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Canonical entropy of three-dimensional BTZ black hole

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Abstract

Recently, Hawking radiation of the black hole has been studied using the tunnel effect method. It is found that the radiation spectrum of the black hole is not a strictly pure thermal spectrum. How does the departure from pure thermal spectrum affect the entropy? This is a very interesting problem. In this Letter, we calculate the partition function by energy spectrum obtained from tunnel effect. Using the partition function, we compute the black hole entropy and derive the expression of the black hole entropy after considering the radiation. And we derive the entropy of charged black hole. In our calculation, we consider not only the correction to the black hole entropy due to fluctuation of energy but also the effect of the change of the black hole charges on entropy. There is no other hypothesis. Our result is more reasonable. According to the fact that the black hole entropy is not divergent, we obtain the lower limit of Banados–Teitelboim–Zanelli black hole energy. That is, the least energy of Banados–Teitelboim–Zanelli black hole, which satisfies the stationary condition in thermodynamics.

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Keywords: Thermal fluctuation; Canonical ensemble; Quantum correction; Black hole entropy

1. Introduction

Black holes have an event horizon from which any matter or information cannot escape. The loss of the information in horizon region shows that the horizon has a property of entropy. There are many methods to calculate the entropies of thermodynamic quantities. And each method means that taking the horizon area of the black hole as an entropy is self-consistent. It is caused by their similarity [1,2]. When the horizon area of the black hole has been taken as entropy, the energy and temperature of the black hole satisfy thermodynamic law. Since Hawking radiation has been discovered, this similarity is described quantificationally [3]. But, how to measure the microstate of the black hole by entropy? This problem has not been solved. At present, discussing the entropy of black hole–matter coupling system becomes a meaningful problem. This problem may provide a way for solving the difficulty of quantum gravitation. Many researchers have expressed a vested interest in fixing the relation between statistical mechanic entropy and thermodynamic entropy [4–8].

In recent years, string theory and loop quantum gravity both had succeeded statistically in explaining the black hole entropy–area law [9,10]. However, which one is perfect? It is expected to choose it by discussing the quantum correction term of the black hole entropy. Therefore, studying the black hole entropy correction value becomes the focus of attention. Many ways of discussing the black hole entropy correction value have emerged [11–18]. Based on string theory and loop quantum gravity, the relationship of the black hole entropy–area is given by [19].

$$S = \frac{A}{4L_p^2} + \rho \ln \frac{A}{4L_p^2} + O\left(\frac{L_p^2}{A}\right), \tag{1}$$

where $A = 16\pi L_p^2 M^2$ is the area of the black hole horizon, $L_p = \sqrt{\hbar G}$ is Planck length. Ref. [13] obtained $\rho = -1/2$ in four dimension space–time.

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On the other hand, there has been much attention devoted to the lower-dimensional gravitation theory region. Recently, the study of two-dimensional black hole thermodynamics shows that entropy satisfies the area relation and the second law of thermodynamics [20–22]. In 1992 Banados, Teitelboim and Zanelli (BTZ) [23,24] showed that $(2+1)$-dimensional gravity has a black hole solution. This black hole is described by two (gravitational) parameters, the mass M and the angular momentum J. It is locally AdS and thus it differs from the Schwarzschild and Kerr solutions since it is an asymptotically anti-de Sitter instead of a flat space–time. Additionally, it has no curvature singularity at the origin. AdS black holes are members of this two-parametric family of BTZ black holes, and they are very interesting in the framework of string theory and black hole physics [25,26].

Recently, a new explanation for Hawking radiation process of the black hole is tunnel process. Based on it, the radiation spectrum is derived. However, this radiation spectrum has a departure from pure thermal spectrum. How does it affect the black hole entropy? In this Letter, we calculate the partition function of BTZ black hole using the radiation spectrum obtained in tunnel process. Furthermore, we derive the entropy of canonical black hole. In our calculation, there is no the other hypothesis. We provide a new way for discussing the entropy of canonical black hole. We take the simple function form of temperature ($c = K_B = 1$).

2. Canonical partition function

Parikh and Wilczek [27] discussed Hawking radiation by tunnel effect. They thought that tunnels in the process of the particle radiation had no potential barrier before particles radiated. Potential barrier is produced by radiation particles itself. That is, during the process of tunnel effect creation, the energy of the black hole decreases and the radius of the black hole horizon reduces. The horizon radius becomes a new value that is smaller than the original value. The decrease of radius is determined by the value of energy of radiation particles. There is a classical forbidden band-potential barrier between original radius and the one after the black hole radiates. Parikh and Wilczek skillfully obtained the radiation spectrum of Schwarzschild and Reissner–Nordström black holes. Refs. [28–37] developed the method proposed by Parikh and Wilczek. They derived the radiation spectrum of the black hole in all kinds of space–time. Refs. [38–41] obtained radiation spectrum of Hawking radiation after considering the generalized uncertainty relation. And Angheben, Nadalini, Vanzo and Zerbini have computed the radiation spectrum of the arbitrary-dimensional black hole and obtained the energy spectrum of radiation particles of general black hole [37,42]

$$\rho_s \propto e^{\Delta S}, \quad \text{where} \tag{2}$$

$$\begin{aligned}\Delta S &= S_{\mathrm{MC}}(E-E_s) - S_{\mathrm{MC}}(E)\\ &= \sum_{k=1} \frac{1}{k!}\left(\frac{\partial^k S_{\mathrm{MC}}(E_b)}{\partial E_b^k}\right)_{E_s=0} (-E_s)^k\\ &= -\beta E_s + \beta_2 E_s^2 + \cdots,\end{aligned} \tag{3}$$

where, $E_b = E - E_s$, E_s is energy of radiated particle, $S_{\mathrm{MC}}(E-E_s)$ is entropy of microcanonical ensemble with energy $(E-E_s)$ (namely Bekenstein–Hawking entropy of black hole with energy $(E-E_s)$). According to the relation of thermodynamics, β should be the inverse of the temperature.

$$\beta_k = \frac{1}{k!}\left(\frac{\partial^k \ln \Omega}{\partial E_b}\right)_{E_s=0} = \frac{1}{k!}\left(\frac{\partial^k S_{\mathrm{MC}}}{\partial E_b}\right)_{E_s=0}. \tag{4}$$

Normalizing the distribution function ρ_s, we obtain $\rho_s = \frac{1}{Z_c} e^{\Delta S}$, where

$$Z_c = \sum_{E_s} \rho(E-E_s) e^{S_{\mathrm{MC}}(E-E_s) - S_{\mathrm{MC}}(E)} \tag{5}$$

is the partition function. For the semi-classical thermal equilibrium system, the canonical partition function can be expressed as

$$Z_c(\beta) = \int_0^\infty e^{\Delta S}\, dE_s\, \rho(E-E_s), \tag{6}$$

where $\rho(E-E_s)$ is a density of state of ensemble (black hole) with energy $(E-E_s)$. From Ref. [43], we have $\rho(E-E_s) \equiv e^{S_{\mathrm{MC}}(E-E_s)}$.

The canonical entropy is expanded a Taylor series near energy E,

$$S_{\mathrm{MC}}(E-E_s) = S_{\mathrm{MC}}(E) - \beta E_s + \beta_2 E_s^2 + \cdots. \tag{7}$$

When we neglect the higher-order small term, Eq. (6) can be rewritten as

$$\begin{aligned}Z_c(\beta) &= \int_0^\infty e^{-\beta E_s + \beta_2 E_s^2}\, dE_s\, e^{S_{\mathrm{MC}}(E-E_s)}\\ &= e^{S_{\mathrm{MC}}(E)} \int_0^\infty dE_s\, e^{-2\beta E_s + 2\beta_2 E_s^2}\\ &= e^{S_{\mathrm{MC}}(E)} \left[\frac{1}{2}\sqrt{\frac{\pi}{-2\beta_2}} \exp\left(\frac{\beta^2}{-2\beta_2}\right)\right.\\ &\quad \left.\times \left(1 - \mathrm{erf}\left(\frac{\beta}{\sqrt{-2\beta_2}}\right)\right)\right],\end{aligned} \tag{8}$$

where

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt$$

is error integral.

3. Canonical entropy

According to the relation between the partition function and entropy

$$S = \ln Z - \beta \frac{\partial \ln Z}{\partial \beta}, \tag{9}$$

we obtain that the entropy of the canonical system is

$$S_C(E) = S_{\mathrm{MC}}(E) + \Delta_S, \tag{10}$$

where

$$\Delta_S = \ln f(\beta) - \beta \frac{\partial \ln f(\beta)}{\partial \beta}, \tag{11}$$

$$f(\beta) = \frac{1}{2}\sqrt{\frac{\pi}{-2\beta_2}} \exp\left(\frac{\beta^2}{-2\beta_2}\right)\left[1 - \mathrm{erf}\left(\frac{\beta}{\sqrt{-2\beta_2}}\right)\right]. \tag{12}$$

According to the asymptotic expression of the error function

$$\mathrm{erf}(z) = 1 - \frac{e^{-z^2}}{\sqrt{\pi} z}\left[1 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{(2z^2)^k}\right], \quad |z| \to \infty,$$

we have

$$f(\beta) = \frac{1}{2\beta}\left[1 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{2^k}\left(\frac{\sqrt{-2\beta_2}}{\beta}\right)^{2k}\right]. \tag{13}$$

Substituting (13) into (11), we derive

$$\Delta_S = \ln\left[\frac{1}{2\beta} + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{2^k 2\beta}\left(\frac{\sqrt{-2\beta_2}}{\beta}\right)^{2k}\right] + \frac{1 + \sum_{k=1}^{\infty} (-1)^k (2\sqrt{-2\beta_2})^{2k} \frac{(2k+1)(2k-1)!!}{2^k (2\beta)^{2k}}}{1 + \sum_{k=1}^{\infty} (-1)^k (2\sqrt{-2\beta_2})^{2k} \frac{(2k-1)!!}{2^k (2\beta)^{2k}}}. \tag{14}$$

The thermal capacity of the system is

$$C \equiv -\beta^2\left(\frac{\partial E}{\partial \beta}\right), \tag{15}$$

and

$$\beta_2 = -\frac{1}{2}\frac{\beta^2}{C}. \tag{16}$$

Then Eq. (14) can be expressed as

$$\Delta_S = \ln\left[T + T\sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{2^k C^k}\right] + \frac{1 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k+1)(2k-1)!!}{2^k C^k}}{1 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{2^k C^k}}, \tag{17}$$

where T is the temperature of the system. When we only consider the logarithmic correction term

$$\Delta_S \approx \ln\left[T + T\sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{2^k C^k}\right]. \tag{18}$$

In error function, we take the sum k from one to n as the approximate value of the series. When z is a real number, its error does not exceed the absolute value of the first term neglected in the series. Therefore, when $C < -1$ or $C > 1$, the first term in Δ_S is not divergent.

4. Canonical entropy of BTZ black hole

For the non-rotating Banados–Teitelboim–Zanelli (BTZ) black hole [23]

$$ds^2 = -\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)dt^2 + \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{-1} dr^2 + r^2\left(d\theta - \frac{J}{2r^2}dt\right)^2, \tag{19}$$

where, M is the Arnowitt–Deser–Misner (ADM) mass, J is the angular momentum (spin) of the BTZ black hole, $l^2 = 1/\Lambda^2$ and Λ is the cosmological constant.

The outer and inner horizon, i.e., r_+ (henceforth simply the black hole horizon) and r_- respectively, concerning the positive mass black hole spectrum with spin ($J \neq 0$) of the line element (19) are given by

$$r_\pm^2 = \frac{l^2}{2}\left(M \pm \sqrt{M^2 - \frac{J^2}{l^2}}\right), \tag{20}$$

and therefore, in terms of the inner and outer horizons, the black hole mass and the angular momentum are given, respectively, by

$$M = \frac{r_+^2}{l^2} + \frac{J^2}{4r_+^2}, \quad J = \frac{2r_+ r_-}{l}. \tag{21}$$

The Hawking temperature T_H of the black hole horizon is [44]

$$T_H = \frac{1}{2\pi r_+}\sqrt{\left(\frac{r_+^2}{l^2} + \frac{J^2}{4r_+^2}\right)^2 - \frac{J^2}{l^2}} = \frac{1}{2\pi r_+}\left(\frac{r_+^2}{l^2} - \frac{J^2}{4r_+^2}\right). \tag{22}$$

In two space–time dimensions we do not have an area law for the black hole entropy; however, one can use a thermodynamic reasoning to define the entropy [44]

$$S_{\mathrm{MC}} = 4\pi r_+. \tag{23}$$

The specific heat of the black hole is given by [45]

$$C = \frac{dE}{dT_H} = \frac{dM}{dT_H} = 4\pi r_+\left(\frac{r_+^2 - r_-^2}{r_+^2 + 3r_-^2}\right) = S_{\mathrm{MC}}\left(\frac{r_+^2 - r_-^2}{r_+^2 + 3r_-^2}\right). \tag{24}$$

When $r_+ \gg r_-$, $C \approx S_{\mathrm{MC}}$, based on the Eq. (22) $T_H \approx \frac{S_{\mathrm{MC}}}{8\pi^2 l^2}$, then Eq. (18) can be rewritten as

$$\Delta_S \approx \ln S_{\mathrm{MC}} + \ln\left[1 + \sum_{k=1} (-1)^k \frac{(2k-1)!!}{2^k S_{\mathrm{MC}}^k}\right] + \mathrm{const.} \tag{25}$$

Based on Eq. (25), when $S_{\mathrm{MC}} > 1$, the entropy is not divergent. Therefore, we can obtain that the energy of BTZ black hole satisfies the following condition.

$$r_+ > \frac{1}{4\pi}. \tag{26}$$

From Eq. (21), when $r_+ \gg r_-$, $r_+^2 \approx Ml^2$. So when the black hole energy satisfies $M > (\frac{1}{4\pi l})^2$, the entropy is not divergent.

Otherwise, the entropy is divergent. It means that the black hole is not at the thermodynamic stationary state. That is, in universe there is no BTZ black hole with energy that is smaller than this energy.

According to Eq. (24), when $r_+ \to r_-$, $C \to 0$, the logarithmic term in entropy is divergent. So we obtain that the black hole is mechanic unstable in this case. BTZ black hole cannot become an extreme black hole by adjusting the value of J. The extreme black hole exists at the very start of the universe. This is consistent with the view of Refs. [46,47].

5. Conclusion and discussion

Ref. [11] obtained the following result, when they discussed the correction to entropy of Schwarzschild black hole by the generalized uncertainty relation.

$$S = \frac{A}{4} - \frac{\pi\alpha^2}{4}\ln\left(\frac{A}{4}\right) + \sum_{n=1}^{\infty} C_n \left(\frac{A}{4}\right)^{-n} + \text{const.} \tag{27}$$

According to Eq. (27), there is a uncertain factor α^2 in the logarithmic term in the correction to entropy. However, in our result, there is no uncertain factor in Eq. (25).

After considering the correction to the black hole thermodynamic quantities due to thermal fluctuation, the expression of entropy is [48–50]

$$S = \ln\rho = S_{\text{MC}} - \frac{1}{2}\ln\left(CT^2\right) + \cdots. \tag{28}$$

There is a limitation in the above result. That is the thermal capacity of Schwarzschild black hole is negative. This leads to the logarithmic correction term divergent given by Eq. (28). So this relation is not valid to Schwarzschild black hole. However, for general four-dimensional curved space–times, when we take a proper approximation or limit, they can return to Schwarzschild space–times. This implies that Eq. (28) has not universality. However, in our result we only request the thermal capacity satisfies $C < -1$ or $C > 1$. According to the discussion to Schwarzschild black hole, we obtain that this condition may be the condition that the black hole exists.

In addition, the research of the black hole entropy is based on the fact that the black hole has thermal radiation and the radiation spectrum is a pure thermal spectrum. However, Hawking obtained that the radiation spectrum is a pure thermal spectrum only under the condition that the background of space–time is invariable. During this radiation process, there exist the information loss. The information loss of the black hole means that the pure quantum state will disintegrate to a mixed state. This violates the unitarity principle in quantum mechanics. When we discuss the black hole radiation by the tunnel effect method, after considering the conversation of energy and the change of the horizon, we derive that the radiation spectrum is no longer a strict pure thermal spectrum. This method can avoid the limit of Hawking radiation and point out that the self-gravitation provides the potential barrier of quantum tunnel.

Our discussion is based on the quantum tunnel effect of the black hole radiation. So our discussion is very reasonable. We provide a way for studying the quantum correction to Bekenstein–Hawking entropy. Based on our method, we can further check the string theory and single Loop quantum gravity and determine which one is perfect. When the thermal capacity of the black hole satisfies $0 \leqslant C \leqslant 1$, the logarithmic correction term of the black hole may be divergent. For general black hole, it needs further discuss that this divergent implies physics characteristic.

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