Existence and construction of nonnegative matrices with prescribed spectrum

Ricardo L. Soto*

Departamento de Matemáticas, Universidad Católica del Norte, Casilla 1280, Antofagasta, Chile
Received 21 December 2001; accepted 3 December 2002
Submitted by R.A. Brualdi

Abstract

We consider the following inverse spectrum problem for nonnegative matrices: given a set of real numbers \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \), find necessary and sufficient conditions for the existence of an \( n \times n \) nonnegative matrix \( A \) with spectrum \( \sigma \). In particular, by the use of a relevant theorem of Brauer we obtain new simple sufficient conditions for the problem to have a solution. Moreover, we can always construct a solution matrix, which is nonnegative generalized stochastic.

© 2003 Elsevier Science Inc. All rights reserved.
AMS classification: 15A18; 15A51
Keywords: Inverse spectrum problem; Nonnegative generalized stochastic matrices

1. Introduction

Definition 1. A matrix \( A = (a_{ij}) \) of order \( n \) is said to be nonnegative if \( a_{ij} \geq 0 \), \( i, j = 1, 2, \ldots, n \).

The following inverse spectrum problem has been studied by many authors: given a set \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) of complex numbers, find necessary and sufficient conditions for the existence of an \( n \times n \) nonnegative matrix \( A \) with spectrum \( \sigma \). If there exists a nonnegative matrix \( A \) with spectrum \( \sigma \), we shall say that \( \sigma \) is realized by \( A \).

—

* Fax: +56-55-355599.
E-mail address: rsoto@ucn.cl. (R.L. Soto).
In this work, we shall consider the nonnegative inverse spectrum problem when 
\[ \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \]
is a set of real numbers with
\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0 > \lambda_{p+1} \geq \cdots \geq \lambda_n. \] (1)

By the use of a relevant theorem of Brauer [2], we find some new simple sufficient conditions for the problem to have a solution. Moreover, we can always compute a solution matrix \( A \), explicitly.

This problem, called nonnegative inverse spectrum problem, is a difficult one and it remains unsolved. In the general case, when \( \sigma \) is a set of complex numbers, the problem has only been solved for \( n = 3 \) by Loewy and London [8]. The cases \( n = 4 \) and 5 have been solved for matrices of trace zero, by Reams [12] and Laffey and Meehan [6], respectively. A necessary and sufficient condition for the existence of \( n \times n \) circulant nonnegative matrices with complex spectrum has been recently obtained in [13].

For the nonnegative inverse spectrum problem we have three basic necessary conditions. If \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) is the spectrum of a nonnegative matrix \( A \), then:

(i) \( \overline{\sigma} = \{\overline{\lambda_1}, \overline{\lambda_2}, \ldots, \overline{\lambda_n}\} = \sigma \),

(ii) \[ \max_j |\lambda_j| \in \sigma, \] (2)

(iii) \[ s_m(\sigma) = \sum_{j=1}^{n} \lambda_j^m \geq 0, \quad m = 1, 2, \ldots \]

The most important necessary condition for the inverse spectrum problem for nonnegative matrices is given by the following theorem, due to Loewy and London [8]:

**Theorem 2.** Let \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) be the spectrum of a nonnegative matrix of order \( n \). Then
\[ (s_k(\sigma))^m \leq n^{m-1} s_{km}(\sigma), \] (3)
for any positive integers \( k \) and \( m \).

When \( n \) is odd and the nonnegative matrix \( A \) have trace zero, Laffey and Meehan [7] proved the following refinement of the inequality (3):
\[ (n - 1)s_4(\sigma) \geq (s_2(\sigma))^2. \] (4)

The following example, given in [8], shows that the basic necessary conditions (2) are not sufficient. In fact, let \( \sigma = \{\sqrt{2}, i, -i\} \). Then, (i), (ii) and (iii) are satisfied. However, the Loewy and London condition (3) is not satisfied for \( k = 1 \) and \( m = 2 \). \( \sigma = \{5, 4, -3, -3, -3\} \) satisfies all three basic necessary conditions and also satisfies the Loewy and London condition (3). However, the refinement of Laffey and Meehan (4) is not satisfied by \( \sigma \). Therefore, \( \sigma \) is not realized by a nonnegative matrix.
Important progresses have been done in the case of a real prescribed spectrum by Suleimanova [16], Perfect [10], Salzmann [14], Kellogg [5], Fiedler [3], Soules [15], Borobia [1] and Radwan [11].

**Definition 3.** A nonnegative $n \times n$ matrix $A$ is called row stochastic, or simply stochastic, if all its rows sum 1.

**Definition 4.** A matrix $A = (a_{kj})$ of order $n$ is said to be generalized stochastic if
\[\sum_{j=1}^{n} a_{kj} = s, \quad k = 1, 2, \ldots, n.\]

Given a set $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, the problem of finding a nonnegative matrix with spectrum $\sigma$ is equivalent to the problem of finding a nonnegative generalized stochastic matrix with spectrum $\sigma$. Johnson [4] has proved the equivalence between both problems.

The following theorem due to Wielandt [17] gives a property of the spectrum of an irreducible nonnegative matrix.

**Theorem 5.** The spectrum of an irreducible matrix of imprimitivity index $h$ is invariant under a rotation through $2\pi / h$, but not through a positive angle smaller than $2\pi / h$.

As a consequence we have that if $A$ is a nonnegative irreducible matrix of order $n$ with real spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and index $h = 2$, then, its spectrum is of the form
\[\lambda_1, \lambda_2, \ldots, \lambda_p, -\lambda_p, \ldots, -\lambda_2, -\lambda_1,\]
when $n$ is even and it is of the form
\[\lambda_1, \lambda_2, \ldots, \lambda_p, 0, -\lambda_p, \ldots, -\lambda_2, -\lambda_1,\]
when $n$ is odd.

Since we shall consider real numbers satisfying (1), then we shall have the following two basic necessary conditions:

(i) $\sum_{k=1}^{n} \lambda_k \geq 0$ and (ii) $\lambda_1 \geq \max_{2 \leq k \leq n} |\lambda_k|$.

It is known that for $n \leq 4$ and $\sigma$ real, necessary conditions (5) are also sufficient. However, for $n \geq 5$, necessary conditions (5) are not sufficient. In fact, let $\sigma = \{2, 1, \frac{1}{2}, -1, -2\}$. If there exists a nonnegative irreducible matrix $A$ with spectrum $\sigma$, then it has index of imprimitivity $h = 2$ and from the Wielandt Theorem 5, $\sigma$ must be invariant under a rotation through an angle $\pi$, which is impossible. Then $A$ is reducible and $\sigma$ must split into two subsets, each of one being the spectrum of a nonnegative matrix, which is also impossible.
2. Preliminary results

The following theorems of Brauer ([2], Theorems 27 and 33) are relevant for the study of the nonnegative inverse spectrum problem. They will be stated here as Theorems Brauer A and Brauer B. In particular, theorem Brauer B plays a fundamental role not only to derive sufficient conditions for the problem to have a solution, but also to compute a solution.

Theorem 6 (Brauer A). If \( A = (a_{kj}) \) is a generalized stochastic matrix of order \( n \) with row sum \( s \) and \( a_{kj} = bj, j = 2, 3, \ldots, n, k < j, \) then \( A \) has eigenvalues \( s, a_{22} - b_2, a_{33} - b_3, \ldots, a_{nn} - b_n. \)

Theorem 7 (Brauer B). Let \( A \) be an \( n \times n \) arbitrary matrix with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n. \) Let \( v = (v_1, v_2, \ldots, v_n)^T \) an eigenvector of \( A \) associated with the eigenvalue \( \lambda_k \) and let \( q \) be any \( n \)-dimensional vector. Then the matrix \( A + vq^T \) has eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k + v^T q, \lambda_{k+1}, \ldots, \lambda_n, k = 1, 2, \ldots, n. \)

Corollary 8. If \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) is realized by a nonnegative matrix, then \( \sigma + \{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\}, \epsilon > 0, \) is also realized by a nonnegative matrix.

Proof. Let \( A \) be a nonnegative matrix with spectrum \( \sigma. \) Then, there exists a nonnegative vector \( x = (x_1, x_2, \ldots, x_n)^T \) such that \( Ax = \lambda_1 x. \) Let the \( p \)th entry of \( x \) be positive, that is, \( x_p > 0 \) and

\[
q = \left( 0, \ldots, 0, \frac{\epsilon}{x_p}, 0, \ldots, 0 \right)^T,
\]

where \( \epsilon/x_p \) is in the \( p \)th position. Then,

\[
A_\epsilon = A + xq^T
\]

has the spectrum \( \sigma_\epsilon. \) \( \Box \)

Corollary 9. If \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) is realized by a positive matrix \( A = (a_{ij}) \) and \( A(x_1, \ldots, x_n)^T = \lambda_1(x_1, \ldots, x_n)^T, \) then for \( \epsilon < n \min_{i,j} \{x_j/x_i a_{ij}\} \) the set \( \sigma_\epsilon = \{\lambda_1 - \epsilon, \lambda_2, \ldots, \lambda_n\} \) is realized by a positive matrix.

Proof. If \( A = (a_{ij}) \) is positive with spectrum \( \sigma \) and \( Ax = \lambda_1 x, \) then \( x = (x_1, \ldots, x_n)^T \) is a positive vector. Let \( D = \text{diag}(x_1, x_2, \ldots, x_n). \) Then, if \( e = (1, 1, \ldots, 1)^T, \)

\[
D^{-1}ADe = \lambda_1 e.
\]

Let

\[
q = \left( -\frac{\epsilon}{n}, -\frac{\epsilon}{n}, \ldots, -\frac{\epsilon}{n} \right)^T,
\]
where
\[ \varepsilon < n \min_{i,j} \left\{ \frac{x_j}{x_i} a_{ij} \right\}. \]

Then the matrix
\[ B = D^{-1} AD + eq^T \]
is positive with spectrum \( \sigma_{\varepsilon} \).

We will say that a partition of \( \sigma \), \( \sigma = \bigcup_{k=1}^r \sigma_k \), is realized by a nonnegative matrix when each set \( \sigma_k \) of the partition is realized by a nonnegative matrix \( A_k \). If this is the case then \( \sigma \) is realized by \( A = A_1 \oplus A_2 \oplus \cdots \oplus A_r \).

In 1949 Suleimanova [16] stated and in 1953 Perfect [10] proved that if \( \sigma = \{ \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \} \) is a set of real numbers satisfying
\[ \lambda_1 + \lambda_2 + \cdots + \lambda_n \geq 0, \quad \lambda_k < 0, \quad k = 2, 3, \ldots, n, \quad (6) \]
then \( \sigma \) is realized by an \( n \times n \) nonnegative matrix. The proof (see [9], Theorem 2.3, p. 183) shows that the companion matrix of the polynomial \( p(\lambda) = \prod_{j=1}^n (\lambda - \lambda_j) \) is nonnegative.

Both theorems of Brauer, A and B, allow us to give easy proofs of the Suleimanova condition (6).

**Proof (Suleimanova A).** The set of real numbers \( \sigma = \{ \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \} \), satisfying the Suleimanova condition (6) is realized by the \( n \times n \) nonnegative generalized stochastic matrix
\[
A = \begin{pmatrix}
S & -\lambda_2 & -\lambda_3 & -\lambda_4 & \cdots & -\lambda_n \\
S - \lambda_2 & 0 & -\lambda_3 & -\lambda_4 & \cdots & -\lambda_n \\
S - \lambda_2 & -\lambda_3 & 0 & -\lambda_4 & \cdots & -\lambda_n \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
S - \lambda_2 & -\lambda_3 & -\lambda_4 & \cdots & -\lambda_n & 0 \\
\end{pmatrix},
\]
where \( S = \sum_{j=1}^n \lambda_j \). \[ \square \]

**Proof (Suleimanova B).** Consider the matrix
\[
B = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
-\lambda_2 & \lambda_2 & 0 & \cdots \\
0 & -\lambda_3 & \lambda_3 & \cdots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots \\
& & & 0 & -\lambda_n \\
0 & 0 & \cdots & -\lambda_n & \lambda_n \\
\end{pmatrix}
\]
\( B \) has an eigenvalue zero with corresponding eigenvector \( e = (1, 1, \ldots, 1)^T \). Let
\[ q = (S, -\lambda_2, -\lambda_3, \ldots, -\lambda_n)^T, \]
where \( S = \sum_{j=1}^{\lambda_j} \lambda_j \). Then
\[
A = B + \mathbf{eq}^T
\]
is nonnegative with spectrum \( \lambda_2, \ldots, \lambda_n \) and \( S - \sum_{j=2}^{\lambda_j} \lambda_j = \lambda_1 \). If \( \lambda_1 > -\sum_{j=2}^{\lambda_j} \lambda_j \), then \( S > 0 \) and we may obtain a positive matrix with spectrum \( \sigma \). In fact, take
\[
q = \left( \frac{S}{n}, \frac{-\lambda_2}{n}, \ldots, \frac{-\lambda_n}{n}, \frac{S(n+1)/2}{n} \right)
\]
and obtain \( B + \mathbf{eq}^T > 0 \). □

Note that we may easily compute a nonnegative matrix \( A \) having the Suleimanova spectrum.

Salzmann [14] generalized the results of Suleimanova and Perfect in the following theorem:

**Theorem 10** (Salzmann). If \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) are real numbers such that
\[
\sum_{k=1}^{\lambda_k} \lambda_k \geq 0 \tag{7}
\]
and
\[
\lambda_k + \lambda_{n-k+1} \leq \frac{2}{n} \sum_{k=1}^{\lambda_k}, \quad k = 2, 3, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor \tag{8}
\]
then \( \sigma \) is realized by a nonnegative diagonalizable matrix. Furthermore, if the inequalities (7) and (8) are strict, the matrix is obtained positive.

No partition of the set \( \sigma = \{7, 5, 1, -3, -4, -6\} \) satisfies Suleimanova nor Salzmann conditions. However, it will be shown in Section 3 that \( \sigma \) is realized by a nonnegative matrix, which we are able to compute it.

3. Main results

**Theorem 11.** Let \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) be a set of real numbers, such that
\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0 > \lambda_{p+1} \geq \cdots \geq \lambda_n.
\]
If
\[
\lambda_1 \geq -\lambda_n - \sum_{S_k < 0} S_k \tag{9}
\]
where \( S_k = \lambda_k + \lambda_{n-k+1} \), \( k = 2, 3, \ldots, \lfloor n/2 \rfloor \) and \( S_{(n+1)/2} = \min(\lambda_{(n+1)/2}, 0) \) for \( n \) odd, then \( \sigma \) is realized by a nonnegative generalized stochastic matrix \( A \).
Proof. First, we consider the case $\lambda_2 \leq -\lambda_n$. Let $n = 2m$ even. Define the matrices

$$B_{11} = \begin{pmatrix} 0 & 0 \\ -\lambda_n & \lambda_n \end{pmatrix}$$

and

$$B_{kk} = \begin{pmatrix} 0 & \lambda_k \\ -\lambda_{n-k+1} & \lambda_k + \lambda_{n-k+1} \end{pmatrix},$$

$k = 2, 3, \ldots, [n/2]$. Now, we form the initial matrix

$$B = \begin{pmatrix} B_{11} & 0 & \cdots & 0 \\ B_{21} & B_{22} & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ B_{[n/2]1} & 0 & \cdots & B_{[n/2][n/2]} \end{pmatrix},$$

where the zeroes are matrices of order 2 and the blocks $B_{kk}, k = 2, 3, \ldots, [n/2]$, are of the form

$$B_{kk} = \begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_{n-k+1} \end{pmatrix}$$

in such a way that all rows of $B$ sum zero. If $\lambda_k$ and $\lambda_{n-k+1}$ are both positive, then the corresponding block $B_{kk}$ changes to

$$B_{kk} = \begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_{n-k+1} \end{pmatrix}$$

and its associated first column block $B_{1k}$ changes to

$$B_{1k} = \begin{pmatrix} 0 & -\lambda_k \\ 0 & -\lambda_{n-k+1} \end{pmatrix}.$$ 

We define the vector $q = (q_1, q_2, \ldots, q_n)$, where

$$q_2 = -\lambda_n,$$

$$q_{2k} = -\min\{\lambda_k + \lambda_{n-k+1}, 0\}, \quad k = 2, 3, \ldots, \frac{n}{2},$$

$$q_{2k-1} = 0, \quad k = 1, 2, \ldots, \frac{n}{2}.$$ 

Then, the matrix

$$A = B + eq^T$$

is nonnegative generalized stochastic with eigenvalues

$$\beta_1 = \sum_{k=1}^{n} q_k = -\lambda_n - \sum_{(\lambda_k+\lambda_{n-k+1})<0} (\lambda_k + \lambda_{n-k+1}) = -\lambda_n - \sum_{S_k<0} S_k$$

and $\lambda_2, \lambda_3, \ldots, \lambda_n$. Now we can apply Corollary 8 to conclude that, as $\lambda_1 \geq \beta_1$, the set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is realized by a nonnegative matrix.

If $\lambda_1 > \beta_1$, we also may to increase properly the entries $q_j$ of the vector $q$ in order to fit $\lambda_1 = \sum_{k=1}^{n} q_k > -\lambda_n - \sum_{S_k<0} S_k$, directly.
If \( n \) is an odd number, \( n = 2m + 1 \), then, to form the matrix \( B \) we need to add on the main diagonal the \( 1 \times 1 \) block \( B_{(n+1)/2,(n+1)/2} = (\lambda_{(n+1)/2}) \) in position \( (n, n) \), and the block \( B_{(n+1)/2,1} = (0, -\lambda_{(n+1)/2}) \) in the positions \( (n, 1) \) and \( (n, 2) \). In this case the last entry of the vector \( q \) is \( q_n = -\min[\lambda_{(n+1)/2}, 0] \).

Now we consider the case \( \lambda_2 > -\lambda_n \). We will suppose that \( \lambda_n < 0 \), otherwise \( \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \) is a nonnegative matrix with spectrum \( \sigma \). In this case we take the nonnegative block

\[
A_1 = \begin{pmatrix}
0 & \lambda_2 \\
-\lambda_n & \lambda_2 + \lambda_n
\end{pmatrix},
\]

with eigenvalues \( \lambda_2 \) and \( \lambda_n \) apart and renumber the rest of eigenvalues as \( \lambda'_1, \lambda'_2, \ldots, \lambda'_{n-2} \), where \( \lambda'_1 = \lambda_1 \) and \( \lambda'_k = \lambda_{k+1}, k = 2, 3, \ldots, n - 2 \). Then we proceed as before. If \( \lambda'_2 > -\lambda'_{n-2} \) we repeat the process until the module of the second eigenvalue on the list is less or equal than the module of the last eigenvalue on the list. A solution matrix \( A \) will be a direct sum of the \( 2 \times 2 \) matrices \( A_1, A_1', \ldots \) and a solution matrix of \( \sigma = \{\lambda_2, \lambda'_2, \ldots, \lambda'_{n-2}, \lambda_n\} \). \( \Box \)

**Theorem 12.** If \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) satisfies the Salzmann conditions (7) and (8), then \( \sigma \) also satisfies the condition of Theorem 11.

**Proof.** We only need to show the statement for \( \sigma \) satisfying \( \sum_{k=1}^{n} \lambda_k = 0 \). In fact, if \( \sum_{k=1}^{n} \lambda_k > 0 \) with \( \lambda_k + \lambda_{n-k+1} \leq (2/n) \sum_{k=1}^{n} \lambda_k, k = 2, 3, \ldots, [n + 1)/2], \) then consider the set \( \{\mu_1, \mu_2, \ldots, \mu_n\} \) with \( \mu_k = \lambda_k - \alpha \), where \( \alpha = 1/n \sum_{k=1}^{n} \lambda_k \). Clearly, \( \sum_{k=1}^{n} \mu_k = 0 \), and

\[
\mu_k + \mu_{n-k+1} = \lambda_k + \lambda_{n-k+1} - 2\alpha \leq 0.
\]

Now, suppose Salzmann conditions are satisfied with \( \sum_{k=1}^{n} \lambda_k = 0 \). Then \( S_k = \lambda_k + \lambda_{n-k+1} \leq 0 \) for \( k = 2, 3, \ldots, [n/2] \) and \( S_{(n+1)/2} = \lambda_{(n+1)/2} \leq 0 \) for \( n \) odd. Then,

\[
\lambda_1 + \lambda_n \geq -\sum_{k=2}^{n-1} \lambda_k = -\sum_{k=2}^{n} S_k < 0
\]

and (9) follows. \( \Box \)

The following example shows that Theorem 11 improves the sufficient conditions of the Salzmann Theorem 10.

**Example 13.** \( \sigma = \{4, 2, 1, -3, -3\} \) does not satisfy Salzmann sufficient conditions. No partition of \( \sigma \) satisfies Salzmann conditions. Since \( \lambda_1 \geq -(3) - (-1) \) for \( \lambda_1 \geq 4, \sigma \) satisfies conditions of Theorem 11 and we have a solution, which may be obtained by following the proof of Theorem 11 as:
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
3 & -3 & 0 & 0 & 0 \\
0 & -2 & 0 & 2 & 0 \\
0 & -2 & 3 & -1 & 0 \\
0 & -1 & 0 & 0 & 1 \\
\end{pmatrix} + \begin{pmatrix}
1 \\
1 \\
1 \\
0 \\
1 \\
\end{pmatrix} \begin{pmatrix}
0 & 3 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 3 \\
0 & 1 & 3 & 0 & 0 \\
0 & 2 & 0 & 1 & 1 \\
\end{pmatrix}

The following result is an important sufficient condition obtained by Kellogg [5] (1971), which also improved the Salzmann sufficient condition.

**Theorem 14** (Kellogg). Let \( \sigma = \{\lambda_0, \lambda_1, \ldots, \lambda_N\} \) be a set of real numbers with \( \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_N \) and let \( M \) be the greatest index \( j \) \((0 \leq j \leq N)\) for which \( \lambda_j \geq 0 \). Let the set of indices

\[
K = \left\{ i : \lambda_i \geq 0 \text{ and } \lambda_i + \lambda_{N-i+1} < 0, \ i \in \left\{1, 2, \ldots, \left[\frac{N}{2}\right]\right\} \right\}.
\]

If

\[
\lambda_0 + \sum_{i \in K, i < k} (\lambda_i + \lambda_{N-i+1}) + \lambda_{N-k+1} \geq 0 \quad \text{for all } k \in K,
\]

and

\[
\lambda_0 + \sum_{i \in K} (\lambda_i + \lambda_{N-i+1}) + \sum_{j=M+1}^{N-M} \lambda_j \geq 0,
\]

provided that \( N \geq 2M + 1 \), then \( \sigma \) is realized by an \((N+1) \times (N+1)\) nonnegative matrix.

Fiedler [3] proved that the Kellogg conditions are sufficient for \( \sigma \) to be the spectrum of some \((N+1) \times (N+1)\) symmetric nonnegative matrix. Soules [15] also gave sufficient conditions for the real set \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) to be realized by a symmetric nonnegative matrix having Perron eigenvector \( x \), that is, \( Ax = \lambda_1 x \). In particular for \( x = e = (1, 1, \ldots, 1)^T \), he proved:

**Theorem 15** (Soules). If \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) and

\[
\frac{1}{n} \lambda_1 + \frac{n-m-1}{n(n+1)} \lambda_2 + \sum_{k=1}^{m} \frac{\lambda_{n-2k+2}}{(k+1)k} \geq 0
\]
holds, where \( n = 2m + 2 \) (\( n \) even) or \( n = 2m + 1 \) (\( n \) odd), then there exists a symmetric generalized doubly stochastic matrix \( A \) with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \).

Borobia [1] obtained a generalization of the Kellogg result, as follows:

**Theorem 16** (Borobia). Let \( \sigma = \{\lambda_0, \lambda_1, \ldots, \lambda_N\} \) be a set of real numbers with \( \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_N \) and let \( M \) be the greatest index \( j \) (\( 0 \leq j \leq N \)) for which \( \lambda_j \geq 0 \). If there exists a partition \( J_1 \cup J_2 \cup \cdots \cup J_s \) of \( J = \{\lambda_{M+1}, \lambda_{M+2}, \ldots, \lambda_N\} \), for some \( 1 \leq S \leq N - M \), such that

\[
\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_M \geq \sum_{\lambda \in J_1} \lambda \geq \sum_{\lambda \in J_2} \lambda \geq \cdots \geq \sum_{\lambda \in J_s} \lambda
\]  

(13)

satisfies the Kellogg condition, then \( \sigma \) is the spectrum of some nonnegative matrix of order \((N + 1) \times (N + 1)\).

Radwan [11] proved that the Borobia conditions are sufficient for the existence of some symmetric nonnegative matrix with a prescribed spectrum \( \sigma \). The Borobia sufficient condition appear to be the most general sufficient condition for the real nonnegative inverse spectrum problem.

The following result is an extension of the Theorem 11. Here we show that under certain conditions, the set \( \sigma \) can be partitioned into subsets \( \sigma_k \), each one of them being realized by a submatrix \( A_k \), not necessarily nonnegative and still obtain a nonnegative matrix \( A \) with spectrum \( \sigma \). We consider only the case in which at least \( \lambda_n < 0 \), because if \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \), then the problem is solved by \( A = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \).

**Theorem 17.** Let \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) be a set of real numbers such that

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0 > \lambda_{p+1} \geq \cdots \geq \lambda_n.
\]

If there exists a partition \( \sigma = \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_t \) with

\[
\sigma_k = \{\lambda_{k_1}, \lambda_{k_2}, \ldots, \lambda_{k_{p_k}}\}, \quad k = 1, 2, \ldots, t, \quad \lambda_{k_1} = \lambda_1,
\]

\[
\lambda_{k_1} \geq 0, \quad \lambda_{k_1} \geq \lambda_{k_2} \geq \cdots \geq \lambda_{k_{p_k}},
\]

\[
S_{kj} = \lambda_{kj} + \lambda_{k_{p_j} - j + 1}, \quad j = 2, 3, \ldots, \left[ \frac{k_{p_k}}{2} \right]
\]

(14)

and for \( k_{p_k} \) odd

\[
S_{kj} = \min \left\{ \lambda_{j, p_j + 1}, 0 \right\}, \quad j = 1, 2, \ldots, t,
\]

\[
T_k = \lambda_{k_1} + \lambda_{k_{p_k}} + \sum_{S_{kj} < 0} S_{kj}, \quad k = 1, 2, \ldots, t
\]  

(15)
and

\[ \lambda_M = \max \left\{ \lambda_M - \sum_{S_1 < 0} S_{1j} - \max_{2 \leq k \leq t} \{ \lambda_{kj} \} \right\}, \quad (16) \]

satisfying

\[ \lambda_1 \geq \lambda_M - \sum_{T_k < 0, k=2}^t T_k, \quad (17) \]

then \( \sigma \) is realized by a nonnegative generalized stochastic matrix.

**Proof.** We can assume that for \( k = 1, 2, \ldots, t \), we have \( \lambda_{k2} \leq -\lambda_{kp_k} \). If not, replace \( \sigma_k \) by \( \sigma'_k = \sigma_k - \{ \lambda_{k2}, \lambda_{kp_k} \} \). Then the set \( \{ \lambda_{k2}, \lambda_{kp_k} \} \) is realizable by a nonnegative matrix \( A_{k2} \), and the partition

\[ \sigma' = \sigma - \{ \lambda_{k2}, \lambda_{kp_k} \} = \sigma_1 \cup \cdots \cup \sigma'_k \cup \cdots \cup \sigma_t \]

still satisfies the conditions of Theorem 17 (note that if \( \lambda_{k2} > -\lambda_{kp_k} \) then \( T'_k \geq T_k \)). In this case a solution will be of the form \( A = A_{k2} \oplus A' \), where \( A' \) is nonnegative with spectrum \( \sigma' = \{ \lambda_{k2}, \lambda_{kp_k} \} \). We can also assume that for \( k = 2, 3, \ldots, t \) we have \( T_k < 0 \). If not, then by Theorem 11 there exists a nonnegative matrix \( A_k \) with spectrum \( \sigma_k \) and we can reduce the problem to a problem of size \( n - k_{p_k} \). In this case a solution will be of the form \( A = A_k \oplus \tilde{A} \), where \( \tilde{A} \) is nonnegative with spectrum \( \sigma - \sigma_k \).

Then, for \( k_{p_k} \) even, we define the initial matrix

\[
B_k = \begin{pmatrix}
B_{11} & 0 & \cdots & 0 \\
B_{21} & B_{22} & 0 & \cdots \\
\vdots & 0 & \ddots & \vdots \\
B_{k1} & & & B_{k2} & [k_{p_k}]
\end{pmatrix}
\]

of order \( k_{p_k} \), where

\[
B_{11} = \begin{pmatrix} 0 & 0 \\ -\lambda_{kp_k} & -\lambda_{kp_k} \end{pmatrix}, \quad B_{jj}^k = \begin{pmatrix} 0 & \lambda_{kj} \\ -\lambda_{kp_k-j+1} & \lambda_{kp_k-j+1} \end{pmatrix}
\]

and the blocks \( B_{jj}^k, j = 2, 3, \ldots, [k_{p_k}/2] \), are of the form

\[
B_{jj}^k = \begin{pmatrix} \lambda_{kj} & 0 \\ 0 & -\lambda_{kj} \end{pmatrix},
\]

in such a way that all rows of \( B_k \) sum to zero. If \( \lambda_{kj} \) and \( \lambda_{kp_k-j+1} \) are both positive then the corresponding block \( B_{jj}^k \) changes to

\[
B_{jj}^k = \begin{pmatrix} \lambda_{kj} & 0 \\ 0 & \lambda_{kp_k-j+1} \end{pmatrix}
\]
and its associated first column block $B^k_{j_1}$ changes to
\[
B^k_{j_1} = \begin{pmatrix}
0 & -\lambda_{k_{j_1}} \\
0 & -\lambda_{k_{p_k} - j + 1}
\end{pmatrix}.
\]
Thus, $B_k$ is a block lower triangular matrix with eigenvalues $0, \lambda_{k_2}, \ldots, \lambda_{k_{p_k}}$. Next, for $k = 2, 3, \ldots, t$, we define the nonnegative vector $q_k = (q_{k_1}, q_{k_2}, \ldots, q_{k_{p_k}})^T$ satisfying
\[
\sum_{k_j = k_1}^{k_{p_k}} q_{k_j} = \lambda_{k_1}
\]
and
\[
q_{k_2} \leq -\lambda_{k_{p_k}},
\]
\[
q_{k_{j+1}} \leq -\min\{\lambda_{k_{j}} + \lambda_{k_{p_k} - j + 1}, 0\}, \quad j = 2, 3, \ldots, \frac{k_{p_k}}{2}.
\]

Then the matrix $A_k = B_k + e_{q_k}^T$, $k = 2, 3, \ldots, t$, which is not necessarily nonnegative, has the spectrum $\sigma_k$.

If $k_{p_k}$, $k = 2, 3, \ldots, t$, is an odd number then the matrix $B_k$ will have the $1 \times 1$ block $B_{k(1_{p_k} + 1)/2, (1_{p_k} + 1)/2} = [\lambda_{k(1_{p_k} + 1)/2}]$ in position $(k_{p_k}, k_{p_k})$ and the block $B_{(1_{p_k} + 1)/2, 1} = (0, -\lambda_{(1_{p_k} + 1)/2})$ in positions $(k_{p_k}, 1)$ and $(k_{p_k}, 2)$. In this case, for $k = 2, 3, \ldots, t$, the last entry of the vector $q_k$ is
\[
q_{k_{p_k}} = -\min\{\lambda_{k_{p_k}/2}, 0\}.
\]

Now we consider the set $\sigma_1 = \{\lambda_1, \lambda_1, \ldots, \lambda_{1_{p_1}}\}$. As it was explained above, the matrix $B_1$ has eigenvalues $0, \lambda_{1_{p_1}}, \ldots, \lambda_{1_{p_1}}$. For $1_{p_1}$ even, we define the vector $q_1 = (q_{1_1}, q_{1_2}, \ldots, q_{1_{p_1}})^T$, as follows:
\[
q_{1_2} = -\lambda_{1_{p_1}};
\]
\[
q_{1_{j+1}} = -\min\{\lambda_{1_{j}}, \lambda_{1_{p_1} - j + 1}, 0\}, \quad j = 2, 3, \ldots, \frac{1_{p_1}}{2},
\]
\[
q_{1_{j-1}} = 0, \quad j = 2, 3, \ldots, \frac{1_{p_1}}{2}
\]
and
\[
q_{i_1} = \max\left\{\max_{2 \leq k \leq t} \{\lambda_{k_1}\} + \left(\lambda_{1_{p_1}} + \sum_{S_j < 0} S_j\right), 0\right\}.
\]

If $1_{p_1}$ is odd then the matrix $B_1$ will have the $1 \times 1$ block $B_{1_{p_1} + 1/2, (1_{p_1} + 1)/2} = [\lambda_{1_{p_1} + 1/2}]$ in position $(1_{p_1}, 1_{p_1})$ and the block $B_{(1_{p_1} + 1)/2, 1} = (0, -\lambda_{(1_{p_1} + 1)/2})$ in positions $(1_{p_1}, 1)$ and $(1_{p_1}, 2)$ and the last entry of the vector $q_1$ is
Then the matrix $A_1 = B_1 + e q_1^T$ is nonnegative generalized stochastic with eigenvalues $\beta_1 = \lambda_M, \lambda_{12}, \ldots, \lambda_{1p_1}$. It is clear from (17) that $\lambda_M \leq \lambda_1$.

Now we form the general initial matrix

$$M = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ A_{t1} & 0 & \cdots & A_{tt} \end{pmatrix},$$

where $A_{jj} = A_j$, $j = 1, 2, \ldots, t$ and the matrices $A_{j1}$, $j = 2, 3, \ldots, t$, are $j_{p_1} \times 1_{p_1}$ matrices of the form

$$A_{j1} = \begin{pmatrix} 0 & 0 & \lambda_M - \lambda_{j1} \\ 0 & 0 & \lambda_M - \lambda_{j1} \\ \vdots & \vdots & \vdots \\ 0 & 0 & \lambda_M - \lambda_{j1} \end{pmatrix},$$

in such a way that all rows of $M$ sum equal to $\lambda_M$. Since the matrices $A_k = A_{kk}$ in $M$ have been constructed as in the proof of Theorem 11 and subject to $\sum_{k=1}^{t} q_{kj} = \lambda_{k1}$, $k = 2, 3, \ldots, t$, then from (9), if $T_k \geq 0$, $A_k$ can be obtained as nonnegative, while if $T_k < 0$, the corresponding matrix $A_k$ is not nonnegative. Finally, (17) guarantees that we may choose $q = (q_1, q_2, \ldots, q_t)^T$ such that

$$A = M + e q_1^T$$

is nonnegative generalized stochastic with spectrum $\sigma$. \qedsymbol

**Example 18.** In [11], Radwan presents the following sets

$$\sigma = \{5, 3, -2, -2, -2, -2\} \quad \text{and} \quad \tau = \{5, 3, -2, -2, -4\}$$

and observes that $\sigma$ satisfies Soules but not Kellogg, while $\tau$ satisfies Kellogg but not Soules. Here we note that both sets, $\sigma$ and $\tau$, satisfy conditions of Theorem 17. We compute a solution matrix by partitioning $\sigma$ into $\sigma_1 = \{5, -2, -2\}$ and $\sigma_2 = \{3, -2, -2\}$. Then

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & -2 & 0 \\ 0 & 2 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 4 & -1 \end{pmatrix} \quad \text{for} \ \sigma_2$$
and
\[
\begin{pmatrix}
0 & 0 & 0 \\
2 & -2 & 0 \\
0 & 2 & -2
\end{pmatrix} + \begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 1
\end{pmatrix} = \begin{pmatrix}
0 & 2 & 2 \\
2 & 0 & 2 \\
0 & 4 & 0
\end{pmatrix}
\]
for \(\sigma_1\)

and a solution matrix, nonnegative generalized stochastic, is
\[
A = \begin{pmatrix}
0 & 2 & 2 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 1 \\
0 & 0 & 1 & 2 & 0 & 1 \\
0 & 0 & 1 & 0 & 4 & -1
\end{pmatrix} + eq^T = \begin{pmatrix}
0 & 2 & 2 & 0 & 0 & 1 \\
2 & 0 & 2 & 0 & 0 & 1 \\
0 & 4 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 2 & 2 \\
0 & 0 & 1 & 2 & 0 & 2 \\
0 & 0 & 1 & 0 & 4 & 0
\end{pmatrix},
\]
where \(e = (1, 1, 1, 1, 1)^T\) and \(q^T = (0, 0, 0, 0, 1)\). In the same way, for \(\tau = \{5, -4\} \cup \{3, -2, -2\}\) we obtain
\[
A = \begin{pmatrix}
0 & 4 & 0 & 0 & 1 \\
4 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 & 2 \\
0 & 1 & 2 & 0 & 2 \\
0 & 1 & 0 & 4 & 0
\end{pmatrix}.
\]

Remark 19. Since the submatrices \(A_k\) in the proof of Theorem 17 need not be non-negative, the corresponding subsets \(\sigma_k\) of the partition \(\sigma = \bigcup_{k=1}^{n} \sigma_k\) need not satisfy basic necessary conditions (5), as it is the case with the set \(\{3, -2, -2\}\) in the above example. For the set \(\tau\) above, we may use the partition \(\tau = \{5, -2, -2\} \cup \{3, -4\}\) and still to obtain a solution. For the set \(\sigma = \{7, 5, 1, -3, -4, -6\}\) given at the end of Section 2 the partition \(\sigma = \{7, -6\} \cup \{5, 1, -3, -4\}\) satisfies conditions of Theorem 17.

The following example shows that conditions of Theorem 17 may be satisfied, although the Borobia conditions are not. Moreover, we can always compute a solution matrix, explicitly.

Example 20. Let \(\sigma = \{8.05, 6, 2.05, 2, -4, -4.1, -5, -5\}\). Borobia conditions are not satisfied. The partition \(\sigma = \sigma_1 \cup \sigma_2\), where \(\sigma_1 = \{8.05, 2.05, -5, -5\}\) and \(\sigma_2 = \{6, 2, -4, -4.1\}\) satisfies conditions of Theorem 17. Therefore, we have a solution matrix (nonnegative generalized stochastic)
\[
A = \begin{pmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{pmatrix} + eq^T,
\]
where \( q = (0, 0, 0, 0, 0, 0, 0, 0.1)^T \),

\[
A_{11} = \begin{pmatrix}
0 & 5 & 0 & 2.95 \\
5 & 0 & 0 & 2.95 \\
0 & 2.95 & 0 & 5 \\
0 & 2.95 & 5 & 0
\end{pmatrix},
A_{22} = \begin{pmatrix}
0 & 4.1 & 0 & 1.9 \\
4.1 & 0 & 0 & 1.9 \\
0 & 2.1 & 0 & 3.9 \\
0 & 2.1 & 4 & -0.1
\end{pmatrix},
A_{21} = \begin{pmatrix}
0 & 0 & 0 & 1.95 \\
0 & 0 & 0 & 1.95 \\
0 & 0 & 0 & 1.95 \\
0 & 0 & 0 & 1.95
\end{pmatrix}.

Observe that \( \sigma_2 \) cannot be the spectrum of a nonnegative matrix and there is not a partition of \( \sigma \) with the corresponding subsets satisfying Borobia conditions.

Acknowledgement

The author thanks Professor A. Borobia and J. Moro for helpful discussion and encouragement and the anonymous referee for comments and suggestions which greatly improved the presentation of the paper.

References