)$, $k \in \mathbb{N}$, be orthonormal such that $H_n = \text{linear span}\{e_1, \dots, e_n\}$, $n \in \mathbb{N}$. Hence $\{e_k\colon k \in \mathbb{N}\}$ is an orthonormal basis of $(H, \langle\cdot,\cdot\rangle)$. Let $\pi_n : H \to H_n$ be the orthogonal projection with respect to $(H, \langle\cdot,\cdot\rangle)$. So, we can define

$$A_n := \pi_n A_{|H_n} \big(= A_{|H_n} \text{ by (H5)(ii)}\big) \tag{3.1}$$

and, furthermore

$$F_n := \pi_n F_{0|H_n}, \qquad \sigma_n := \pi_n \sigma_{|H_n}.$$

Obviously, $\sigma_n : H_n \to H_n$ is a self-adjoint, positive definite linear operator on H_n. Furthermore, σ_n is bijective, since it is one-to-one. To see the latter, one simply picks an orthonormal basis $\{e_1^\sigma, \dots, e_n^\sigma\}$ of H_n with respect to the inner product $\langle\cdot,\cdot\rangle_\sigma$ defined by $\langle x, y\rangle_\sigma := \langle \sigma x, y\rangle$. Then if $x \in H_n$ is such that $\sigma_n x = \pi_n \sigma x = 0$, it follows that

$$\big\langle x, e_i^\sigma \big\rangle_\sigma = \big\langle \sigma x, e_i^\sigma \big\rangle = 0, \quad \forall 1 \leqslant i \leqslant n.$$

But $x = \sum_{i=1}^n \langle x, e_i^\sigma\rangle_\sigma e_i^\sigma$, hence $x = 0$.

Now fix $n \in \mathbb{N}$ and on H_n consider the stochastic equation

$$\begin{cases} dX_n(t) = \big(A_n X_n(t) + F_n\big(X_n(t)\big)\big)\,dt + \sigma_n\,dW_n(t), \\ X_n(0) = x \in H_n, \end{cases} \tag{3.2}$$

where $W_n(t) = \pi_n W(t) = \sum_{i=1}^n \langle e_k, W(t)\rangle e_k$.

(3.2) has a unique strong solution $X_n(t,x)$, $t \geqslant 0$, for every initial condition $x \in H_n$ which is pathwise continuous $\mathbb{P}$-a.s. Consider the associated transition semigroup defined as before by

$$P_t^n f(x) = \mathbb{E}\big[f\big(X_n(t,x)\big)\big], \quad t > 0,\ x \in H_n, \tag{3.3}$$

where $f \in B_b(H_n)$.

Below we shall prove the following:

Proposition 3.1. *Assume that* (H1)–(H5) *hold. Then*:

(i) *For all* $f \in C_b(H)$ *and all* $t > 0$,

$$\lim_{n\to\infty} P_t^n f(x) = P_t f(x), \quad \forall x \in H_{n_0},\ n_0 \in \mathbb{N}.$$

(ii) *For all nonnegative* $f \in B_b(H)$ *and all* $n \in N$, $p \in (1,\infty)$

$$\big(P_t^n f(x)\big)^p \leqslant P_t^n f^p(y) \exp\left[\big\|\sigma^{-1}\big\|^2 \frac{p\omega|x-y|^2}{(p-1)(1-e^{-2\omega t})}\right], \quad t>0,\ x,y \in H_n. \tag{3.4}$$

Proof. (i) Define

$$W_{A,\sigma}(t) := \int_0^t e^{(t-s)A}\sigma\, dW(s), \quad t \geqslant 0.$$

Note that by (H2)(i) we have that $W_{A,\sigma}(t)$, $t \geqslant 0$, is well defined and pathwise continuous. For $x \in H_{n_0}$, $n_0 \in \mathbb{N}$ fixed, let $Z(t)$, $t \geqslant 0$, be the unique variational solution (with triple $V \subset H \subset V^*$ as in Remark 1.5(ii), see e.g. [13]) to

$$\begin{cases} dZ(t) = \big[AZ(t) + F_0\big(Z(t) + W_{A,\sigma}(t)\big)\big]\,dt, \\ Z(0) = x, \end{cases} \tag{3.5}$$

which then automatically satisfies

$$\mathbb{E} \sup_{t\in[0,T]} \big|Z(t)\big|^2 < +\infty. \tag{3.6}$$

Then we have (see [8]) that $Z(t) + W_{A,\sigma}(t)$, $t \geqslant 0$, is a mild solution to (1.1) (with F_0 Lipschitz), hence by uniqueness

$$X(t,x) = Z(t) + W_{A,\sigma}(t), \quad t \geqslant 0. \tag{3.7}$$

Clearly, since

$$\mathbb{E} \sup_{t\in[0,T]} \big|W_{A,\sigma}(t)\big|^2 < +\infty, \tag{3.8}$$

we have

$$\pi_n W_{A,\sigma}(t) \to W_{A,\sigma}(t) \quad \text{as } n\to\infty \text{ in } L^2(\Omega,\mathcal{F},\mathbb{P}),\ \forall t \geqslant 0.$$

We set $X_n(t) := X_n(t,x)$ (= solution of (3.2)). Defining

$$W_{A_n,\sigma_n}(t) = \int_0^t e^{(t-s)A_n}\sigma_n\, dW_n(t), \quad t \geqslant 0,$$

and

$$Z_n(t) := X_n(t) - W_{A_n,\sigma_n}(t), \quad n \in \mathbb{N},\ t \geqslant 0,$$

it is enough to show that

$$Z_n(t) \to Z(t) \quad \text{as } n \to \infty \text{ in } L^2(\Omega, \mathcal{F}, \mathbb{P}),\ \forall t \geqslant 0, \tag{3.9}$$

because then by (3.7)

$$X_n(t) \to X(t) \quad \text{as } n \to \infty \text{ in } L^2(\Omega, \mathcal{F}, \mathbb{P}),\ \forall t \geqslant 0,$$

and the assertion follows by Lebesgue's dominated convergence theorem. To show (3.9) we first note that by the same argument as above

$$dZ_n(t) = \left[A_n Z_n(t) + F_n\big(Z_n(t) + W_{A_n,\sigma_n}(t)\big)\right] dt$$

and thus (in the variational sense), since $A = A_n$ on H_n by (3.1)

$$d\big(Z(t) - Z_n(t)\big) = \left[A\big(Z(t) - Z_n(t)\big) + F_0\big(X(t)\big) - F_n\big(X_n(t)\big)\right] dt.$$

Applying Itô's formula we obtain that for some constant $c > 0$

$$\begin{aligned}
\frac{1}{2}\big|Z(t) - Z_n(t)\big|^2 &\leqslant \int_0^t \Big[(\omega + 1/2)\big|Z(s) - Z_n(s)\big|^2 \\
&\quad + \big|F_0\big(X(s)\big) - F_0\big(X_n(s)\big)\big|^2 + \big|(1 - \pi_n)F_0\big(X(s)\big)\big|^2\Big] ds \\
&\leqslant c \int_0^t \big|Z(s) - Z_n(s)\big|^2 ds + c \int_0^t \big|W_{A,\sigma}(s) - W_{A_n,\sigma_n}(s)\big|^2 ds \\
&\quad + \int_0^t \big|(1 - \pi_n)F_0\big(X(s)\big)\big|^2 ds.
\end{aligned}$$

Now (3.9) follows by the linear growth of F_0, (3.6)–(3.8) and Gronwall's lemma, if we can show that

$$\int_0^T \mathbb{E}\big|W_{A,\sigma}(s) - W_{A_n,\sigma_n}(s)\big|^2 ds \to 0 \quad \text{as } n \to \infty. \tag{3.10}$$

To this end we first note that a straightforward application of Duhamel's formula yields that

$$e^{tA}|_{H_n} = e^{tA_n} \quad \forall t \geqslant 0.$$

Therefore

$$W_{A,\sigma}(s) - W_{A_n,\sigma_n}(s) = \int_0^s e^{(t-r)A}(\sigma - \pi_n \sigma \pi_n)\, dW(r),$$

and thus

$$\mathbb{E}\big|W_{A,\sigma}(s) - W_{A_n,\sigma_n}(s)\big|^2 = \int_0^s \big\|e^{(t-r)A}(\sigma - \pi_n\sigma\pi_n)\big\|^2\, dr$$
$$= \sum_{i=1}^{\infty} \int_0^s \big|e^{rA}(\sigma - \pi_n\sigma\pi_n)e_i\big|^2\, dr.$$

Since for any $i \in \mathbb{N}$, $r \in [0, s]$, the integrands converge to 0, Lebesgue's dominated convergence theorem implies (3.10).

(ii) Fix $T > 0$, $n \in \mathbb{N}$ and $x, y \in H_n$. Let $\xi^T \in C^1([0,\infty))$ be defined by

$$\xi^T(t) := \frac{2\omega e^{-\omega t}|x-y|}{1 - e^{-2\omega T}}, \quad t \geqslant 0.$$

Consider for $X_n(t) = X_n(t,x)$, $t \geqslant 0$, see the proof of (i), the stochastic equation

$$\begin{cases} dY_n(t) = \Big[A_n Y_n(t) + F_n\big(Y_n(t)\big) + \xi^T(t)\dfrac{X_n(t)-Y_n(t)}{|X_n(t)-Y_n(t)|}\mathbb{1}_{X_n(t)\neq Y_n(t)}\Big]dt \\ \quad + \sigma_n\, dW_n(t), \\ Y_n(0) = y. \end{cases} \tag{3.11}$$

Since

$$z \to \frac{X_n(t) - z}{|X_n(t) - z|}\mathbb{1}_{X_n(t)\neq z}$$

is dissipative on H_n for all $t \geqslant 0$ (cf. [17]), (3.11) has a unique strong solution $Y_n(t) = Y_n(t,y)$, $t \geqslant 0$, which is pathwise continuous $\mathbb{P}$-a.s.

Define the first coupling time

$$\tau_n := \inf\big\{t \geqslant 0\colon\ X_n(t) = Y_n(t)\big\}. \tag{3.12}$$

Writing the equation for $X_n(t) - Y_n(t)$, $t \geqslant 0$, applying the chain rule to $\phi_\epsilon(z) := \sqrt{z+\epsilon^2}$, $z \in (-\epsilon^2, \infty)$, $\epsilon > 0$, and letting $\epsilon \to 0$ subsequently, we obtain

$$\frac{d}{dt}\big|X_n(t) - Y_n(t)\big| \leqslant \omega\big|X_n(t) - Y_n(t)\big| - \xi^T(t)\mathbb{1}_{X_n(t)\neq Y_n(t)} \quad t \geqslant 0,$$

which yields

$$d\big(e^{-\omega t}\big|X_n(t) - Y_n(t)\big|\big) \leqslant -e^{-\omega t}\xi^T(t)\mathbb{1}_{X_n(t)\neq Y_n(t)}\, dt, \quad t \geqslant 0. \tag{3.13}$$

In particular, $t \mapsto e^{-\omega t}|X_n(t) - Y_n(t)|$ is decreasing, hence $X_n(T) = Y_n(T)$ for all $T \geqslant \tau_n$. But by (3.13) if $T \leqslant \tau_n$ then

$$\big|X_n(T) - Y_n(T)\big|e^{-\omega T} \leqslant |x-y| - |x-y| \int_0^T \frac{2\omega e^{-2\omega t}}{1 - e^{-2\omega T}}\, dt = 0.$$

So, in any case

$$X_n(T) = Y_n(T), \quad \mathbb{P}\text{-a.s.} \tag{3.14}$$

Let

$$\begin{aligned} R := \exp\Bigg[&- \int_0^{T\wedge\tau_n} \frac{\xi^T(t)}{|X_n(t) - Y_n(t)|} \big\langle X_n(t) - Y_n(t), \sigma^{-1} dW_n(t)\big\rangle \\ &- \frac{1}{2} \int_0^{T\wedge\tau_n} \frac{(\xi^T(t))^2 |\sigma^{-1}(X_n(t) - Y_n(t))|^2}{|X_n(t) - Y_n(t)|^2}\, dt \Bigg]. \end{aligned}$$

By (3.14) and Girsanov's theorem for $p > 1$,

$$\begin{aligned} \big(P_T^n f(y)\big)^p &= \big(\mathbb{E}\big[f\big(Y_n(T)\big)\big]\big)^p = \big(\mathbb{E}\big[Rf\big(X_n(T)\big)\big]\big)^p \\ &\leqslant \big(P_T^n f^p(x)\big)\big(\mathbb{E}\big[R^{p/(p-1)}\big]\big)^{p-1}. \end{aligned} \tag{3.15}$$

Let

$$\begin{aligned} M_p = \exp\Bigg[&-\frac{p}{p-1} \int_0^{T\wedge\tau_n} \frac{\xi^T(t)}{|X_n(t) - Y_n(t)|} \big\langle X_n(t) - Y_n(t), \sigma^{-1} dW_n(t)\big\rangle \\ &- \frac{p^2}{2(p-1)^2} \int_0^{T\wedge\tau_n} \frac{(\xi^T(t))^2 |\sigma^{-1}(X_n(t) - Y_n(t))|^2}{|X_n(t) - Y_n(t)|^2}\, dt \Bigg]. \end{aligned}$$

We have $\mathbb{E} M_p = 1$ and hence,

$$\begin{aligned} \mathbb{E} R^{p/(p-1)} &= \mathbb{E}\Bigg\{ M_p \exp\Bigg[\frac{p}{2(p-1)^2} \int_0^{T\wedge\tau_n} \frac{(\xi^T(t))^2 |\sigma^{-1}(X_n(t) - Y_n(t))|^2}{|X_n(t) - Y_n(t)|^2}\, dt \Bigg]\Bigg\} \\ &\leqslant \sup_{\Omega} \exp\Bigg[\frac{p}{2(p-1)^2} \int_0^{T\wedge\tau_n} \big(\xi^T(t)\big)^2 \big\|\sigma^{-1}\big\|^2\, dt \Bigg] \\ &\leqslant \exp\Bigg[\big\|\sigma^{-1}\big\|^2 \frac{p\omega |x-y|^2}{(p-1)^2(1 - e^{-2\omega T})} \Bigg]. \end{aligned}$$

Combining this with (3.15) we get the assertion (with T replacing t). □

4. Proof and consequences of Theorem 1.6

On the basis of Propositions 3.1 and 2.1 we can now easily prove Theorem 1.6.

Proof of Theorem 1.6. Let $f \in \mathrm{Lip}_b(H)$, $f \geqslant 0$. By Proposition 3.1(i) it then follows that (3.4) holds with $P_t f$ replacing $P_t^n f$ provided F is Lipschitz. Using that $\bigcup_{n\in\mathbb{N}} H_n$ is dense in H and that $P_t f(x)$ is continuous on x (cf. [8]) we obtain (3.4) for all $x, y \in H$. In particular, this is true for $P_t^{\alpha_n,\beta_n} f$ from Proposition 2.1.

Now fix $t > 0$ and $k \in \mathbb{N}$, let

$$\chi_k(s) := \frac{1}{k}\mathbb{1}_{[t,t+1/k]}(s), \quad s \geqslant 0.$$

Using (3.4) for $P_t^{\alpha_n,\beta_m} f$, (1.6), Proposition 2.1 and Jensen's inequality, we obtain for $x, y \in K$

$$\begin{aligned}
p_t^\mu f(x) &= \lim_{k\to\infty} \frac{1}{k} \int_t^{t+1/k} p_s^\mu f(x)\,ds \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \lim_{m\to\infty} \int_0^{t+1} \chi_k(s) P_s^{\alpha_n,\beta_m} f(x)\,dx \\
&\leqslant \lim_{k\to\infty} \lim_{n\to\infty} \lim_{m\to\infty} \int_0^{t+1} \chi_k(s) \big(P_s^{\alpha_n,\beta_m} f^p(y)\big)^{1/p} \exp\left[\|\sigma^{-1}\|^2 \frac{\omega|x-y|^2}{(p-1)(1-e^{-2\omega s})}\right] ds \\
&\leqslant \lim_{k\to\infty} \lim_{n\to\infty} \lim_{m\to\infty} \left(\int_0^{t+1} \chi_k(s) P_s^{\alpha_n,\beta_m} f^p(y) \exp\left[\|\sigma^{-1}\|^2 \frac{p\omega|x-y|^2}{(p-1)(1-e^{-2\omega s})}\right] ds\right)^{1/p} \\
&= \big(p_t^\mu f^p(y)\big)^{1/p} \exp\left[\|\sigma^{-1}\|^2 \frac{\omega|x-y|^2}{(p-1)(1-e^{-2\omega t})}\right],
\end{aligned}$$

where we note that we have to choose the sequences (α_n), (β_n) such that (2.7) holds both for f and f^p instead of f. Since K is dense in H_0, (1.10) follows for $f \in C_b(H)$, for all $x, y \in H_0$, since $p_t^\mu f$ is continuous on H_0 by (1.4).

Let now $f \in B_b(H)$, $f \geqslant 0$. Let $f_n \in C_b(H)$, $n \in \mathbb{N}$, such that $f_n \to f$ in $L^p(H,\mu)$ as $n \to \infty$, $p \in (1,\infty)$ fixed. Then, since μ is invariant for p_t^μ, $t > 0$, selecting a subsequence if necessary, it follows that there exists $K_1 \in \mathcal{B}(H)$, $\mu(K_1) = 1$, such that

$$p_t^\mu f_n(x) \to p_t^\mu f(x) \quad \text{as } n \to \infty, \quad \forall x \in K_1.$$

Taking this limit in (1.10) we obtain (1.10) for all $x, y \in K_1$. Taking into account that p_t^μ is continuous and that K_1 is dense in $H_0 = \operatorname{supp}\mu$, (1.10) follows for all $x, y \in H_0$. □

Corollary 1.7 immediately follows from Theorem 1.6 and the following general result:

Proposition 4.1. *Let E be a topological space and P a Markov operator on $B_b(E)$. Assume that for any $p > 1$ there exists a continuous function η_p on $E \times E$ such that $\eta_p(x,x) = 0$ for all $x \in E$ and*

$$P|f|(x) \leqslant \big(P|f|^p(y)\big)^{1/p} e^{\eta_p(x,y)} \quad \forall x, y \in E,\ f \in B_b(E). \tag{4.1}$$

Then P is strong Feller, i.e. maps $B_b(E)$ into $C_b(E)$. Furthermore, for any σ-finite measure μ on $(E, \mathcal{B}(E))$ such that

$$\int_E |Pf|\,d\mu \leqslant C \int_E |f|\,d\mu, \quad \forall f \in B_b(E), \tag{4.2}$$

for some $C > 0$, P uniquely extends to $L^p(E,\mu)$ with $PL^p(E,\mu) \subset C(E)$ for any $p > 1$.

Proof. Since P is linear, we only need to consider $f \geqslant 0$. Let $f \in B_b(E)$ be nonnegative. By (4.1) and the property of η_p we have

$$\limsup_{x \to y} Pf(x) \leqslant \big(Pf^p(y)\big)^{1/p}, \quad p > 1.$$

Letting $p \downarrow 1$ we obtain $\limsup_{x\to y} Pf(x) \leqslant Pf(y)$. Similarly, using $f^{1/p}$ to replace f and replacing x with y, we obtain

$$\big(Pf^{1/p}(y)\big)^p \leqslant \big(Pf(x)\big)e^{p\eta_p(y,x)}, \quad \forall x, y \in E,\ p > 1.$$

First letting $x \to y$ then $p \to 1$, we obtain $\liminf_{x\to y} Pf(x) \geqslant Pf(y)$. So $Pf \in C_b(E)$. Next, for any nonnegative $f \in L^p(E,\mu)$, let $f_n = f \wedge n$, $n \geqslant 1$. By (4.2) and $f_n \to f$ in $L^p(E,\mu)$ we have $P|f_n - f_m|^p \to 0$ in $L^1(E,\mu)$ as $n,m \to \infty$. In particular, there exists $y \in E$ such that

$$\lim_{n,m\to\infty} P|f_n - f_m|^p(y) = 0. \tag{4.3}$$

Moreover, by (4.1), for $B_N := \{x \in E\colon\ \eta_p(x,y) < N\}$

$$\sup_{x \in B_N} \big|Pf_n(x) - Pf_m(x)\big|^p \leqslant \sup_{x\in B_N} \big(P|f_n - f_m|(x)\big)^p \leqslant \big(P|f_n - f_m|^p(y)\big)e^{pN}.$$

Since by the strong Feller property $Pf_n \in C_b(E)$ for any $n \geqslant 1$ and noting that $C_b(B_N)$ is complete under the uniform norm, we conclude from (4.3) that Pf is continuous on B_N for any $N \geqslant 1$, and hence, $Pf \in C(H)$. □

Proof of Corollary 1.8. Let μ_1, μ_2 be probability measures on $(H, \mathcal{B}(H))$ satisfying (H4). Define $\mu := \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$. Then μ satisfies (H4) and $\mu_i = \rho_i \mu$, $i = 1, 2$, for some $\mathcal{B}(H)$-measurable $\rho_i : H \to [0, 2]$. Let $i \in \{1, 2\}$.

Since ρ_i is bounded, by (H4)(iii) and Theorem 1.2 it follows that

$$\int_H L_\mu u\,d\mu_i = 0, \quad \forall u \in D(L_\mu).$$

Hence

$$\frac{d}{dt}\int_H e^{tL_\mu}u\,d\mu_i = \int_H L_\mu\big(e^{tL_\mu}u\big)\,d\mu_i = 0, \quad \forall u \in D(L_\mu),$$

i.e.

$$\int_H p_t^\mu u\,d\mu_i = \int_H u\,d\mu_i \quad \forall u \in \mathcal{E}_A(H).$$

Since $\mathcal{E}_A(H)$ is dense in $L^1(H,\mu_i)$, μ_i is (p_t^μ)-invariant. But as mentioned before, by Theorem 1.6 it follows that (p_t^μ) is irreducible on H_0 (see [9]) and it is strong Feller on H_0 by Corollary 1.7. So, since $\mu_i(H_0)=1$, $\mu_i=\mu$. □

Proof of Corollary 1.10. Let

$$\tilde{A} := A - \omega I, \qquad D(\tilde{A}) := D(A),$$
$$\tilde{F}_0 := F_0 + \omega I.$$

By (H2), $\tilde{A}$ has discrete spectrum. Let $e_k \in H$, $-\lambda_k \in (-\infty, 0]$, be the corresponding orthonormal eigenvectors, eigenvalues respectively.

For $k \in \mathbb{N}$ define

$$\varphi_k(x) := \langle e_k, x\rangle, \quad x \in H.$$

We note that by a simple approximation (1.5) also holds for any Lipschitz function on H and thus (cf. the proof of [6, Proposition 5.7(iii)]) also (1.6) holds for such functions, i.e. in particular, for all $k \in \mathbb{N}$

$$[0,\infty) \ni t \mapsto p_t\varphi_k(x) \quad \text{is continuous for all } x \in H_0. \tag{4.4}$$

Since any compactly supported smooth function on $\mathbb{R}^N$ is the Fourier transform of a Schwartz test function, by approximation it easily follows that setting

$$\mathcal{F}C_b^\infty\big(\{e_k\}\big) := \big\{g\big(\langle e_1,\cdot\rangle,\dots,\langle e_N,\cdot\rangle\big)\colon N \in \mathbb{N},\ g \in C_b^\infty\big(\mathbb{R}^N\big)\big\},$$

we have $\mathcal{F}C_b^\infty(\{e_k\}) \subset D(L_\mu)$ and for $\varphi \in \mathcal{F}C_b^\infty(\{e_k\})$

$$L_\mu\varphi(x) = \frac{1}{2}\operatorname{Tr}\big[D^2\varphi(x)\big] + \big\langle x, AD\varphi(x)\big\rangle + \big\langle F_0(x), D\varphi(x)\big\rangle, \quad x \in H.$$

Then by approximation it is easy to show that

$$\varphi_k, \varphi_k^2 \in D(L_\mu) \quad \text{and} \quad L_\mu\varphi_k = -\lambda_k\varphi_k + \langle e_k, \tilde{F}_0\rangle,$$
$$L_\mu\varphi_k^2 = -2\lambda_k\varphi_k^2 + 2\varphi_k\langle e_k, \tilde{F}_0\rangle + 2, \quad \forall k \in \mathbb{N}. \tag{4.5}$$

Since we assume that $|F_0|$ is in $L^2(H,\mu)$, by [3, Theorem 1.1] we are in the situation of [15, Chapter II]. So, we conclude that by [15, Chapter II, Theorem 1.9] there exists a normal (that is $\mathbb{P}_x[X(0)=x]=1$) Markov process $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, (X(t))_{t\geqslant 0}, (\mathbb{P}_x)_{x\in H_0})$ with state space H_0 and $M\in\mathcal{B}(H_0)$, $\mu(M)=1$, such that $X(t)\in M$ for all $t\geqslant 0$ $\mathbb{P}_x$-a.s. for all $x\in M$ and which has continuous sample paths $\mathbb{P}_x$-a.s for all $x\in M$ and for which by the proof of [6, Proposition 8.2] and (4.4), (4.5) we have that for all $k\in\mathbb{N}$

$$\beta_k^x(t):=\varphi_k\big(X(t)\big)-\varphi_k(x)-\int_0^t L_\mu\varphi_k\big(X(s)\big)\,ds,\quad t\geqslant 0,$$

$$M_k^x(t):=\varphi_k^2\big(X(t)\big)-\varphi_k^2(x)-\int_0^t L_\mu\varphi_k^2\big(X(s)\big)\,ds,\quad t\geqslant 0,\tag{4.6}$$

are continuous local $(\mathcal{F}_t)$-martingales with $\beta_k^x(0)=M_k(0)=0$ under $\mathbb{P}_x$ for all $x\in M$. Fix $x\in M$. Below $\mathbb{E}_x$ denotes expectation with respect to $\mathbb{P}_x$. Since for $T>0$

$$\begin{aligned}&\int_H\int_0^T \mathbb{E}_x\big(1+\big|X(s)\big|^2\big)\big(1+\big|F_0\big(X(s)\big)\big|\big)\,ds\,\mu(dx)\\&\quad=T\int_H\big(1+|x|^2\big)\big(1+\big|F_0(x)\big|\big)\mu(dx)<\infty,\end{aligned}$$

making M smaller if necessary, by (H4)(ii) we may assume that

$$\mathbb{E}_x\int_0^T\big(1+\big|X(s)\big|^2\big)\big(1+\big|F_0\big(X(s)\big)\big|\big)\,ds<\infty.\tag{4.7}$$

By standard Markov process theory we have for their covariation processes under $\mathbb{P}_x$,

$$\big\langle\beta_k^x,\beta_{k'}^x\big\rangle_t=\int_0^t\big\langle D\varphi_k\big(X(s)\big),D\varphi_{k'}\big(X(s)\big)\big\rangle\,ds=t\delta_{k,k'},\quad t\geqslant 0.\tag{4.8}$$

Indeed, an elementary calculation shows that for all $k\in\mathbb{N}$, $t\geqslant 0$,

$$\begin{aligned}&\beta_k^x(t)^2-\int_0^t\big|D\varphi_k\big(X(s)\big)\big|^2\,ds\\&\quad=M_k^x(t)-2\varphi_k(x)\beta_k^x(t)-\int_0^t\big(\beta_k^x(t)-\beta_k^x(s)\big)L_\mu\varphi_k\big(X(s)\big)\,ds,\end{aligned}\tag{4.9}$$

where all three summands on the right-hand side are martingales. Since we have a similar formula for finite linear combinations of $\varphi_k' s$ replacing a single φ_k, by polarization we get (4.8). Note that by (4.5) and (4.7) all integrals in (4.6), (4.9) are well defined.

Hence, by (4.8) β_k^x, $k \in \mathbb{N}$, are independent standard $(\mathcal{F}_t)$-Brownian motions under $\mathbb{P}_x$. Now it follows by [11, Theorem 13] that, with $W^x = (W^x(t))_{t \geqslant 0}$, being the cylindrical Wiener process on H given by $W^x = (\beta_k^x e_k)_{k \in \mathbb{N}}$, we have for every $t \geqslant 0$,

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A} F_0\big(X(s)\big)\, ds + \int_0^t e^{(t-s)A} dW^x(s), \quad \mathbb{P}\text{-a.s.}, \tag{4.10}$$

that is, the tuple $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P}_x, W^x, X)$ is a solution to

$$\begin{cases} Y(t) = e^{tA}Y(0) + \displaystyle\int_0^t e^{(t-s)A} F_0\big(Y(s)\big)\, ds + \int_0^t e^{(t-s)A}\, dW(s), \quad \mathbb{P}\text{-a.s., } \forall t \geqslant 0, \\ \text{law } Y(0) = \delta_x \quad (:= \text{ Dirac measure in } x), \end{cases} \tag{4.11}$$

in the sense of [11, p. 4].

We note that the zero set in (4.10) is indeed independent of t, since all terms are continuous in t $\mathbb{P}_x$-a.s. because of (H2)(ii) and (4.7).

Claim. *We have X-pathwise uniqueness for Eq.* (4.11) (*in the sense of* [11, *p.* 98]).

For any given cylindrical $(\mathcal{F}_t')$-Wiener process W on a stochastic basis $(\Omega', \mathcal{F}', (\mathcal{F}_t')_{t \geqslant 0}, \mathbb{P}')$ let $Y = Y(t)$, $Z = Z(t)$, $t \geqslant 0$, be two solutions of (4.11) such that $\operatorname{law}(Z) = \operatorname{law}(Y) = \operatorname{law}(X)$ and $Y(0) = Z(0)$ $\mathbb{P}'$-a.s. Then by (4.7)

$$\mathbb{E}' \int_0^T \big|F_0\big(Y(s)\big)\big|\, ds = \mathbb{E}' \int_0^T \big|F_0\big(Z(s)\big)\big|\, ds = \mathbb{E}_x \int_0^T \big|F_0\big(X(s)\big)\big|\, ds < \infty \tag{4.12}$$

(which, in particular implies by (4.11) and by (H2)(i) that both Y and Z have $\mathbb{P}'$-a.s. continuous sample paths). Hence applying [11, Theorem 13] again (but this time using the dual implication) we obtain for all $k \in \mathbb{N}$

$$\begin{aligned} \big\langle e_k, Y(t) - Z(t)\big\rangle &= -\lambda_k \int_0^t \big\langle e_k, Y(s) - Z(s)\big\rangle\, ds \\ &\quad + \int_0^t \big\langle e_k, \tilde{F}_0\big(Y(s)\big) - \tilde{F}_0\big(Z(s)\big)\big\rangle\, ds, \quad t \geqslant 0,\ \mathbb{P}'\text{-a.s.} \end{aligned}$$

Therefore, by the chain rule for all $k \in \mathbb{N}$,

$$\langle e_k, Y(t) - Z(t)\rangle^2 = -2\lambda_k \int_0^t \langle e_k, Y(s) - Z(s)\rangle^2\, ds + 2\int_0^t \langle e_k, Y(s) - Z(s)\rangle\langle e_k, \tilde{F}_0\big(Y(s)\big) - \tilde{F}_0\big(Z(s)\big)\rangle\, ds, \quad t \geqslant 0,\ \mathbb{P}'\text{-a.s.}$$

Dropping the first term on the right-hand side and summing up over $k \in \mathbb{N}$ (which is justified by (4.11) and the continuity of Y and Z), we obtain from (H3) that

$$\big|Y(t) - Z(t)\big|^2 \leqslant 2\int_0^t \langle Y(s) - Z(s), \tilde{F}_0\big(Y(s)\big) - \tilde{F}_0\big(Z(s)\big)\rangle\, ds \leqslant 2\omega \int_0^t \big|Y(s) - Z(s)\big|^2\, ds, \quad t \geqslant 0,\ \mathbb{P}'\text{-a.s.}$$

Hence, by Gronwall's lemma $Y = Z$ $\mathbb{P}'$-a.s. and the Claim is proved.

By the Claim we can apply [11, Theorem 10, (1) $\Leftrightarrow$ (3)] and then [11, Theorem 1] to conclude that Eq. (4.11) has a strong solution (see [11, Definition 1]) and that there is one strong solution with the same law as X, which hence by (4.7) has continuous sample paths a.s. Now all conditions in [11, Theorem 13.2] are fulfilled and, therefore, we deduce from it that on any stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ with $(\mathcal{F}_t)$-cylindrical Wiener process W on H and for $x, y \in M$ there exist pathwise unique continuous strong solutions $X(t,x)$, $X(t,y)$, $t \geqslant 0$, to (4.11) such that

$$\mathbb{P} \circ X(\cdot, x)^{-1} = \mathbb{P}_x \circ X^{-1}$$

and

$$\mathbb{P} \circ X(\cdot, y)^{-1} = \mathbb{P}_y \circ X^{-1},$$

in particular, $X(0,x) = x$ and $X(0,y) = y$ and

$$\begin{aligned} \mathbb{P} \circ X(t,x)^{-1}(dz) &= p_t(x, dz), \quad t \geqslant 0, \\ \mathbb{P} \circ X(t,y)^{-1}(dz) &= p_t(y, dz), \quad t \geqslant 0. \end{aligned} \tag{4.13}$$

In particular, we have proved (i). To prove (ii), below for brevity we set $X := X(\cdot, x)$, $X' := X(\cdot, y)$. Then proceeding as in the proof of the Claim, by (1.13) and noting that $s^{-1}\Phi(s) \to \infty$ as $s \to \infty$, we obtain

$$\frac{d}{dt}\big|X(t) - X'(t)\big|^2 \leqslant a - \Phi_0\big(\big|X(t) - X'(t)\big|^2\big) \tag{4.14}$$

for some constant $a > 0$, only depending on ω and Φ, where $\Phi_0 = \frac{1}{2}\Phi$.

Now we consider two cases.

Case 1. $|x-y|^2 \leqslant \Phi_0^{-1}(2a)$.

Define $f(t) := |X(t) - X'(t)|^2$, $t \geqslant 0$, and suppose there exists $t_0 \in (0, \infty)$ such that

$$f(t_0) > \Phi_0^{-1}(a).$$

Then we can choose $\delta \in [0, t_0]$ maximal such that

$$f(t) > \Phi_0^{-1}(a), \quad \forall t \in (t_0 - \delta, t_0].$$

Hence, because by (4.14) f is decreasing on every interval where it is larger than $\Phi_0^{-1}(a)$, we obtain that

$$f(t_0 - \delta) \geqslant f(t_0) > \Phi_0^{-1}(a).$$

Suppose $t_0 - \delta > 0$. Then $f(t_0 - \delta) \leqslant \Phi_0^{-1}(a)$ by the continuity of f and the maximality of δ. So, we must have $t_0 - \delta = 0$, hence

$$f(t_0) \leqslant f(t_0 - \delta) = f(0) = |x - y|^2 \leqslant \Phi_0^{-1}(2a).$$

So,

$$\left|X(t) - X'(t)\right|^2 \leqslant \Phi_0^{-1}(2a), \quad \forall t > 0.$$

Case 2. $|x-y|^2 > \Phi_0^{-1}(2a)$.

Define $t_0 = \inf\{t \geqslant 0\colon\ |X(t) - X'(t)|^2 \leqslant \Phi_0^{-1}(2a)\}$. Then by Case 1, starting at $t = t_0$ rather than $t = 0$ we know that

$$\left|X(t) - X'(t)\right|^2 \leqslant \Phi_0^{-1}(2a), \quad \forall t \geqslant t_0. \tag{4.15}$$

Furthermore, it follows from (4.14) that

$$d\left|X(t) - X'(t)\right|^2 \leqslant -\frac{1}{2}\Phi_0\left(\left|X(t) - X'(t)\right|^2\right)dt, \quad \forall t \leqslant t_0.$$

This implies

$$\Psi\left(\left|X(t) - X'(t)\right|^2\right) \geqslant \frac{1}{2}\int\limits_{|X(t)-X'(t)|^2}^{|x-y|^2} \frac{dr}{\Phi_0(r)} \geqslant \frac{t}{4}, \quad \forall t \leqslant t_0.$$

Therefore,

$$\left|X(t) - X'(t)\right|^2 \leqslant \Psi^{-1}(t/4), \quad \forall t \leqslant t_0. \tag{4.16}$$

Combining Case 1, (4.15) and (4.16) we conclude that

$$\left|X(t) - X'(t)\right|^2 \leqslant \Psi^{-1}(t/4) + \Phi_0^{-1}(2a), \quad \forall t > 0. \tag{4.17}$$

Combining (4.17) with Theorem 1.6 for all $f \in B_b(H)$ we obtain

$$\left(p_{t/2}|f|\left(X(t/2)\right)\right)^2 \leqslant \left(p_{t/2}f^2\left(X'(t/2)\right)\right)\exp\left[\frac{\lambda(1+\Psi^{-1}(t/8))}{(1-\varepsilon^{-\omega t/2})^2}\right], \quad \forall t > 0$$

for some constant $\lambda > 0$. By Jensen's inequality and approximation it follows that for all $f \in L^2(H,\mu)$

$$\begin{aligned}\left(p_t|f|(x)\right)^2 &\leqslant \mathbb{E}\left(p_{t/2}|f|\left(X(t/2)\right)\right)^2 \\ &\leqslant \left(p_t f^2(y)\right)\exp\left[\frac{\lambda(1+\Psi^{-1}(t/8))}{(1-\varepsilon^{-\omega t/2})^2}\right], \quad \forall t > 0,\ \forall x, y \in M. \end{aligned}\tag{4.18}$$

But since $H_0 = \operatorname{supp}\mu$, M is dense in H_0, hence by the continuity of $p_t f$ (cf. Corollary 1.7) (4.18) holds for all $x \in H_0$, $y \in M$. Since $\mu(M) = 1$ this completes the proof by integrating both sides with respect to $\mu(dy)$. □

Remark 4.2. We would like to mention that by using [2] instead of [15] we can drop the assumption that $|F_0| \in L^2(H,\mu)$. So, by (4.9) and the proof above we can derive (4.8) avoiding to assume the usually energy condition

$$\int_0^t \left|F_0\left(X(s)\right)\right|^2 ds < \infty, \quad \mathbb{P}_x\text{-a.s.}$$

Details will be included in a forthcoming paper. We would like to thank Tobias Kuna at this point from whom we learnt identity (4.9) by private communication.

5. Existence of measures satisfying (H4)

To prove existence of invariant measures we need to strengthen some of our assumptions. So, let us introduce the following conditions.

(H1)′ $(A, D(A))$ is self-adjoint satisfying (1.2).
(H6) There exists $\eta \in (\omega, \infty)$ such that

$$\left\langle F_0(x) - F_0(y), x - y\right\rangle \leqslant -\eta|x-y|^2, \quad \forall x, y \in D(F).$$

Remark 5.1. (i) Clearly, (H1)′ implies (H1) and (H5). (H1)′ and (H2)(i) imply that $(A, D(A))$ and thus also $(1+\omega-A, D(A))$ has a discrete spectrum. Let $\lambda_i \in (0,\infty)$, $i \in \mathbb{N}$, be the eigenvalues of the latter operator. Then by (H2)

$$\sum_{i=1}^{\infty} \lambda_i^{-1} < \infty. \tag{5.1}$$

(ii) If we assume (5.1), i.e. that $(1+\omega-A)^{-1}$ is trace class, then all what follows holds with (H2) replaced by (H2)(i). So, $\sigma^{-1}\in L(H)$ is not needed in this case.

Let F_α, $\alpha<0$, be as in Section 2. Then e.g. by [5, Theorem 3.2] equation (1.1) with F_α replacing F_0 has a unique mild solution $X_\alpha(t,x)$, $t\geqslant 0$. Since there exist $\tilde\eta\in(\omega,\infty)$ and $\alpha_0>0$ such that each F_α, $\alpha\in(0,\alpha_0)$, satisfies (H6) with $\tilde\eta$ replacing η, by [5, Section 3.4] X_α has a unique invariant measure μ_α on $(H,\mathcal{B}(H))$ such that for each $m\in\mathbb{N}$

$$\sup_{\alpha\in(0,\alpha_0)}\int_H |x|^m\mu_\alpha(dx)<\infty. \tag{5.2}$$

That these moments are indeed uniformly bounded in α, follows from the proof of [5, Proposition 3.18] and the fact that $\tilde\eta\in(\omega,\infty)$.

Let N_Q denote the centered Gaussian measure on $(H,\mathcal{B}(H))$ with covariance operator Q defined by

$$Qx:=\int_0^\infty e^{tA}\sigma e^{tA}x\,dt,\quad x\in H,$$

which by (H2)(ii) is trace class.

Let $W^{1,2}(H,N_Q)$ be defined as usual, that is as the completion of $\mathcal{E}_A(H)$ with respect to the norm

$$\|\varphi\|_{W^{1,2}}:=\left(\int_H\left(\varphi^2+|D\varphi|^2\right)dN_Q\right)^{1/2},\quad \varphi\in\mathcal{E}_A(H),$$

where D denotes first Fréchet derivative. By [7] we know that

$$W^{1,2}(H,N_Q)\subset L^2(H,N_Q),\quad \text{compactly.} \tag{5.3}$$

Theorem 5.2. *Assume that* (H1)′, (H2), (H3) *and* (H6) *hold and let* $\mu_\alpha,\alpha\in(0,\alpha_0)$ *be as above. Suppose that there exists a lower semi-continuous function* $G:H\to[0,\infty]$ *such that*

$$\{G<\infty\}\subset D(F),\quad |F_0|\leqslant G\quad \text{on } D(F)\quad \text{and}\quad \sup_{\alpha\in(0,\alpha_0)}\int_H G^2\,d\mu_\alpha<\infty. \tag{5.4}$$

Then $\{\mu_\alpha\colon\ \alpha\in(0,\alpha_0)\}$ *is tight and any limit point* μ *satisfies* (H4) *and hence by Corollary* 1.8 *all of these limit points coincide. Furthermore, for all* $m\in\mathbb{N}$

$$\int_H\left(\left|F_0(x)\right|^2+|x|^m\right)\mu(dx)<\infty \tag{5.5}$$

and there exists $\rho:H\to[0,\infty)$, $\mathcal{B}(H)$*-measurable, such that* $\mu=\rho N_Q$ *and* $\sqrt{\rho}\in W^{1,2}(H,\mu)$.

Proof. We recall that by [3, Theorem 1.1] for each $\alpha \in (0, \alpha_0)$

$$\mu_\alpha = \rho_\alpha N_Q; \quad \sqrt{\rho_\alpha} \in W^{1,2}(H, N_Q) \tag{5.6}$$

and as is easily seen from its proof, that

$$\int_H |D\sqrt{\rho_\alpha}|^2 \, dN_Q \leqslant \frac{1}{4} \int_H |F_\alpha|^2 \, d\mu_\alpha. \tag{5.7}$$

But by (2.3) and (5.4) the right-hand side of (5.7) is uniformly bounded in α. Hence by (5.3) there exists a zero sequence $\{\alpha_n\}$ such that

$$\sqrt{\rho_{\alpha_n}} \to \sqrt{\rho} \quad \text{in } L^2(H, N_Q) \text{ as } n \to \infty,$$

for some $\sqrt{\rho} \in W^{1,2}(H, N_Q)$ and therefore, in particular,

$$\rho_{\alpha_n} \to \rho \quad \text{in } L^1(H, N_Q) \text{ as } n \to \infty. \tag{5.8}$$

Define $\mu := \rho N_Q$ and $\rho_n := \rho_{\alpha_n}$, $n \in \mathbb{N}$. Since G is lower semi-continuous and $\mu_{\alpha_n} \to \mu$ as $n \to \infty$ weakly, (5.2) and (5.4) imply

$$\int_H \big(G^2(x) + |x|^m\big)\mu(dx) < \infty, \quad \forall m \in \mathbb{N}. \tag{5.9}$$

Hence by (5.4) both (H4)(i) and (H4)(ii) follow. So, it remains to prove (H4)(iii).

Since σ is independent of α, to show (5.9) it is enough to prove that for all $\varphi \in C_b(H)$, $h \in D(A)$,

$$\lim_{n\to\infty} \int_H F^h_{\alpha_n}(x)\varphi(x)\mu_{\alpha_n}(dx) = \int_H F^h_0(x)\varphi(x)\, \mu(dx), \tag{5.10}$$

where $F^h_\alpha := \langle h, F_\alpha \rangle$, $\alpha \in [0, \alpha_0)$. We have

$$\begin{aligned} &\left| \int_H F^h_{\alpha_n}\varphi \, d\mu_{\alpha_n} - \int_H F^h_0 \varphi \, d\mu \right| \\ &\quad \leqslant \|\varphi\|_\infty \int_H \big|F^h_{\alpha_n} - F^h_0\big| \rho_n \, dN_Q + \int_H \big|F^h_0 \varphi\big| |\rho_n - \rho| \, dN_Q. \end{aligned} \tag{5.11}$$

But by (2.3) and (5.4) we have

$$\begin{aligned} \int_H \big|F^h_{\alpha_n} - F^h_0\big|\rho_n \, dN_Q \leqslant &\int_{\{|G| \leqslant M\}} \big|F^h_{\alpha_n} - F^h_0\big|\rho_n \, dN_Q \\ &+ \frac{2|h|}{M} \sup_{\alpha \in (0,\alpha_0)} \int_H G^2 \, d\mu_\alpha. \end{aligned}$$

Hence first letting $n \to \infty$ then $M \to \infty$ by (2.2), (5.4) and (5.8) Lebesgue's generalized dominated convergence theorem implies that the first term on the right-hand side of (5.11) converges to 0. Furthermore, for every $\delta \in (0,1)$

$$\left| \int_H F_0^h \varphi \, d\mu_{\alpha_n} - \int_H F_0^h \varphi \, d\mu \right| \leqslant \left| \int_H \frac{F_0^h}{1+\delta|F_0^h|} \varphi(\rho_n - \rho) \, dN_Q \right| + \delta \|\varphi\|_\infty \left(\int_H |F_0^h|^2 \, d\mu_{\alpha_n} + \int_H |F_0^h|^2 \, d\mu \right). \tag{5.12}$$

Since by (2.3) and (5.4)

$$\sup_{\alpha \in (0,\alpha_0)} \int_H |F_0^h|^2 \, d\mu_\alpha < \infty,$$

(H4)(iii) follows from (5.12) by letting first $n \to \infty$ and then $\delta \to 0$, since for fixed $\delta > 0$ the first term in the right-hand side converges to zero by (5.8). □

Example 5.3. Let $H = L^2(0,1)$, $Ax = \Delta x$, $x \in D(A) := H^2(0,1) \cap H_0^1(0,1)$. Let $f : \mathbb{R} \to \mathbb{R}$ be decreasing such that for some $c_3 > 0$, $m \in \mathbb{N}$,

$$|f(s)| \leqslant c_3 (1 + |s|^m), \quad \forall s \in \mathbb{R}. \tag{5.13}$$

Let $s_i \in \mathbb{R}$, $i \in \mathbb{N}$, be the set of all arguments where f is not continuous and define

$$\bar{f}(s) = \begin{cases} [f(s_{i+}), f(s_{i-})], & \text{if } s = s_i \text{ for some } i \in \mathbb{N}, \\ f(s), & \text{else.} \end{cases}$$

Define

$$F : D(F) \subset H \to 2^H, \quad x \mapsto \bar{f} \circ x,$$

where

$$D(F) = \{x \in H \colon \ \bar{f} \circ x \subset H\}.$$

Then F is m-dissipative. Let F_0 be defined as in Section 2.

Since $A \leqslant \omega$ for some $\omega < 0$, it is easy to check that all conditions (H1)′, (H2), (H3), (H6) with $\eta = 0$ hold for any $\sigma \in L(H)$ such that $\sigma^{-1} \in L(H)$. Define

$$G(x) := \begin{cases} \left(\int_0^1 |x(\xi)|^{2m} d\xi \right)^{1/2} & \text{if } x \in L^{2m}(0,1), \\ +\infty & \text{if } x \notin L^{2m}(0,1). \end{cases}$$

Then $\{G < \infty\} \subset D(F)$ and $|F_0| = |F_0|_{L^2(0,1)} \leqslant G$ on $D(F)$. Furthermore, by [6, (9.3)]

$$\sup_{\alpha \in (0,\alpha_0)} \int_H G^2 \, d\mu_\alpha < \infty. \tag{5.14}$$

Note that from [6, Hypothesis 9.5] only the first inequality, which clearly holds by (5.13) in our case, was used to prove [6, (9.3)]. Hence all assumptions of Theorem 5.2 above hold and we obtain the existence of the desired unique probability measure μ satisfying (H4) in this case. We emphasize that no continuity properties of f and F_0 are required. In particular, then all results stated in Section 1 except for Corollary 1.10(ii) hold in this case.

If moreover there exists an increasing positive convex function Φ on $[0,\infty)$ satisfying (1.12) such that

$$\big(f(s)-f(t)\big)(s-t) \leqslant c-\Phi\big(|s-t|^2\big), \quad s,t\in\mathbb{R},$$

then by Jensen's inequality (1.13) holds. Hence, by Corollary 1.10 one obtains an explicit upper bound for $\|p_t\|_{L^2(H,\mu)\to L^\infty(H,\mu)}$. A natural and simple choice of Φ is $\Phi(s)=s^m$ for $m>1$.

One can extend these results to the case, where $(0,1)$ above is replaced by a bounded open set in $\mathbb{R}^d$, $d=2$ or 3 for $\sigma=(-\Delta)^\gamma$, $\gamma\in(\frac{d-2}{4},\frac{1}{2})$, based on Remark 1.1(iv).

Before to conclude we want to present a condition in the general case (i.e for any Hilbert space H as above) that implies (5.4), hence by Theorem 5.2 ensures the existence of a probability measure satisfying (H4) so that all results of Section 1 apply also to this case. As will become clear from the arguments below, such condition is satisfied if the eigenvalues of A grow fast enough in comparison with $|F_0|$. To this end we first note that by (5.1) for $i\in\mathbb{N}$ we can find $q_i\in(0,\lambda_i)$, $q_i\uparrow\infty$ such that $\sum_{i=1}^\infty q_i^{-1}<\infty$ and $\frac{q_i}{\lambda_i}\to 0$ as $i\to\infty$. Define $\Theta: H\to[0,\infty]$ by

$$\Theta(x) := \sum_{i=1}^\infty \frac{\lambda_i}{q_i}\langle x,e_i\rangle^2, \quad x\in H, \tag{5.15}$$

where $\{e_i\}_{i\in N}$ is an eigenbasis of $(1+\omega-A, D(A))$ such that e_i has eigenvalue λ_i. Then Θ has compact level sets and $|\cdot|^2\leqslant\Theta$.

Below we set

$$H_n := \text{lin span}\,\{e_1,\dots,e_n\}, \qquad \pi_n := \text{projection onto } H_n,$$

$$\tilde{A} := A-(1+\omega)I, \qquad D(\tilde{A}) := D(A), \tag{5.16}$$

$$\tilde{F}_0 := F_0+(1+\omega)I. \tag{5.17}$$

We note that obviously $H_n\subset\{\Theta<+\infty\}$ for all $n\in\mathbb{N}$.

Theorem 5.4. *Assume that* (H1)$'$, (H2)*,* (H3) *and* (H6) *hold and let* μ_α, $\alpha\in(0,\alpha_0)$*, be as above. Suppose that* $\{\Theta<+\infty\}\subset D(F)$ *and that for some* $C\in(0,\infty)$*,* $m\in\mathbb{N}$

$$\big|F_0(x)\big| \leqslant C\big(1+|x|^m+\Theta^{1/2}(x)\big), \quad \forall x\in D(F). \tag{5.18}$$

Then

$$\sup_{\alpha\in(0,\alpha_0)}\int_H \Theta\,d\mu_\alpha<\infty \tag{5.19}$$

and (5.4) *holds, so Theorem* 5.2 *applies.*

Proof. Consider the Kolmogorov operator L_α corresponding to $X_\alpha(t,x)$, $t \geqslant 0$, $x \in H$, which for $\varphi \in \mathcal{F}C_b^2(\{e_n\})$, i.e., $\varphi = g(\langle e_1, \cdot\rangle, \dots, \langle e_N, \cdot\rangle)$ for some $N \in \mathbb{N}$, $g \in C_b^2(\mathbb{R}^N)$, is given by

$$L_\alpha\varphi(x) := \frac{1}{2}\operatorname{Tr}\bigl[\sigma^2 D^2\varphi(x)\bigr] + \bigl\langle x, AD\varphi(x)\bigr\rangle + \bigl\langle F_\alpha(x), D\varphi(x)\bigr\rangle, \quad x \in H, \tag{5.20}$$

where D^2 denotes the second Fréchet derivative. Then, an easy application of Itô's formula shows that the $L^1(H,\mu_\alpha)$-generator of (P_t^α) (given as before by $P_t^\alpha f(x) = \mathbb{E}[f(X_\alpha(t,x))]$) is given on $\mathcal{F}C_b^2(\{e_n\})$ by L_α. In particular,

$$\int_H L_\alpha\varphi\, d\mu_\alpha = 0, \quad \forall\varphi \in \mathcal{F}C_b^2\bigl(\{e_n\}\bigr).$$

By a simple approximation argument and (5.2) we get for $\alpha \in (0,\alpha_0)$ and

$$\varphi_n(x) := \sum_{i=1}^n q_i^{-1}\langle x, e_i\rangle^2, \quad x \in H,\ n \in \mathbb{N},$$

that also

$$\int_H L_\alpha\varphi_n\, d\mu_\alpha = 0. \tag{5.21}$$

But for all $x \in H$, with $\tilde F_\alpha$ defined as $\tilde F_0$ in (5.17), we have

$$\begin{aligned}
L_\alpha\varphi_n(x) &= -2\sum_{i=1}^n \frac{\lambda_i}{q_i}\langle x, e_i\rangle^2 + 2\sum_{i=1}^n q_i^{-1}\bigl\langle \tilde F_\alpha(x), e_i\bigr\rangle\langle x, e_i\rangle \\
&\quad + \sum_{i,j=1}^n q_i^{-1}\langle \sigma_n e_i, \sigma_n e_j\rangle \\
&\leqslant -2\Theta(\pi_n x) + 2\left(\sum_{i=1}^n q_i^{-1}\bigl\langle \tilde F_\alpha(x), e_i\bigr\rangle^2\right)^{1/2}\left(\sum_{i=1}^n q_i^{-1}\langle x, e_i\rangle^2\right)^{1/2} \\
&\quad + \sum_{i=1}^n q_i^{-1}|\sigma_n e_i|^2 \\
&\leqslant -2\Theta(\pi_n x) + c_1\bigl(1 + |x|^{m+1} + \Theta^{1/2}(x)|x|\bigr) + \|\sigma\|^2\sum_{i=1}^\infty q_i^{-1},
\end{aligned} \tag{5.22}$$

for some constant c_1 independent of n and α. Here we used (2.3) and (5.18). Now (5.21), (5.2) and (5.22) immediately imply that for some constant $\tilde c_1$

$$\sup_{\alpha\in(0,\alpha_0)}\int_H \Theta(x)\mu_\alpha(dx) \leqslant \sup_{\alpha\in(0,\alpha_0)} \tilde c_1\left(1 + \int_H |x|^{m+2}\mu_\alpha(dx)\right) + \|\sigma\|^2\sum_{i=1}^\infty q_i^{-1} < \infty.$$

So, (5.19) is proved, which by (5.18) implies (5.4) and the proof is complete. □

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