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Polynomial Characterizations of (H, F)-Invariant Subspaces with Applications

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ABSTRACT

A direct study of (H, F)-invariant subspaces associated with the polynomial fractional system model $Z = PQ^{-1}R + W$ is undertaken. The earlier results on the polynomial characterization of these subspaces are extended to cover the general case. For full rank transfer matrices, the smallest unobservability subspace containing $\operatorname{im} G$ and the largest reachability subspace in $\ker H$ are described in terms of factors of the polynomial system matrix. A new polynomial model and its superspaces play the central role in all these characterizations. The results obtained are applied to two measurement feedback problems in linear control theory, yielding geometric interpretations for two rational matrix equations.

1. INTRODUCTION

Motivated by the fundamental work of Fuhrmann [3,4], where a theory of polynomial models and natural realizations has been developed, and a series of papers that followed [7,2,6,5,9,10], here we have taken up a study of (H, F)-invariant (conditionally invariant) subspaces from a polynomial model viewpoint.

Let us first summarize the results pertaining to the dual concept of (F, G)-invariant (controlled invariant) subspaces: The first elegant polynomial characterizations for (F, G)-invariant subspaces were obtained by Emre and Hautus [2]. These involved the solvability of certain polynomial matrix equations. The neatest characterization turned out to be possible for the largest (F, G)-invariant subspace in $\ker H$; it was shown that this subspace is given by a polynomial model X_S involving the polynomial system matrix S

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associated with the fractional representation. In Proposition 2.3, the reader will find a summary of the results of [2] on (F, G)-invariant subspaces in $\ker H$. A somewhat different route was followed by Fuhrmann and Willems [6], where the main idea was to relate the (F, G)-invariant subspaces to factors of the numerator (and the denominator) polynomial matrices appearing in the fractional representation. This was done under certain restrictive assumptions and by the extensive use of bicausal isomorphisms in the characterizations. Here, as in [2], the polynomial models of the type X_S played a central role. Later [9], Khargonekar and Emre reemphasized the role of the system matrix S in their characterizations of stabilizability subspaces. They also clarified the link between the different polynomial characterizations obtained by Fuhrmann and Willems [6] and Emre and Hautus [2] for the case of right-fractional representations $Z = PQ^{-1}$. (We should also mention that [6] and [9] also contain interesting results related to $\mathbb{R}[z]$-module structure on (F, G)-invariant subspaces. In this paper, however, our main concern is with the $\mathbb{R}$-linear structure.)

It is well known that the concepts of reachability and observability, although dual concepts, have received an almost equal amount of attention in system theory literature. This remark alone suffices to motivate a thorough study of (H, F)-invariant subspaces in their own right. Thus, in [5], a duality theory has been developed for polynomial models with the purpose of studying (H, F)-invariant subspaces. This has made it possible to obtain counterparts to most of the results of [6] on (F, G)-invariant subspaces. These results again demonstrated the relation between certain (H, F)-invariant subspaces and nonsingular factors of numerator polynomial matrices by the use of bicausal isomorphisms. The results of Fuhrmann [5] were obtained for state space realizations associated with a transfer matrix of the form $Z = Q^{-1}R$; hence, they were essentially for observable systems. The principle motivation for this paper has been to extend the results of [5] to the most general fractional representations $Z = PQ^{-1}R + W$, the natural realization of which may not be observable. This is done in Section 3.

The main contribution of this paper is in finding a useful concrete representation for the dual space to X_S, which is denoted by X^S. Thus, in the notation of [5], $X^S = (X_{S'})^{\perp}$, where orthogonality is with respect to an appropriate dual state space. This being the case, the exposition of the results of this paper can be based on the duality theory of [5]. In the Appendix we have outlined how such an approach can be followed, by proving a central result via the duality theory of [5]. Because the direct use of definitions requires much less machinery and background, here we follow that approach. The dual relationship between various results will be rather obvious in the course of our development.

The paper is organized as follows: In Section 3, we review certain results of Fuhrmann [3,4] and Emre and Hautus [2]. Section 3 contains a characterization of (H, F)-invariant subspaces in terms of some polynomial matrix equations and an explicit representation for the smallest (H, F)-invariant subspace containing $\operatorname{im} G$. We also show how this representation covers the corresponding results of Fuhrmann [5]. The next section is devoted to a study of subspaces of the type X^S and its superspaces. We show that the representation for v_* obtained in Section 3 is essentially X^S, and for full column rank transfer matrices obtain a concrete description for the smallest unobservability subspace containing $\operatorname{im} G$. The considerations of this section also lead to a generalization of a result of Fuhrmann [5] on the largest reachability subspace in $\ker H$. Finally, we apply the results of Sections 3 and 4 to two problems in control theory in Section 5. This section is in the spirit of Khargonekar, Georgiou, and Özgüler [10], and yields geometric interpretations for the two matrix equations that arise as the solvability conditions for the control problems.

2. NATURAL REALIZATIONS AND (F, G)-INVARIANCE

In this section, we briefly review certain results developed in [3,4] and [2]. For details on notation and terminology the reader is referred to [10].

Let $\mathbb{R}((z^{-1}))^{k\times l}$ denote the $k \times l$ matrices of real truncated Laurent series in z^{-1}. Let $\mathbb{R}[z]^{k\times l}$ and $\mathbb{R}(z)^{k\times l}$ denote the set of polynomial and rational matrices, and let $z^{-1}\mathbb{R}[[z^{-1}]]^{k\times l}$ be the set of strictly proper (rational) matrices. Any element X in $\mathbb{R}((z^{-1}))^{k\times l}$ can uniquely be written as $X = (X)_+ + (X)_-$, where *the polynomial part* $(X)_+$ *of* X is in $\mathbb{R}[z]^{k\times l}$, and the *strictly proper part* $(X)_-$ *of* X is in $z^{-1}\mathbb{R}[[z^{-1}]]^{k\times l}$. By $(X)_{-n}$ we denote the coefficient of z^n in the Laurent series expansion of X. A result we will frequently use is on the division of polynomial matrices: Given N in $\mathbb{R}[z]^{k\times l}$ and a nonsingular M in $\mathbb{R}[z]^{k\times k}$, there exist unique K $[=(M^{-1}N)_+]$ and L $[=M(M^{-1}N)_+]$ such that $N = MK + L$. If $M = zI - A_1$ for A_1 in $\mathbb{R}^{k\times k}$ (i.e., for constant A_1), then L is a constant matrix.

Let a (strictly proper) transfer matrix Z in $\mathbb{R}(z)^{p\times m}$ be represented in the form

$$Z = PQ^{-1}R + W, \tag{2.1}$$

where P, Q, R, and W are polynomial matrices with Q in $\mathbb{R}[z]^{r\times r}$ nonsingular. With Q let us associate an $\mathbb{R}$-linear set of r-polynomial vectors as follows:

$$X_Q = \{x \text{ in } \mathbb{R}[z]^r : (Q^{-1}x)_+ = 0\},$$

which turns out to be finite dimensional with $\dim X_Q = \deg(\det Q)$, where $\deg(\cdot)$ denotes the (causality) degree of its argument. The vector space X_Q is actually the image of the projection

$$\pi_Q : \mathbb{R}[z]^r \to \mathbb{R}[z]^r : \alpha \mapsto Q(Q^{-1}\alpha)_- .$$

With the fractional representation (2.1) of Z, let us further associate

$$F: X_Q \to X_Q : x \mapsto \pi_Q(zx),$$

$$G: \mathbb{R}^m \to X_Q : u \mapsto \pi_Q(Ru),$$

$$H: X_Q \to \mathbb{R}^p : x \mapsto (PQ^{-1}x)_{-1}.$$

Also let D be a greatest right common factor of P and Q, and let $\tilde{P} := PD^{-1}$, $\tilde{Q} := QD^{-1}$, which are polynomial matrices. Similarly, let E be a greatest left common factor of Q and R, and let $\hat{Q} := E^{-1}Q$, $\hat{R} := E^{-1}R$, which are polynomial matrices.

The following result of Fuhrmann [3,4] associates a natural realization with the fractional representation (2.1).

PROPOSITION 2.2 (Fuhrmann [3,4]). *The linear system $\Sigma(P,Q,R,W) := (F,G,H,X_Q)$ is a realization of the transfer matrix Z of* (2.1). *The unobservable subspace η and the reachable subspace $\mathcal{R}_0$ associated with $\Sigma(P,Q,R,W)$ are given by*

$$\eta = \tilde{Q}X_D = \{x = \tilde{Q}\tilde{x} : \tilde{x} \textit{ is in } X_D\},$$

$$\mathcal{R}_0 = EX_{\hat{Q}} = \{x = E\hat{x} : \hat{x} \textit{ is in } X_{\hat{Q}}\}.$$

If in the fractional representation (2.1) $P = I$ and $W = 0$, then we write $\Sigma(Q,R)$ instead of $\Sigma(I,Q,R,0)$, and similarly, if $R = I$ and $W = 0$, we write $\Sigma(P,Q)$ instead of $\Sigma(P,Q,I,0)$.

The result of Proposition 2.2 thus establishes a sound basis for a study of the interrelation between the geometric and polynomial fractional approaches to linear system theory. Further results in this direction have been obtained by Emre and Hautus [2]. We present some of these results in a slightly rephrased manner.

Let M and N be polynomial matrices of sizes $k \times k$ and $k \times l$ with M nonsingular. Consider the $\mathbb{R}$-linear set defined by

$$X_N(M) := \{x \text{ in } \mathbb{R}[z]^k : x = \pi_M(Ny) \text{ for some strictly proper } y \text{ such that } (Ny)_- = 0\}.$$

(In the notation of [2]: $X_N(M) = \pi_M(X_N)$, where $X_N := \{x \text{ in } \mathbb{R}[z]^k : \text{there exist strictly proper } y \text{ such that } (Ny)_- = 0\}$. We prefer the notation $X_N(M)$ to establish uniformity with the notation we will use for its dual space in Section 4.) It is easy to see that $X_M(M) = X_M$ and hence $X_N(M)$ is an extention of the definition of X_M to rectangular matrices N. Also, for any $k \times l$ matrix N, we have $X_N(M) \subseteq X_M$, by definition. The subspaces $X_N(M)$ of X_M play a central role in polynomial characterizations of (F,G)-invariant subspaces. This is illustrated by the following result, essentially that of Emre and Hautus [2].

PROPOSITION 2.3. *Given the fractional representation* (2.1) *of* Z, *let* $\Sigma(P,Q,R,W) = (F,G,H,X_Q)$ *and define*

$$S := \begin{bmatrix} Q & R \\ -P & W \end{bmatrix}, \qquad \Theta := \begin{bmatrix} Q & 0 \\ -P & I \end{bmatrix}.$$

(i) *A subspace* v *of* X_Q *is an* (F,G)*-invariant subspace in* $\ker H$ *if and only if there exist constant matrices* A_1, C_1 *and polynomial matrices* C, D *such that*

$$\begin{bmatrix} V \\ D \end{bmatrix}(zI - A_1) = S\begin{bmatrix} C \\ C_1 \end{bmatrix},$$

where V *in a basis matrix of* v.

(ii) *If* $Z = Q^{-1}R$, *then the largest* (F,G)*-invariant subspace in* $\ker H$ *associated with* $\Sigma(Q,R)$ *is given by*

$$v^* = X_R(Q) \quad [= \pi_Q(X_R)].$$

(iii) *In the general case,* v^* *associated with* $\Sigma(P,Q,R,W)$ *is given by*

$$v^* = \not{p}X_S(\Theta) \quad (= \pi_Q[\not{p}(X_S)]),$$

where $\not{p} : \mathbb{R}[z]^{r+m} \to \mathbb{R}[z]^r : [\alpha' : \beta']' \mapsto \alpha$ *is the natural projection.*

Proof. For the case where $Q^{-1}R$ is strictly proper, the proposition is contained in Section 8 of Emre and Hautus [2]. This assumption, however, can easily be removed by the application of the projection π_Q, as we have indicated in the parentheses. (This observation is due to P. P. Khargonekar.) Hence, we omit the details of the proof. ■

The following result will be frequently used in the following sections.

LEMMA 2.4 (Emre and Hautus [2]). *A polynomial matrix B is a basis matrix for the $\mathbb{R}$-linear space X_M if and only if the columns of B are $\mathbb{R}$-linearly independent, M and B are left coprime, and there exist constant matrices C_1, A_1 satisfying*

$$M^{-1}B = C_1(zI - A_1)^{-1}.$$

3. (H, F)-INVARIANT SUBSPACES OF X_Q

Let Z be a $p \times m$ strictly proper transfer matrix in polynomial fractional representation

$$Z = PQ^{-1}R + W. \tag{3.1}$$

In this section, we obtain polynomial characterizations for (H, F)-invariant subspaces associated with the realization $\Sigma(P, Q, R, W) = (F, G, H, X_Q)$.

Recall that a subspace $\mathcal{v}$ of X_Q is (H, F)-invariant iff $F(\mathcal{v} \cap \ker H) \subseteq \mathcal{v}$, or equivalently, iff there exists an $\mathbb{R}$-linear $K_1 : \mathbb{R}^p \to X_Q$ such that $\mathcal{v}$ is $(F - K_1 H)$-invariant. As an immediate consequence of this definition, we have the following result.

LEMMA 3.2. *A subspace $\mathcal{v}$ of X_Q is (H, F)-invariant if and only if there exist constant matrices A_1, C_1, and a polynomial matrix K such that*

$$QC_1 + K(PQ^{-1}V)_{-1} = V(zI - A_1), \tag{3.3i}$$

where V is a basis matrix for $\mathcal{v}$. Further, the subspace $\mathcal{v}$ also contains $\operatorname{im} G$ if and only if there also exist a constant matrix B_1 and a polynomial matrix L such that

$$QL + VB_1 = R. \tag{3.3ii}$$

Proof. If v is (H, F)-invariant, then there exists an $\mathbb{R}$-linear $K_1: R^p \to X_Q$ such that $(F - K_1H)v \subseteq v$. Let $K := K_1(I)$, which is a polynomial matrix satisfying $(Q^{-1}K)_+ = 0$. By the definitions of the maps $F: X_Q \to X_Q$ and $H: X_Q \to \mathbb{R}^p$, the condition $(F - K_1H)v \subseteq v$ is then equivalent to $\pi_Q(zV) - K(PQ^{-1}V)_{-1} = VA_1$ for some constant A_1. This implies (3.3i) for some polynomial matrix C_1. Since $(Q^{-1}K)_+ = 0$ and $(Q^{-1}V)_+ = 0$, it follows by (3.3i) that $C_1 = [(Q^{-1}V)_-(zI - A_1)]_+ = (Q^{-1}V)_{-1}$, i.e., C_1 is actually constant. Conversely, if (3.3i) holds, let $K_1: \mathbb{R}^p \to X_Q: u \mapsto \pi_Q(Ku)$. Then, by (3.3i), $\pi_Q(zV) - \pi_Q[K(PQ^{-1}V)_{-1}] = VA_1$, which in turn is equivalent, by the definitions of the maps F, K_1, and H, to $(F - K_1H)v \subseteq v$.

Note that if v is any subspace of X_Q, then v contains $\operatorname{im} G$ if and only if the columns of $\pi_Q(R)$ are in v, i.e., iff $\pi_Q(R) = VB_1$ for some constant B_1. This is equivalent to (3.3ii) holding for some polynomial L and constant B_1. ∎

REMARK. The maps F, G, H, K_1 and the matrices K, A_1, B_1 are related as follows: the $\mathbb{R}$-span of the columns of $\pi_Q(K)$ is $\operatorname{im} K_1$, B_1 is the matrix representation of the codomain restriction of G to v, and A_1 is the matrix representation of the restriction of $F - K_1H$ to v, with respect to the natural bases in $\mathbb{R}^p$, $\mathbb{R}^m$, and the basis matrix V in v.

If v_1 and v_2 are (H, F)-invariant subspaces of X_Q both containing $\operatorname{im} G$, then the intersection of v_1 and v_2 is also (H, F)-invariant and contains $\operatorname{im} G$. Hence, there is a (unique) *smallest (H, F)-invariant subspace of X_Q which contains* $\operatorname{im} G$. This subspace we will denote by v_*.

The following result yields an explicit description for v_*:

THEOREM 3.4. *The smallest (H, F)-invariant subspace containing* $\operatorname{im} G$ *associated with $\Sigma(P, Q, R, W) = (F, G, H, X_Q)$ is given by*

$$v_* = \{x \text{ in } \mathbb{R}[z]^r: \text{there exists } \alpha \text{ in } \mathbb{R}[z]^m \text{ such that}$$
$$x = \pi_Q(R\alpha) \text{ and } (Z\alpha)_+ = 0\}.$$

Proof. It is easy to see that the set v_* is an $\mathbb{R}$-linear subspace of X_Q. We now show that

$$F(v_* \cap \ker H) \subseteq v_*. \tag{3.5}$$

Let x be in the intersection of v_* and $\ker H$, so that $x = \pi_Q(R\alpha)$, $(PQ^{-1}x)_{-1} = 0$, $(Z\alpha)_+ = 0$. Note that by the first equation, $(PQ^{-1}R\alpha)_{-1} = (Z\alpha)_{-1} =$

$(PQ^{-1}x)_{-1}$. Hence, $(Z\alpha)_{-1}=0$. Considering $F(x)=\pi_Q(zx)$, we have $F(x)=\pi_Q(Rz\alpha)$, and $(Zz\alpha)_+=z(Z\alpha)_+ + [z(Z\alpha)_-]_+ = z(Z\alpha)_+ + (Z\alpha)_{-1}$. It follows that $(Zz\alpha)_+=0$ and hence $F(x)$ is in v_*. This establishes (3.5). Further, if x is in $\operatorname{im} G$, we have $x=\pi_Q(Ru)$, for some constant u. As Z is strictly proper, it immediately follows that $(Zu)_+=0$. Thus, we also have $\operatorname{im} G \subseteq v_*$.

We now show that if v is any other (H, F)-invariant subspace containing $\operatorname{im} G$, then v_* is in v.

Let V be a basis matrix for v. By Lemma 3.2, there exist constant A_1, B_1, C_1 and polynomial K, L such that

$$QC_1 + K(PQ^{-1}V)_{-1} = V(zI - A_1), \tag{3.6i}$$

$$QL + VB_1 = R. \tag{3.6ii}$$

The equation (3.6i) implies $PC_1 + PQ^{-1}K(PQ^{-1}V)_{-1} = PQ^{-1}V(zI-A_1)$, which, on taking the strictly proper part of each term, yields $(PQ^{-1}K)_-(PQ^{-1}V)_{-1} = [PQ^{-1}V(zI-A_1)]_-$. Note that, writing $PQ^{-1}V=(PQ^{-1}V)_- + (PQ^{-1}V)_+$, we have $[PQ^{-1}V(zI-A_1)]_- = [(PQ^{-1}V)_-(zI-A_1)]_- = (PQ^{-1}V)_-(zI-A_1) - [(PQ^{-1}V)_-(zI-A_1)]_+$, where the last term is simply $(PQ^{-1}V)_{-1}$. Hence, $(PQ^{-1}K)_-(PQ^{-1}V)_{-1} = (PQ^{-1}V)_-(zI-A_1) - (PQ^{-1}V)_{-1}$. Since the matrix $\hat{K} = I + (PQ^{-1}K)_-$ is bicausal, we further have

$$(PQ^{-1}V)_{-1}(zI-A_1)^{-1} = \hat{K}^{-1}(PQ^{-1}V)_-. \tag{3.7}$$

Given an element x in v_*, we have $x=\pi_Q(R\alpha)$, $(Z\alpha)_+=0$, for some polynomial α. There exist a polynomial β and constant b_1 such that

$$B_1\alpha = (zI-A_1)\beta + b_1. \tag{3.8}$$

Then, by (3.6ii), we have $x=\pi_Q(R\alpha)=Vb_1+\pi_Q[V(zI-A_1)\beta]$. By (3.6i), we further have

$$x = Vb_1 + \pi_Q\left[K(PQ^{-1}V)_{-1}\beta\right]. \tag{3.9}$$

Multiplying each term in (3.7) on the right by $B_1\alpha$, we have $(PQ^{-1}V)_{-1}(zI-A_1)^{-1}B_1\alpha = \hat{K}^{-1}(PQ^{-1}VB_1)_-\alpha$, which, by (3.8) and (3.6ii), implies $(PQ^{-1}V)_{-1}\beta = \hat{K}^{-1}(PQ^{-1}R)_-\alpha - (PQ^{-1}V)_{-1}(zI-A_1)^{-1}b_1$. Now, in view of $(Z\alpha)_+ = [(PQ^{-1}R)_-\alpha]_+ = 0$, the right hand side of this equation is strictly proper. Thus, $(PQ^{-1}V)_{-1}\beta=0$. Consequently, by (3.9), we can write $x=$

Vb_1, i.e., x is in $\mathcal{v}$. Therefore, $\mathcal{v}_*$ is contained in any other (H, F)-invariant subspace which contains $\operatorname{im} G$, i.e., it is the smallest such subspace. ■

The main results of Fuhrmann [5] on polynomial characterization of (H, F)-invariant subspaces, namely Theorems (3.3) and (3.8) and Corollary (3.9), can be obtained by specializations of our Lemma 3.2 and Theorem 3.4 to the case $Z = Q^{-1}R$. This is the object of the next result, which can also be viewed as establishing the equivalence of alternative characterizations for this special case.

COROLLARY 3.10. *Let Z be a $p \times m$ strictly proper transfer matrix. Let Q and R be polynomial matrices such that $Z = Q^{-1}R$, and let $\Sigma(Q, R) = (F, G, H, X_Q)$.*

(i) *A subspace $\mathcal{v}$ of X_Q is (H, F)-invariant if and only if $\mathcal{v} = DX_E$ for some polynomial matrices D and E such that $Q^{-1}DE$ is bicausal.*

(ii) *A subspace $\mathcal{v}$ of X_Q is (H, F)-invariant and contains* $\operatorname{im} G$ *if and only if $\mathcal{v} = DX_E$ for some polynomial matrices D and E such that $Q^{-1}DE$ is bicausal and $D^{-1}R$ is polynomial.*

(iii) *If Z is of full row rank, then $\mathcal{v}_* = DX_E$, where D and E are polynomial matrices such that $Q^{-1}DE$ is bicausal and $D^{-1}R$ is a right unimodular polynomial matrix.*

Proof. Specialization of Lemma 3.2 and Theorem 3.4 to the case $Z = Q^{-1}R$ yields the following:

(a) A subspace $\mathcal{v}$ of X_Q is (H, F)-invariant iff there exist a constant A_1 and polynomial K such that

$$[Q + \pi_Q(K)](Q^{-1}V)_{-1} = V(zI - A_1). \tag{3.11}$$

(b) A subspace $\mathcal{v}$ of X_Q is (H, F)-invariant and contains $\operatorname{im} G$ iff there exist constant A_1 and B_1 and polynomial K such that (3.11) holds and

$$VB_1 = R. \tag{3.12}$$

(c) The smallest (H, F)-invariant subspace containing $\operatorname{im} G$ is given by $\mathcal{v}_* = \{x \text{ in } X_Q\text{: there exist polynomial } \alpha \text{ such that } x = R\alpha\}$.

[Verification of (a), (b), and (c) is quite straightforward and hence it is omitted.]

(i): Let $\mathcal{V}$ be (H, F)-invariant, so that for a basis matrix V of $\mathcal{V}$, (3.11) holds for some A_1 and K. Let E and S be left coprime polynomial matrices such that

$$(Q^{-1}V)_{-1}(zI - A_1)^{-1} = E^{-1}S. \tag{3.13}$$

It follows by (3.11) that, with $\hat{K} := Q + \pi_Q(K)$, we have

$$\hat{K}E^{-1}S = V.$$

This by left coprimeness of E and S implies that

$$\hat{K} = DE, \qquad V = DS \tag{3.14}$$

for some polynomial matrix D. Note that $Q^{-1}DE = Q^{-1}\hat{K} = I + (Q^{-1}K)_-$ and is hence bicausal. Further, the columns of S are $\mathbb{R}$-linearly independent, as the columns of $V = DS$ are. Now, (3.13), where E and S are left coprime, implies that S is a basis matrix for X_E. Therefore, $\mathcal{V} = DX_E$, as desired.

Conversely, suppose $\mathcal{V} = DX_E$ with $Q^{-1}DE$ bicausal. Let S be a basis matrix for X_E, so that for some constant C_1 and A_1 we have $E^{-1}S = C_1(zI - A_1)^{-1}$. Clearly, $V := DS$ is a basis matrix for $\mathcal{V}$. Hence, $DEC_1 = V(zI - A_1)$. Let us set $K := DEC_0^{-1} - Q$, where $C_0 := (Q^{-1}DE)_0$ is nonsingular, as $Q^{-1}DE$ is bicausal. Note that $Q^{-1}K$ is strictly proper and hence $K = \pi_Q(K)$. Further, $(Q + K)C_0C_1 = V(zI - A_1)$. From this equation it also follows that $C_0C_1 = (Q^{-1}V)_{-1}$. Consequently, (3.11) holds.

(ii): Suppose (3.11) and (3.12) hold. Then, there exist polynomial matrices D, E, and S satisfying (3.13) and (3.14) with $\mathcal{V} = DX_E$. But then (3.12) implies $R = DSB_1$, i.e., $D^{-1}R$ is polynomial. Conversely, suppose $\mathcal{V} = DX_E$ where $Q^{-1}DE$ is bicausal and $D^{-1}R := \tilde{R}$ is polynomial. Since $E^{-1}\tilde{R} = (DE)^{-1}QQ^{-1}R$, where $(DE)^{-1}Q$ is bicausal and $Q^{-1}R$ is strictly proper, it follows that the columns of $\tilde{R}$ are in X_E. This immediately implies that the columns of $R = D\tilde{R}$ are in $\mathcal{V} = DX_E$. Hence, (3.12) holds for some constant B_1.

(iii): By part (ii) it follows that DX_E is (H, F)-invariant and contains $\operatorname{im} G$. Let x be in DX_E, so that $x = D\tilde{x}$ for some $\tilde{x}$ in X_E. Now, $Q^{-1}x = Q^{-1}DEE^{-1}\tilde{x}$ is strictly proper, as $Q^{-1}DE$ is bicausal and $E^{-1}\tilde{x}$ is strictly proper. Thus, x is in X_Q. Since, $D^{-1}R$ is right unimodular, there exists a polynomial U satisfying $D^{-1}RU = I$. It follows that $x = RU\tilde{x}$, and hence x is in $\mathcal{V}_*$ of (c). Therefore, $DX_E \subseteq \mathcal{V}_*$; since DX_E itself is (H, F)-invariant and contains $\operatorname{im} G$, this means $DX_E = \mathcal{V}_*$. ■

REMARK 3.15. Consider the rational extended input-output map (see [7])

$$f:\mathbb{R}[z]^m \to \mathbb{R}(z)^p : \alpha \mapsto PQ^{-1}\alpha,$$

induced by the transfer matrix $Z = PQ^{-1}$ in right-fractional representation. Let $f^{-1}(V)$ denote the inverse image of a subspace V of $\mathbb{R}(z)^p$, and let V/W denote the quotient space of V modulo its subspace W. Note that the state space X_Q can be identified with $\mathbb{R}[z]^m/QR[z]^m$. It can easily be shown that $\eta = f^{-1}(\mathbb{R}[z]^p)/Q\mathbb{R}[z]^m$, $v_* = f^{-1}(z^{-1}\mathbb{R}[[z^{-1}]]^p)/QR[z]^m$, $v_* \cap \eta = f^{-1}(\{0\})/QR[z]^m$. Therefore, for $\Sigma(P, Q)$, the unobservable subspace is the set of states that cause zero future outputs, the smallest (H, F)-invariant subspace containing $\operatorname{im} G$ is the set of states that cause zero past outputs, and their intersection is the set of states causing zero outputs. Theorem 3.4 yields an extension of the above interpretation for v_* to the general case $Z = PQ^{-1}R + W$; thus v_* is the set of states that cause zero past outputs and that can be reached using past input sequences. [In the general case, one can write similar formulas for η and v_* and can give $\mathbb{R}[z]$-submodule characterizations for (H, F)-invariant subspaces via the use of extended input-output maps induced by the meta-right-fractional representation that we introduce in the next section. However, we do not intend to explore these ideas further here.]

4. THE SUBSPACE $X^N(M)$ AND ITS SUPERSPACES

In this section, we introduce a new polynomial model $X^N(M)$ and examine its various properties. The main point is that such models are most convenient for a concrete representation of v_* and $\mathcal{N}_*$, the smallest unobservability subspace containing $\operatorname{im} G$. The symmetry between the representations $X_N(M)$ and $X^N(M)$ is another motivation for the introduction of these polynomial models.

Our guideline in the polynomial characterization of this section is as follows: We consider the right-fractional representation $Z = PQ^{-1}$ and obtain expressions for v_* and $\mathcal{N}_*$ in this special case in terms of the polynomial model $X^N(M)$. We then introduce the *meta-right-fractional representation*

$$\hat{Z} = ST^{-1} = \begin{bmatrix} Q & R \\ -P & W \end{bmatrix} \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -PQ^{-1} & Z \end{bmatrix}$$

to essentially reduce v_* and $\mathcal{N}_*$ characterizations of the general case $Z = PQ^{-1}R + W$ to those of the right-fractional representations. In Section 2, we have actually followed a similar procedure for the dual case; we first considered the left-fractional representation $Z = Q^{-1}R$ and then *the meta-left-fractional representation*

$$\tilde{Z} := \Theta^{-1}S = \begin{bmatrix} Q & 0 \\ -P & I \end{bmatrix}^{-1} \begin{bmatrix} Q & R \\ -P & W \end{bmatrix} = \begin{bmatrix} I & Q^{-1}R \\ 0 & Z \end{bmatrix}$$

for polynomial characterizations of $\mathscr{v}^*$, reducing the general case $Z = PQ^{-1}R + W$ to the case $Z = Q^{-1}R$. In this section, we apply the same idea to the characterization of the largest reachability subspace in $\ker H$, $\mathscr{R}^*$, in the general case. Thus, the main merit of meta-fractions consists in their use in reducing the general characterizations to those for which the underlying state space is either reachable or observable. This idea (although in a raw form) is also present in Khargonekar and Emre [9]. Note that if PQ^{-1} is strictly proper (which can always be assumed, by the strict system equivalence of Fuhrmann [4]), then the rational matrix $\hat{Z}$ is proper and can be considered as a transfer matrix; the underlying natural realization of $\hat{Z}$ is observable iff the natural realization $\Sigma(P, Q, R, W)$ is. However, the former is always reachable, whereas the latter is not if Q and R have a nontrivial left factor. Our results in this section reinforce the feeling that as long as one is interested in (H, F)-invariant subspaces of $\Sigma(P, Q, R, W)$, the meta-fraction $\hat{Z} = ST^{-1}$ can replace $Z = PQ^{-1}R + W$. Similar remarks apply to $\tilde{Z} = \Theta^{-1}S$. Another important feature of the meta-fractional representations $\hat{Z} = ST^{-1}$ and $\tilde{Z} = \Theta^{-1}S$ is that the polynomial system matrix S is the numerator matrix for both representations. Thus, via meta-fractions, we illustrate the significant relation between the system matrix (and its right or left factors) and the subspaces $\mathscr{v}_*$, $\mathscr{N}_*$, $\mathscr{v}^*$, and $\mathscr{R}^*$. [We also remark that the meta-fractions play a central role in $\mathbb{R}[z]$-submodule characterizations of (F, G) and (H, F)-invariant subspaces. This we intend to pursue elsewhere.] Finally, the following results indicate that the use of bicausal isomorphisms in the polynomial characterizations of these subspaces can be avoided by choosing an appropriate (left or right) fractional representation for the invariant subspace at hand; the use of bicausal isomorphisms, however, does provide a convenient setup for a study of state feedback and output injection groups on polynomial models.

Let N and M be a pair of $l \times k$ and $k \times k$ polynomial matrices with M nonsingular. Consider an $\mathbb{R}$-linear set defined by

$$(4.1) \quad X^N(M) := \left\{ x \text{ in } X_M : (NM^{-1}x)_+ = N\alpha \text{ for some polynomial } \alpha \right\}.$$

Note that for all polynomial matrices N, $X^N(M)$ is, by definition, a subspace of X_M. In particular, $X^I(M) = X_M$ and $X^M(M) = X_I = \{0\}$. Also, if N is nonsingular and N^{-1} is proper (which is automatically satisfied if N is either row or column proper), then $X^N(M) = \{x \text{ in } X_M : (NM^{-1}x)_+ = 0\}$. The following lemma, which is basically a restatement of Theorem 3.4, stimulates our interest in subspaces of the form (4.1).

LEMMA 4.2. *Let $Z = PQ^{-1}R + W$ be a $p \times m$ strictly proper transfer matrix in polynomial fractional representation. Let v_* denote the smallest (H, F)-invariant subspace containing* $\operatorname{im} G$ *associated with the realization $\Sigma(P, Q, R, W) = (F, G, H, X_Q)$.*

(i) *In the special case of $R = I$ and $W = 0$, i.e., $Z = PQ^{-1}$, we have $v_* = X^P(Q)$.*

(ii) *In the general case, we have $v_* = \not{p}X^S(T)$, where*

$$S := \begin{bmatrix} Q & R \\ -P & W \end{bmatrix}, \qquad T := \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix},$$

and $\not{p} : \mathbb{R}[z]^{r \times m} \to \mathbb{R}[z]^r : [\alpha_1' : \alpha_2']' \mapsto \alpha_1$.

Proof. (i): By Theorem 3.4, we have $v_* = \{x = \pi_Q(\alpha) : (PQ^{-1}\alpha)_+ = 0, \alpha \text{ in } \mathbb{R}[z]^m\}$, for the case $Z = PQ^{-1}$. If x is in v_*, then $(PQ^{-1}x)_+ = [P(Q^{-1}\alpha)_-]_+ = -P(Q^{-1}\alpha)_+$, where we have used $(PQ^{-1}\alpha)_+ = 0$. Hence, x is in $X^P(Q)$. Conversely, let x be in $X^P(Q)$; then $(PQ^{-1}x)_+ = P\beta$ for some polynomial β. With $\alpha := x - Q\beta$ we have $x = \pi_Q(\alpha)$ and $(PQ^{-1}\alpha)_+ = (PQ^{-1}x)_+ - P\beta = 0$. Consequently x is in v_*. This establishes $v_* = X^P(Q)$ for the case $Z = PQ^{-1}$.

(ii): Let x be in v_*. By Theorem 3.4, there exists a polynomial α such that, with $\beta := -(Q^{-1}R\alpha)_+$, we have $x = Q\beta + R\alpha$, $(Z\alpha)_- = 0$. Consider $\hat{x} := [x' : 0]'$, which clearly satisfies $\not{p}(\hat{x}) = x$. Also, since

$$T^{-1}\hat{x} = \begin{bmatrix} Q^{-1} & -Q^{-1}R \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} Q^{-1}x \\ 0 \end{bmatrix}$$

is strictly proper, we have $\hat{x}$ in X_T. We further have

$$(ST^{-1}\hat{x})_+ = \begin{bmatrix} x \\ -(PQ^{-1}x)_+ \end{bmatrix},$$

where, as $(Z\alpha)_+ = 0$, $(PQ^{-1}x)_+ = P\beta + (PQ^{-1}R\alpha)_+ = P\beta - W\alpha$. We can then let $\hat{\alpha} := [\beta' : \alpha']'$ to write $(ST^{-1}\hat{x})_+ = S\hat{\alpha}$. Consequently, $\hat{x}$ is in $X^S(T)$ and $x = \not{p}(\hat{x})$ is in $\not{p}X^S(T)$. This shows that v_* is contained in $\not{p}X^S(T)$.

To see the reverse inclusion, let $\hat{x} := [x_1' : x_2']'$ be in $X^S(T)$. As $\hat{x}$ is in X_T, it follows that $x_2 = 0$. Also as $(ST^{-1}\hat{x})_+ = S\hat{\alpha}$ for some polynomial $\hat{\alpha} = [\alpha_1' : \alpha_2']'$, we have

$$\left(\begin{bmatrix} I & 0 \\ -PQ^{-1} & Z \end{bmatrix}\begin{bmatrix} x_1 \\ 0 \end{bmatrix}\right)_+ = \begin{bmatrix} Q & R \\ -P & W \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}.$$

This equation implies $x_1 = Q\alpha_1 + R\alpha_2$, $(PQ^{-1}x_1)_+ = P\alpha_1 - W\alpha_2$. We now have, from these, $(Z\alpha_2)_+ = (PQ^{-1}R\alpha_2)_+ + W\alpha_2 = 0$. It follows that $x_1 = \not{p}(\hat{x})$ is in v_*, establishing the reverse inclusion. Therefore, $v_* = \not{p}X^S(T)$. ■

The preceding lemma demonstrates that the subspaces $X^P(Q)$ and $X^S(T)$ are basic for (H, F)-invariant subspaces, just as $X_R(Q)$ and $X_S(\Theta)$ are basic for (F, G)-invariant subspaces. We now proceed to examine further properties of the subspaces $X^N(M)$.

Consider N, M, and $X^N(M)$ associated with them. Let $N = \tilde{N}D$ be a factorization of N into a square nonsingular polynomial matrix D and a polynomial matrix $\tilde{N}$. It is easy to see that the set

$$X^{\tilde{N}}(N, M) = \left\{x \text{ in } X_M : (NM^{-1}x)_+ = \tilde{N}\alpha \text{ for some polynomial } \alpha\right\}$$

is $\mathbb{R}$-linear and $X^N(M) \subseteq X^{\tilde{N}}(N, M) \subseteq X_M$. In fact, if $N = \tilde{N}\tilde{D} = \hat{N}\hat{D}$ are two such factorizations of N and if $\hat{D}\tilde{D}^{-1}$ is polynomial, then $X^N(M) \subseteq X^{\tilde{N}}(N, M) \subseteq X^{\hat{N}}(N, M) \subseteq X_M$. Thus, in this manner we create certain "superspaces" of $X^N(M)$ in X_M. In case N is square nonsingular, a maximal such superspace is $X^I(N, M) = X_M$. When N is of full column rank, a maximal superspace of $X^N(M)$ again exists and is given by $X^U(N, M)$, where $N = UD$ with D a greatest right factor of N (consequently, U is left-unimodular, i.e., for some polynomial U we have $\tilde{U}U = I$). One naturally expects that such maximal superspaces of $X^S(T)$ should be related to the "smallest unobservability subspace containing im$G$." In case the transfer matrix is of full column rank, this can be substantiated.

Recall that the *smallest unobservability subspace containing* im$G$ can be defined by its property

$$\mathcal{N}_* := v_* + v^*,$$

where v_* is the smallest (H, F)-invariant subspace containing im$G$ and v^* is the largest (F, G)-invariant subspace in ker$H$.

THEOREM 4.3. *Let $Z = PQ^{-1}R + W$ be a $p \times m$ strictly proper transfer matrix in polynomial fractional representation. Assume that Z is of full* *column rank. Then, the unobservability subspace $\mathcal{N}_*$ associated with the realization $\Sigma(P, Q, R, W)$ of Z is given by*

$$\mathcal{N}_* = \not{p}X^U(S, T),$$

where U is a left unimodular polynomial matrix that satisfies $S = UD$, $\tilde{U}U = I$ for some polynomial $\tilde{U}$ and nonsingular polynomial D.

Proof. Since Z is of full column rank and T is nonsingular, we see that

$$S = \begin{bmatrix} I & 0 \\ -PQ^{-1} & Z \end{bmatrix} T$$

is also of full column rank. Thus, there does exist a polynomial matrix U as in the statement of the theorem.

By Lemma (4.2), $v_* = \not{p}X^S(T)$. Since $X^S(T)$ is in $X^U(S, T)$, it follows that v_* is also contained in $\not{p}X^U(S, T)$. Let x be in v^*. By Proposition 2.3(iii), there exist polynomial α_1 and α_2 and strictly proper y_1 and y_2 such that $x = \pi_Q(\alpha_1)$ and $[\alpha_1' : \alpha_2']' = S[y_1' : y_2']'$. It follows that $D[y_1' : y_2']' = \tilde{U}[\alpha_1' : \alpha_2']'$ and hence is polynomial. Also note the equalities $x = Qy_1 + Ry_2 - Q(Q^{-1}Ry_2)_+$, $(PQ^{-1}x)_+ = Py_1 - Wy_2 - P(Q^{-1}Ry_2)_+$, which imply, with $\hat{x} := [x' : 0]'$, that

$$(ST^{-1}\hat{x})_+ = \begin{bmatrix} x \\ -(PQ^{-1}x)_+ \end{bmatrix} = \begin{bmatrix} Q & R \\ -P & W \end{bmatrix} \begin{bmatrix} y_1 + (Q^{-1}Ry_2)_+ \\ y_2 \end{bmatrix}.$$

Note here that

$$D\begin{bmatrix} y_1 + (Q^{-1}Ry_2)_+ \\ y_2 \end{bmatrix} = \tilde{U}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} - D\begin{bmatrix} (Q^{-1}Ry_2)_+ \\ 0 \end{bmatrix}$$

is polynomial. Since $\hat{x}$ is in X_T, $x = \not{p}(\hat{x})$, and (as we have shown above) $(ST^{-1}\hat{x})_+ = U\hat{\alpha}$ for some polynomial $\hat{\alpha}$, it follows that x is in $\not{p}X^U(S, T)$. We have thus established that $v_* + v^*$ is contained in $\not{p}X^U(S, T)$. We now show the reverse inclusion. Let $\hat{x}$ be in $X^U(S, T)$. Since $\hat{x}$ is in X_T, we have $\hat{x} = [x' : 0]'$ for some x in X_Q. We further have $(ST^{-1}\hat{x})_+ = U\hat{\alpha}$ for some polynomial $\hat{\alpha}$. There exist polynomial $\hat{\beta}$ and strictly proper $\hat{y}$ such that

$$\hat{\alpha} = D\hat{\beta} + D\hat{y}.$$

This implies $(ST^{-1}\hat{x})_+ = S\hat{\beta} + S\hat{y}$, and also that $S\hat{y}$ is polynomial. Let $x_1 := \pi_Q(\delta_1)$ where $\delta_1 := \not{p}(S\hat{y})$. Then, x_1 is clearly in v^* and $\hat{x}_1 := [x_1' : 0]'$ is such that $\hat{x}_1$ is in X_T and it satisfies

$$\left(ST^{-1}\hat{x}_1\right)_+ = S\hat{y} - S\hat{\phi},$$

where $\hat{\phi} := [(Q^{-1}Ry_2)'_+ : 0]'$. Thus, $\hat{x}_2 := \hat{x} - \hat{x}_1$ is such that $\hat{x}_2$ is in X_T and it satisfies

$$\begin{aligned}\left(ST^{-1}\hat{x}_2\right)_+ &= \left(ST^{-1}\hat{x}\right)_+ - \left(ST^{-1}\hat{x}_1\right)_+ \\ &= S(\hat{\beta} - \hat{\phi}).\end{aligned}$$

It follows that $\hat{x}_2$ is in $X^S(T)$. Also note that $x = \not{p}(\hat{x})$, $x_1 = \not{p}(\hat{x}_1)$, and hence $x = x_1 + \not{p}(\hat{x}_2)$, where $\not{p}(\hat{x}_2)$ is in $\not{p}X^S(T) = v_*$. We have thus shown that any x in $\not{p}X^U(S,T)$ has a decomposition of the form $x = x_1 + x_2$ where x_1 is in v^* and x_2 is in v_*. Therefore, $v_* + v^* = \not{p}X^U(S,T)$. ■

Corollary 4.4. *Let $Z = PQ^{-1}$ be a $p \times m$ strictly proper transfer matrix, and assume that Z is of full column rank. Then, $\mathcal{N}_*$ associated with $\Sigma(P,Q)$ is given by*

$$\mathcal{N}_* = X^V(P,Q),$$

where V is a left unimodular polynomial matrix satisfying $P = VE$, $\tilde{V}V = I$ for some polynomial $\tilde{V}$ and nonsingular polynomial E.

Proof. We specialize the result of Theorem 4.3 to the case $R = I$ and $W = 0$. Then,

$$S = \begin{bmatrix} Q & I \\ -P & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ V & 0 \end{bmatrix} \begin{bmatrix} -E & 0 \\ Q & I \end{bmatrix}$$

is a factorization of S, with

$$U := \begin{bmatrix} 0 & I \\ V & 0 \end{bmatrix}, \qquad D := \begin{bmatrix} -E & 0 \\ Q & I \end{bmatrix},$$

as desired. It is now straightforward to verify the second equality in $\mathcal{N}_* = \not{p}X^U(S,T) = X^V(P,Q)$. ■

In the same manner, we can obtain a polynomial characterization of the "largest reachability subspace in $\ker H$" associated with the realization $\Sigma(P, Q, R, W)$. The result obtained is the precise counterpart of Theorem 4.3 and is a generalization of Theorem (4.1) of Fuhrmann [5]. Recall that the *largest reachability subspace* in $\ker H$ can be defined by $\mathscr{R}^* := \nu_* \cap \nu^*$, where ν_* is the smallest (H, F)-invariant subspace containing $\operatorname{im} G$ and ν^* is the largest (F, G)-invariant subspace in $\ker H$.

Given a nonsingular $k \times k$ polynomial matrix M and a $k \times l$ polynomial matrix N, let $N = E\tilde{N}$ be a factorization of N with E square nonsingular. The set

$$X_{\tilde{N}}(M, N) := \{x \text{ in } \mathbb{R}[z]^k : x = \pi_M(Ny),\ (\tilde{N}y)_- = 0$$

$$\text{for some strictly proper } y\}$$

is $\mathbb{R}$-linear and is easily seen to be a subspace of $X_N(M)$. [The usual—less explicit—notation for $X_{\tilde{N}}(M, N)$ is $EX_{\tilde{N}}$.] Given two such factorizations $N = \tilde{E}\tilde{N} = \hat{E}\hat{N}$ such that $\hat{E}^{-1}\tilde{E}$ is polynomial, we have

$$X_{\tilde{N}}(M, N) \subseteq X_{\hat{N}}(M, N) \subseteq X_N(M) \subseteq X_M.$$

When N is of full row rank, a minimal such subspace exists and, as the following result illustrates, is related to $\mathscr{R}^*$.

THEOREM 4.5. *Let $Z = PQ^{-1}R + W$ be a $p \times m$ strictly proper transfer matrix in polynomial fractional representation. Assume that Z is of full row rank. Then, the reachability subspace $\mathscr{R}^*$ associated with the realization $\Sigma(P, Q, R, W)$ is given by*

$$\mathscr{R}^* = \not p X_U(\Theta, S),$$

where U is a right unimodular polynomial matrix satisfying $S = EU$, $U\hat{U} = I$ for some nonsingular polynomial matrix E and a polynomial matrix $\hat{U}$.

Proof. It is easy to see that Θ is nonsingular and, since Z is of full row rank, S is of full row rank, guaranteeing the existence of the factorization $S = EU$. The proof consists in establishing that $\nu^* \cap \nu_* = \not p X_U(\Theta, S)$ by making use of Proposition 2.3(iii) and Lemma 4.2(ii). Since this idea and the technique used in establishing the equality parallel the proof of Theorem 4.3, we omit the details of the proof. It has also been pointed out to us by P. P.

Khargonekar and the referee that this theorem can also be obtained from Section VI of Khargonekar and Emre [9], where stabilizability subspaces are considered. In doing this, one first recognizes that the assumption that $Q^{-1}R$ is strictly proper can be removed (as we have illustrated in Section 2) and then the fact that a reachability subspace is a stabilizability subspace with respect to any stability region. ■

COROLLARY 4.6 (Fuhrmann [5, Theorem (4.1)]). *Let $Z = Q^{-1}R$ be a strictly proper $p \times m$ full row-rank transfer matrix. Then, $\mathscr{R}^*$ associated with $\Sigma(Q, R)$ is given by $\mathscr{R}(X_V(Q, R)$, where V is a right unimodular polynomial matrix satisfying $R = DV$, $V\hat{V} = I$ for some polynomial $\hat{V}$ and nonsingular polynomial D.*

Proof. In the special case $P = I$ and $W = 0$, we have

$$S = \begin{bmatrix} Q & R \\ I & 0 \end{bmatrix} = \begin{bmatrix} D & Q \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & V \\ I & 0 \end{bmatrix},$$

where

$$E := \begin{bmatrix} D & Q \\ 0 & I \end{bmatrix}, \qquad U := \begin{bmatrix} 0 & V \\ I & 0 \end{bmatrix}$$

is a factorization of S as desired in Theorem 4.5. Hence, $\mathscr{R}^* = \not{p}X_U(\Theta, S)$. It is now routine to verify that $\not{p}X_U(\Theta, S) = X_V(Q, R)$. ■

Returning to the examination of the properties of $X^N(M)$ and its superspaces, we see that when a right factor D of N is also a right factor of M, the superspace $X^{\tilde{N}}(N, M)$ and $X^N(M)$ are related in a simple way.

LEMMA 4.7. *Let D be a common right factor of N and M, and let $\tilde{N} := ND^{-1}$, $\tilde{M} := MD^{-1}$. Then,*

$$X^{\tilde{N}}(N, M) = X^N(M) + \tilde{M}X_D.$$

Proof. If x is in $\tilde{M}X_D$, then $x = \tilde{M}\tilde{x}$ for some $\tilde{x}$ in X_D, and $(NM^{-1}x)_+ = \tilde{N}x$. Thus, $\tilde{M}X_D$ is in $X^{\tilde{N}}(N, M)$. As $X^N(M)$ is also in $X^{\tilde{N}}(N, M)$, it follows that we have one way of the inclusion. Let x be in $X^{\tilde{N}}(N, M)$, so that $(NM^{-1}x)_+ = \tilde{N}\alpha$ for some polynomial α. There exist γ in X_D and a polynomial β such that $\alpha = D\beta + \gamma$. It follows that $x_1 := \tilde{M}\gamma$ is in $\tilde{M}X_D$, and

$x_2 := x - x_1$ satisfies $(NM^{-1}x_2)_+ = N\beta$. Thus, x_2 is in $X^N(M)$. This establishes the reverse inclusion and hence the lemma. ∎

REMARK 4.8. We now discuss the construction of bases for $X_N(M)$ and $X^N(M)$. For simplicity we assume that $M^{-1}N$ (and NM^{-1}) is proper. Note that in this case $X_N(M)$ is given by

$$X_N(M) = \{x \text{ in } \mathbb{R}[z]^k : x = Ny \text{ for some } y \text{ in } z^{-1}\mathbb{R}[[z^{-1}]] \text{ such that } (Ny)_- = 0\}.$$

By Emre and Hautus [2, Section 7], a basis for $X_N(M)$ can be obtained as follows: Let U be a unimodular polynomial matrix such that $UN = [\tilde{N}' : 0]'$, where N is row-proper with row indices $\{\mu_1, \ldots, \mu_t\}$. Then, the columns of the block-diagonal polynomial matrix $T := \text{diag}\{[z^{\mu_i - 1}, z^{\mu_i - 2}, \ldots, z, 1]\}$ is a basis for $X_{UN}(UM)$, and hence the columns of $U^{-1}T$ form a basis for $X_N(M)$. It follows that

$$\dim X_N(M) = \sum_{i=1}^{t} \mu_i,$$

where $\{\mu_i\}$ are the row indices of N defined as above. Since by Corollary A.3(i) we have $\dim X^N(M) = \dim X_M - \dim X_{N'}(M')$, it immediately follows that

$$\dim X^N(M) = \dim X_M - \sum_{i=1}^{t} \mu_i,$$

where $\{\mu_i\}$ are the row indices of N' or, equivalently, the column indices of N. Note that if N is nonsingular, then the sum of the column indices of N is the degree of $\det N$. Consequently, for the case of nonsingular N, $\dim X^N(M) = \deg[\det(NM^{-1})] =$ the degree at infinity of $\det(NM^{-1})$. We construct a basis for $X^N(M)$ first in the special case where $M = \text{diag}\{m_i\}$ and $N = [\tilde{N} : 0]$ with $\tilde{N}$ in column-proper form having column indices $\{\mu_1, \ldots, \mu_t\}$: Let $\deg(m_i) = \nu_i$ for $i = 1, \ldots, k$. We claim that a basis for $X^N(M)$ is given by the nonzero columns of the block diagonal $S = \text{diag}\{[z^{d_i - 1}, z^{d_i - 2}, \ldots, z, 1]\}$, where $d_i := \nu_i - \mu_i$ for $i = 1, \ldots, t$ and $d_i = \nu_i$ for $i = t+1, \ldots, k$, and where the ith block is zero if $d_i < 0$. Let $x := [0, \ldots, 0, z^{d_j - s}, 0, \ldots, 0]$ be a typical nonzero column of S with $s > 0$. Then, $(NM^{-1}x)_+ = (n_j m_j^{-1} z^{d_j - s})_+$, where

n_j is the jth column of N. Since $\deg(z^{d_j-s}/m_j) < -\mu_j - s < -\mu_j = -\deg(n_j)$, it follows that $n_j z^{d_j-s}/m_j$ is strictly proper. Therefore $(NM^{-1}x)_+ = 0$, i.e., x is in $X^N(M)$. Also, given any x in $X^N(M)$, we have $(NM^{-1}x)_+ = N\alpha$ for some polynomial vector α. Let $y := M^{-1}x$, and define $\tilde{y} := [y_1, \ldots, y_t]'$, $\tilde{\alpha} := [\alpha_1, \ldots, \alpha_t]'$. Then $(\tilde{N}\tilde{y})_+ = \tilde{N}\tilde{\alpha}$, where $\tilde{N}$ is column proper and hence admits a proper rational left inverse $\hat{N}$ satisfying $\hat{N}\tilde{N} = I$. Consequently, $\tilde{\alpha} = \tilde{y} - \hat{N}(\tilde{N}\tilde{y})_-$, where the right hand side of the equality is strictly proper. Since $\tilde{\alpha}$ is polynomial, we must have $\tilde{\alpha} = 0$, i.e., $\tilde{N}\tilde{y} = \hat{y}$ for some strictly proper vector $\hat{y}$. This implies that $\max_i\{\deg(y_i) + \mu_i\} < 0$. (Compare the predictable degree property of Forney [16].) Hence, $\deg(y_i) < -\mu_i$ for $i = 1, \ldots, t$. Since $x_i = m_i y_i$, it follows that $\deg(x_i) < \nu_i - \mu_i$ for $i = 1, \ldots, t$. By the fact that x is in X_M, we also have $\deg(x_i) < \nu_i$ for $i = t+1, \ldots, k$. Whenever, $d_i = \nu_i - \mu_i < 0$, the equality $x_i = m_i y_i$ implies, as x_i is polynomial, that $x_i = 0$. Therefore, the nonzero columns of S span $X^N(M)$. This yields, in this special case, a constructive procedure to obtain a basis for $X^N(M)$. We also note that the indices $\{d_i\}$ are precisely the indices at infinity of the rational matrix NM^{-1}. Under a certain condition, the construction of a basis for $X^N(M)$ in the general case can easily be reduced to the above case: Given the polynomial matrices N and M, let V be a unimodular polynomial matrix such that $NV = [\tilde{N}:0] := L$, where $\tilde{N}$ is column proper with column indices $\{\mu_1, \ldots, \mu_t\}$. Let U be another unimodular polynomial matrix such that $K := UMV$ is column-proper with indices $\{\nu_i\}$. *Suppose that among the set of U satisfying this condition, there is one with the further property that $NM^{-1}U^{-1}$ is proper.* Let $\Lambda := \operatorname{diag}\{z^{\nu_1}, \ldots, z^{\nu_k}\}$. Since Λ and K are both column-proper with the same column indices, it follows that the rational matrix $B := K\Lambda^{-1}$ is bicausal. It can be shown that the R-linear map

$$\psi : X^L(\Lambda) \to X^N(M) : x \mapsto \pi_M\left[U^{-1}(Bx)_+\right]$$

is an isomorphism (where one makes use of the italicized assumption above). Note that L and Λ satisfy the requirements of the special case discussed above. Hence, a basis for $X^L(\Lambda)$ is given by the nonzero columns of S. Using the isomorphism ψ, the nonzero columns of $\hat{S} := \pi_M[U^{-1}(BS)_+]$ then constitute a basis for $X^N(M)$.

5. APPLICATIONS TO MEASUREMENT FEEDBACK PROBLEMS

In this section, we present two major applications of the main results of Sections 3 and 4. The first of these is to what we call the "output stabilization problem with measurement feedback," and the second to the "disturbance decoupling problem with measurement feedback," In both cases, we first give the polynomial solvability conditions and, with the help of the characterizations developed for (H, F)-invariant subspaces, obtain geometric interpretations.

Consider a one-input-channel, two-output-channel system model

$$\begin{bmatrix} y_m \\ y \end{bmatrix} = \begin{bmatrix} Z_m \\ Z \end{bmatrix} u, \tag{5.1a}$$

where u represents the control inputs, y_m the measured outputs, and y the outputs to be controlled. The transfer matrices Z_m and Z are of sizes $p \times m$ and $q \times m$, respectively, and they are assumed to be strictly proper. Let $\hat{P}$, $\hat{T}$, and $\hat{Q}$ be $p \times m$, $q \times m$, and $m \times m$ jointly coprime polynomial matrices, with $\hat{Q}$ nonsingular, such that

$$\begin{bmatrix} Z_m \\ Z \end{bmatrix} = \begin{bmatrix} \hat{P} \\ \hat{T} \end{bmatrix} \hat{Q}^{-1}. \tag{5.1b}$$

The *output stabilization problem with measurement feedback* is that of determining a feedback of the form

$$u = -Z_c y_m + v, \tag{5.1c}$$

where v is an external input and Z_c is a proper rational matrix such that in the closed-loop system the transfer matrix from v to y is stable.

We now derive a solvability condition for this problem. For the sake of simplicity, it is assumed that there exist polynomial matrices P, Q, and D with D nonsingular such that P and Q are right coprime and

$$\hat{P} = PD, \qquad \hat{Q} = QD, \tag{5.1d}$$

where $\det D$ has all unstable zeros, which is equivalent to the assumption that the unobservable modes of $\Sigma(\hat{P}, \hat{Q})$ are all unstable.

PROPOSITION 5.2. *The output stabilization problem with measurement feedback of* (5.1) *is solvable if and only if there exist a polynomial matrix X and a rational matrix Y such that*

$$DX + YP = I. \tag{5.3}$$

Proof. Since it is somewhat unrelated to the rest of the contents of the paper, we omit the proof. The interested reader is referred to [11] for a detailed discussion of this type of problems and a proof of the proposition. ∎

Let $\Sigma(\hat{P},\hat{Q}) = (F, G, H, X_Q)$, and let $\hat{\eta}$ be the unobservable subspace and $\hat{\upsilon}_*$ be the smallest (H, F)-invariant subspace containing $\operatorname{im} G$ of $X_{\hat{Q}}$ associated with this realization. The theorem below yields a geometric interpretation for the matrix equation (5.3) in terms of the subspaces $\hat{\eta}$ and $\hat{\upsilon}_*$.

THEOREM 5.4. *The following statements are equivalent*:

(i) *There exist a polynomial* X *and rational* Y *satisfying* (5.3).
(ii) $\dim X^P(\hat{P},\hat{Q}) = \dim X^{\hat{P}}(\hat{Q}) + \dim QX_D$.
(iii) $\hat{\eta} \cap \hat{\upsilon}_* = \{0\}$.

Proof. By Proposition 2.2 and Theorem 3.4, respectively, $\hat{\eta} = QX_D$ and $\hat{\upsilon}_* = X^{\hat{P}}(\hat{Q})$. We first show that (i) implies (iii). Let x be in the intersection of $\hat{\eta}$ and $\hat{\upsilon}_*$, so that $x = Q\tilde{x}$, $(\hat{P}\hat{Q}^{-1}x)_+ = \hat{P}\alpha$ for some $\tilde{x}$ in X_D and some polynomial α. Now, (i) implies $\tilde{x} = DX\tilde{x} + YP\tilde{x}$, where $(PQ^{-1}x)_+ = P\tilde{x} = PD\alpha$ yields $\tilde{x} = DX\tilde{x} + YPD\alpha = D(X\tilde{x} + \alpha - XD\alpha)$. Thus, $D^{-1}x$ is polynomial. But since $\tilde{x}$ is in X_D, it follows that $\tilde{x} = 0$ and hence $x = 0$. Consequently, (iii) holds.

To show that (iii) implies (ii), we use the result of Lemma 4.7 to write

$$X^P(\hat{P},\hat{Q}) = X^{\hat{P}}(\hat{Q}) + QX_D.$$

By (iii),

$$\hat{\eta} \cap \hat{\upsilon}_* = QX_D \cap X^{\hat{P}}(\hat{Q}) = \{0\},$$

and hence (ii) must hold.

Now, we show that (iii) implies (i). We first derive a useful equivalent condition to (i).

Let V be an $m \times m$ unimodular polynomial matrix such that

$$PV = [L:0],$$

where L has full column rank l. Also let U be another unimodular polynomial matrix such that

$$V^{-1}DU =: \begin{bmatrix} K & N \\ 0 & M \end{bmatrix},$$

where it is partitioned so that K is of size $l \times l$. (See [8, Section 6.3] for the existence of such unimodular matrices.) Note that if $l = m$, then P is of full column rank and hence there exists an $m \times p$ rational matrix Y such that $YP = I$, i.e., (i) holds. If $l < m$, M is unimodular, and K is a left factor of N, then letting $\tilde{N} := K^{-1}N$, we can write

$$\begin{bmatrix} K & N \\ 0 & M \end{bmatrix} \begin{bmatrix} 0 & -\tilde{N}M^{-1} \\ 0 & M^{-1} \end{bmatrix} + \begin{bmatrix} \tilde{Y} \\ 0 \end{bmatrix} [L:0] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

where $\tilde{Y}$ is any $l \times p$ rational matrix that satisfies $\tilde{Y}L = I$. Note that, as M is unimodular, $\hat{X}$ below is polynomial and $\hat{Y}$ is clearly rational:

$$X := \begin{bmatrix} 0 & -\tilde{N}M^{-1} \\ 0 & M^{-1} \end{bmatrix}, \qquad \hat{Y} := \begin{bmatrix} \tilde{Y} \\ 0 \end{bmatrix}.$$

Now, letting $X := U\hat{X}V^{-1}$, $Y := V\hat{Y}$, it is easy to check that (i) holds. We have thus established that *if either*

$$\text{(C.1)} \qquad l = m$$

or

$$\text{(C.2)} \qquad M \text{ is unimodular and } K \text{ is a left factor of } N,$$

then the condition (i) *holds*. (The converse of this statement is also true.)

Suppose (i) does not hold, so that (C.1) and (C.2) fail. Thus, $l < m$, and either M is not unimodular or K is not a left factor of N. In both cases, we will show the existence of a nonzero element x in the intersection of $\hat{\eta}$ and $\hat{\nu}_*$.

If M is not unimodular, the space X_M is nonempty. Let m be a nonzero element of X_M, and consider $x := Q\pi_D(V\hat{m})$, $\hat{m} := [0: m']'$. Clearly, x is in $QX_D = \hat{\eta}$. Also, $(\hat{P}\hat{Q}^{-1}x)_+ = P\pi_D(V\hat{m}) = PV\hat{m} - PD\alpha$, where $\alpha := (D^{-1}V\hat{m})_+$. Since $\hat{m}$ satisfies

$$PV\hat{m} = [L:0]\begin{bmatrix} 0 \\ m \end{bmatrix} = 0,$$

it follows that $(\hat{P}\hat{Q}^{-1}x)_+ = -\hat{P}\alpha$ and hence x is also in $X^{\hat{P}}(\hat{Q}) = \hat{\nu}_*$. Consequently, (iii) does not hold.

If K is not a left factor of N, it follows that $(K^{-1}N)_-$ is nonzero. Thus, there exists a constant vector g such that $K(K^{-1}N)_-g$ is nonzero and we can let $x := Q\pi_D(V\hat{k})$, where

$$\hat{k} := \begin{bmatrix} K(K^{-1}N)_{-}g \\ 0 \end{bmatrix}.$$

The vector x is nonzero, and it is clearly in QX_D. Also, $(\hat{P}\hat{Q}^{-1}x)_{+} = P\pi_D(V\hat{k}) = PV\hat{k} - PD\alpha$, where $\alpha := (D^{-1}V\hat{k})_{+}$. The vector $\hat{k}$, on the other hand, is such that $PV\hat{k} = LK(K^{-1}Ng)_{-} = LNg - LK(K^{-1}Ng)_{+}$, where the right hand side can be rewritten as

$$LNg - LK(K^{-1}Ng)_{+} = [L:0]\begin{bmatrix} K & N \\ 0 & M \end{bmatrix}\begin{bmatrix} -(K^{-1}Ng)_{+} \\ g \end{bmatrix}.$$

Consequently, $PV\hat{k} = PD\beta$, where $\beta = U[-(K^{-1}Ng)'_{+} : g']'$. It follows that $(\hat{P}\hat{Q}^{-1}x)_{+} = \hat{P}(\beta - \alpha)$ for polynomials β and α, and hence x is in $X^{\hat{P}}(\hat{Q})$. Therefore, also in this case, the condition (iii) fails. This establishes the fact that (iii) implies (i). ■

The subspace $\hat{\mathscr{V}}_{*}$ represents the smallest subspace of $X_{\hat{Q}}$ that can be made to contain the reachable subspace under a suitable output injection. Since in the above problem the reachable subspace is the state space $X_{\hat{Q}}$ itself, it follows that $\hat{\mathscr{V}}_{*}$ represent the set of all modes that can become reachable after a suitable output injection. With this interpretation of $\hat{\mathscr{V}}_{*}$ in mind, we see that the condition (iii) obtained above as a solvability condition for the output stabilization problem can be read off as: *There is no unobservable mode that may become reachable with output injection.*

Also note that the statement (iii) is precisely the dual of the statement $\hat{\mathscr{V}}^{*} + \hat{\mathscr{R}}_0 = X_{\hat{Q}}$, where $\hat{\mathscr{V}}^{*}$ is the largest (F, G)-invariant subspace in $\ker H$ and $\hat{\mathscr{R}}_0$ is the reachable subspace associated with an appropriate system. The latter is the solvability condition for the output stabilization problem (against disturbances) of Wonham [15] using state feedback.

The second application we consider is to *the disturbance decoupling problem with measurement feedback*, which has been considered by Akashi and Imai [1] and Schumacher [13] in a geometric setup.

Consider the two-input-channel, two-output-channel system model

(5.5a) $$\begin{bmatrix} y_m \\ y \end{bmatrix} = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}\begin{bmatrix} u \\ w \end{bmatrix},$$

where Z_1, Z_2, Z_3, and Z_4 are strictly proper transfer matrices of sizes $p \times m$, $p \times s$, $q \times m$, and $q \times s$, respectively. Here, u represents the control inputs, w the disturbances, y_m the measured outputs, and y the outputs to be controlled. The problem is to determine a proper rational Z_c such that with the feedback of the form $u = -Z_c y_m + v$, where v is a possible external input, the transfer matrix from w to y in the closed-loop system is identically zero, i.e., the output y depends only on the external input v and does not depend on the disturbance w.

Let Q be an $r \times r$ nonsingular polynomial matrix, and $P, T, R, S, W_1, W_2, W_3, W_4$ polynomial matrices of appropriate sizes such that Q^{-1} is proper, PQ^{-1} is strictly proper, $Q^{-1}R$ is strictly proper, and

$$\text{(5.5b)} \qquad \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} = \begin{bmatrix} P \\ T \end{bmatrix} Q^{-1} [R : S] + \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}.$$

(Note that polynomial matrices satisfying (5.5b) always exist. They may also be made to satisfy the extra conditions that Q^{-1} is proper and that PQ^{-1} and $Q^{-1}R$ are strictly proper by suitable transformations; see, e.g., [12].)

Proposition 5.6 (Özgüler and Eldem [12]). *The disturbance decoupling problem with measurement feedback is solvable if and only if there exist an $(r+m) \times (r+p)$ proper rational matrix X satisfying*

$$\text{(5.7)} \qquad \underbrace{\begin{bmatrix} Q & S \\ -T & W_4 \end{bmatrix}}_{\Pi_4} = \underbrace{\begin{bmatrix} Q & R \\ -T & W_3 \end{bmatrix}}_{\Pi_3} X \underbrace{\begin{bmatrix} Q & S \\ -P & W_2 \end{bmatrix}}_{\Pi_2}.$$

We show in the following theorem that the condition (5.7) is equivalent to the geometric condition derived by Akashi and Imai [1] and Schumacher [13].

Let us associate the realizations

$$\Sigma(T, Q, R, W_3) = (F, G, C, X_Q),$$

$$\Sigma(P, Q, S, W_2) = (F, B, H, X_Q)$$

with the transfer matrices Z_3 and Z_2, respectively. Let $v^*(\ker C)$ be the largest (F, G)-invariant subspace contained in $\ker C$, and $v_*(\operatorname{im} B)$ be the smallest (H, F)-invariant subspace containing $\operatorname{im} B$ associated with the above realization.

THEOREM 5.8. *The following statements are equivalent*:

(i) *There exists a proper* X *satisfying* (5.7).
(ii) *There exists a proper* Y *satisfying*

$$Z_4 = Z_3 Y Z_2.$$

(iii) $\upsilon_*(\operatorname{im} B) \subseteq \upsilon^*(\ker C)$.

Proof. If (i) holds, then by suitable manipulations it is easy to see that the lower right $m \times p$ submatrix of X satisfies (ii). Thus, (i) implies (ii).

Suppose now that (ii) holds so that

$$(5.9)\quad \begin{bmatrix} Q & S \\ -T & W_4 \end{bmatrix} = \begin{bmatrix} Q & R \\ -T & W_3 \end{bmatrix} \begin{bmatrix} Q^{-1} - Q^{-1}RYPQ^{-1} & -Q^{-1}RY \\ YPQ^{-1} & Y \end{bmatrix} \times \begin{bmatrix} Q & S \\ -P & W_2 \end{bmatrix}.$$

Since Q^{-1} and Y are proper and PQ^{-1} and $Q^{-1}R$ are strictly proper, it follows that (i) holds. Thus, (ii) implies (i). Let x be in $\upsilon_*(\operatorname{im} B)$. Then, by Theorem 3.4, there exists a polynomial α such that

$$x = \pi_Q(S\alpha), \qquad (Z_2\alpha)_+ = 0.$$

Let $\beta := (Q^{-1}S\alpha)_+$, and multiply (5.9) on the right by $[\beta' : \alpha']'$ to obtain

$$(5.10)\qquad \begin{bmatrix} x \\ \gamma \end{bmatrix} = \begin{bmatrix} Q & R \\ -T & W_3 \end{bmatrix} \begin{bmatrix} (Q^{-1}S\alpha)_- - Q^{-1}RYZ_2\alpha \\ YZ_2\alpha \end{bmatrix},$$

where

$$\gamma := -T\beta + S\alpha$$

is polynomial. Note that, as $(Z_2\alpha)_+ = 0$, $Q^{-1}R$ is strictly proper, and Y is proper, the equation (5.10) is of the form

$$\begin{bmatrix} x \\ \gamma \end{bmatrix} = \begin{bmatrix} Q & R \\ -T & W_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

where y_1 and y_2 are strictly proper. It follows by Proposition 2.3(iii) that x is in $v^*(\ker C)$. Thus, $v_*(\operatorname{im} B)$ is contained in $v^*(\ker C)$. Consequently, (ii) implies (iii).

To complete the proof, we now show that (iii) implies (ii). Suppose (iii) holds. Let V be a basis matrix for $v_*(\operatorname{im} B)$. By Lemma 3.2, there exist constant matrices A_1, B_1, C_1 and polynomial matrices L and K satisfying

$$QC_1 + K(PQ^{-1}V)_- = V(zI - A_1), \tag{5.11}$$

$$QL + VB_1 = S. \tag{5.12}$$

As all columns of V are in $v^*(\ker C)$, by Proposition 2.3(iii) there exist strictly proper matrices Y_1 and Y_2 and a polynomial matrix M such that

$$\begin{bmatrix} V \\ M \end{bmatrix} = \begin{bmatrix} Q & R \\ -T & W_3 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \tag{5.13}$$

which in particular implies that

$$(TQ^{-1}V)_- = (TQ^{-1}R)_- Y_2, \tag{5.14}$$

$$(TQ^{-1}V)_{-1} = 0. \tag{5.15}$$

The equation (5.11) implies $(PQ^{-1}K)_-(PQ^{-1}V)_{-1} = [PQ^{-1}V(zI - A_1)]_- = (PQ^{-1}V)_-(zI - A_1) - (PQ^{-1}V)_{-1}$, which, letting $\hat{K} := I + (PQ^{-1}K)_-$, can be written as

$$(PQ^{-1}V)_{-1}(zI - A_1)^{-1} = \hat{K}^{-1}(PQ^{-1}V)_-. \tag{5.16}$$

Again by Equation (5.11),

$$(TQ^{-1}K)_-(PQ^{-1}V)_{-1} = TQ^{-1}V(zI - A_1) - (TQ^{-1}V)_{-1},$$

which by (5.15) implies

$$(TQ^{-1}K)_-(PQ^{-1}V)_{-1} = (TQ^{-1}V)_-(zI - A_1). \tag{5.17}$$

Let C_2 and C_3 be constant matrices satisfying $(PQ^{-1}V)_{-1} = C_2C_3$, $\tilde{C}_2C_2 = I$, $C_3\tilde{C}_3 = I$ for some constant $\tilde{C}_2$ and $\tilde{C}_3$. It follows that (5.16) and (5.17) can be rewritten as

$$C_3(zI - A_1)^{-1} = \tilde{C}_2\hat{K}^{-1}(PQ^{-1}V)_-, \tag{5.18}$$

$$(TQ^{-1}KC_2)_- = (TQ^{-1}V)_-(zI - A_1)\tilde{C}_3. \tag{5.19}$$

By (5.14) and (5.19) we also have

$$(TQ^{-1}KC_2)_- = (TQ^{-1}R)_- Y_2(zI - A_1)\tilde{C}_3. \tag{5.20}$$

The equations (5.17), (5.18), and (5.20) now yield $(TQ^{-1}V)_- = (TQ^{-1}R)_- Y_2(zI - A_1)\tilde{C}_3\tilde{C}_2\hat{K}^{-1}(PQ^{-1}V)_-$, which, on multiplying on the right by B_1 and employing (5.12), finally yields $(TQ^{-1}S)_- = (TQ^{-1}R)_- Y_2(zI - A_1) \tilde{C}_3\tilde{C}_2K^{-1}(PQ^{-1}S)_-$, i.e., $Z_4 = Z_3YZ_2$, $Y := Y_2(zI - A_1)\tilde{C}_3\tilde{C}_2\hat{K}^{-1}$, where, as $Y_2(zI - A_1)$ and $\hat{K}^{-1}$ are proper, Y is a proper rational matrix. The condition (ii) follows. ■

REMARK 5.21. The result of Theorem 5.8 involves a comparison of the three conditions and the computations involved in checking these three conditions. In [12, Corollary (4.11)], it was shown that condition (i) of Theorem 5.8 holds iff a certain inequality among the column indices of Π_2, the row indices of Π_3, and the degrees of the entries of Π_4 holds. We have shown in Remark 4.8 that the computation of the row and column indices of Π_3 and Π_2 yields bases for $\mathcal{V}^*$ and $\mathcal{V}_*$, respectively. Thus, the inclusion $\mathcal{V}_* \subseteq \mathcal{V}^*$ can be checked by simply checking the inequality among the abovementioned integers. Note that this is an alternative to the computational procedures of the geometric theory for $\mathcal{V}_*, \mathcal{V}^*$, and the checking of the inclusion $\mathcal{V}_* \subseteq \mathcal{V}^*$. It can also be shown that Theorem 5.8(ii) holds iff an inequality is satisfied by the column indices at infinity of Z_2, the row indices at infinity of Z_3, and the degrees at infinity of entries of Z_4. Through the equivalence of Theorem 5.8(ii) and (iii), we further see that the subspaces $\mathcal{V}_*, \mathcal{V}^*$ and the relation $\mathcal{V}_* \subseteq \mathcal{V}^*$ are tightly connected with these indices at infinity—a point we have touched in Remark 4.18 but which certainly requires further elucidation. We finally remark that similar results can be stated for other related control problems such as disturbance decoupling with internal stability and pole placement; these require a similar theory developed for stabilizability and detectability subspaces in a parallel manner to our Section 4.

APPENDIX. DUALITY

In this appendix, we use the duality theory of Fuhrmann [5] for polynomial models to establish the duality between $X_{\tilde{N}}(M, N)$ and $X^{\tilde{N}}(N, M)$. For the details of various unproved facts, the reader is referred to Section II of [5].

Between the r-vectors of the truncated Laurent series $\mathbb{R}((z^{-1}))^r$, a dual pairing is defined as follows: Given $a = \sum_{i=i_a}^{\infty} a_i z^{-i}$ and $b = \sum_{i=i_b}^{\infty} b_i z^{-i}$, let

$$[a, b] := [b(z^{-1})'a(z)]_{-1} = \sum_{i=-\infty}^{\infty} b_i' a_{1-i}.$$

This defines a nondegenerate bilinear form on $\mathbb{R}((z^{-1}))^r \times \mathbb{R}((z^{-1}))^r$ with the following properties: $[a, b] = 0$ for all b implies $a = 0$, $[a, b] = [(a)_+, (b)_-] + [(a)_-, (b)_+]$, $[(a)_+, b] = [a, (b)_-]$, and $[(a)_-, b] = [a, (b)_+]$. For any matrix M in $\mathbb{R}((z^{-1}))^{r \times s}$, a in $\mathbb{R}((z^{-1}))^s$, and b in $\mathbb{R}((z^{-1}))^r$, we have $[Ma, b] = [a, M'b]$.

Given a subset V of $\mathbb{R}((z^{-1}))^r$, the annihilator $V^{\perp}$ of V is defined by $V^{\perp} := \{b \text{ in } \mathbb{R}((z^{-1}))^r : [a, b] = 0 \text{ for all } a \text{ in } V\}$. Let N be a $k \times l$ polynomial matrix, and let

$$Y_N := \{y \text{ in } z^{-1}\mathbb{R}[[z^{-1}]]^l : (Ny)_- = 0\}.$$

LEMMA A.1. $Y_N^{\perp} = N'\mathbb{R}[z]^k$.

Proof. Let y be in Y_N, so that $(Ny)_- = 0$, and let $N'\beta$ be in $N'\mathbb{R}[z]^k$. Then, $[y, N'\beta] = [Ny, \beta] = [(Ny)_-, \beta] = 0$, which establishes $N'\mathbb{R}[z]^k \subseteq Y_N^{\perp}$. To see the reverse inclusion, let y be in $(N'\mathbb{R}[z]^k)^{\perp}$, so that for all $N'\beta$ in $N'\mathbb{R}[z]^k$, we have $[y, N'\beta] = [(Ny)_-, \beta] = 0$. Since $[(Ny)_-, b] = 0$ for all b in $z^{-1}\mathbb{R}[[z^{-1}]]^k$, it follows that $[(Ny)_-, c] = 0$ for all c in $\mathbb{R}((z^{-1}))^k$. Consequently, $(Ny)_- = 0$ establishing $(N'\mathbb{R}[z]^k)^{\perp} \subseteq Y_N$, or $(Y_N)^{\perp} \subseteq N'\mathbb{R}[z]^k$. ■

A basic result, Theorem 2.9, of [5] is that the pairing

$$\langle x, \hat{x} \rangle := [M^{-1}x, \hat{x}] = [x, M'^{-1}\hat{x}]$$

is a dual pairing between the elements x of X_M and $\hat{x}$ of $X_{M'}$. Hence, via

$\langle \cdot , \cdot \rangle$, one can identify $X_{M'}$ as the dual space to X_M. For any subspace V of X_M, the annihilator of V is given by $V^{\perp} = \{\hat{x} \text{ in } X_{M'} : \langle x, \hat{x} \rangle = 0 \text{ for all } x \text{ in } V\}$. We now prove the main result of the Appendix.

THEOREM A.2. *Let N be in $\mathbb{R}[z]^{l \times k}$, and M be nonsingular in $\mathbb{R}[z]^{k \times k}$. Let $N = \tilde{N}D$ be a factorization of N with D nonsingular in $\mathbb{R}[z]^{k \times k}$ and $\tilde{N}$ in $\mathbb{R}[z]^{l \times k}$. Then,*

$$[X^{\tilde{N}}(N, M)]^{\perp} = X_{\tilde{N}'}(M', N').$$

Proof. Let $\hat{x}$ be in $X_{\tilde{N}'}(M', N')$, so that $\hat{x} = \pi_{M'}(N'y)$ for some strictly proper y satisfying $(\tilde{N}'y)_- = 0$. Let x be in $X^{\tilde{N}}(N, M)$ so that x is in X_M and $(NM^{-1}x)_+ = \tilde{N}\alpha$ for some α in $\mathbb{R}[z]^k$. Then, $\langle x, \hat{x} \rangle = [M^{-1}x, \pi_{M'}(N'y)] = [x, (M'^{-1}N'y)_-] = [(NM^{-1}x)_+, y] = [\tilde{N}\alpha, y] = [\alpha, N'y] = 0$, where the last equality follows by the fact that $(\tilde{N}'y)_- = 0$. Therefore, $X_{\tilde{N}'}(M', N')$ is contained in $[X^{\tilde{N}}(N, M)]^{\perp}$. To see the reverse inclusion, first note that for any y in $Y_{\tilde{N}'}$, $\hat{x} = \pi_{M'}(N'y)$ is in $X_{\tilde{N}'}(M', N')$. Let x be in the annihilator of $X_{\tilde{N}'}(M', N')$. Then, for all y in $Y_{\tilde{N}'}$, we have $\langle x, \hat{x} \rangle = [(NM^{-1}x)_+, y] = 0$. This implies, by Lemma A.1, that $(NM^{-1}x)_+$ is an element of $\tilde{N}\mathbb{R}[z]^k$. Therefore, x is in $X^{\tilde{N}}(N, M)$. This establishes $[X_{\tilde{N}'}(M', N')]^{\perp} \subseteq X^{\tilde{N}}(N, M)$, or equivalently, $[X^{\tilde{N}}(N, M)]^{\perp} \subseteq X_{\tilde{N}'}(M', N')$. ■

COROLLARY A.3.

(i) $\dim X^{\tilde{N}}(N, M) = \dim X_M - \dim X_{\tilde{N}'}(M', N')$,
(ii) $X^N(M) = [X_{N'}(M')]^{\perp}$.

Proof. Statement (i) is obvious. For (ii), let $D = I$ in Theorem A.2. ■

REFERENCES

1 H. Akashi and H. Imai, Disturbance localization and output deadbeat control through an observer in discrete-time, linear multivariable systems, *IEEE Trans. Automat. Control*, 24:621–627 (1979).

2 E. Emre and M. L. J. Hautus, A polynomial characterization of (A, B)-invariant and reachability subspaces, *SIAM J. Control Optim.* 18:420–436 (1980).

3 P. Fuhrmann, Algebraic system theory: An analyst's point of view, *J. Franklin Inst.*, 301:521–540 (1976).

4 P. Fuhrmann, On strict system equivalence and similarity, *Internat. J. Control* 25:5–10 (1977).

5 P. Fuhrmann, Duality in polynomial models with some applications to geometric control theory, *IEEE Trans. Automat. Control* 26:284–295 (1981).

6 P. Fuhrmann and J. C. Willems, A study of (A, B)-invariant subspaces via polynomial models, *Internat. J. Control* 31:467–494 (1980).
7 M. L. J. Hautus and M. Heymann, Causal factorization and linear feedback, *SIAM J. Control Optim.* 19:445–468 (1978).
8 T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, N.J., 1980.
9 P. P. Khargonekar and E. Emre, Further results on polynomial characterizations of (F, G)-invariant and reachability subspaces, *IEEE Trans. Automat. Control* 27:352–366 (1982).
10 P. P. Khargonekar, T. T. Georgiou, and A. B. Özgüler, Skew-prime polynomial matrices: The polynomial model approach, *Linear Algebra Appl.* 50:403–435 (1983).
11 A. B. Özgüler, Fundamental equations of the regulator problems, in preparation.
12 A. B. Özgüler and V. Eldem, Disturbance decoupling problems via dynamic output feedback as two-sided model matching, *IEEE Trans. Automat. Control*, to appear.
13 H. Schumacher, Compensator synthesis using (C, A, B)-pairs, *IEEE Trans. Automat. Control* 25:1133–1138 (1980).
14 W. A. Wolovich, *Linear Multivariable Systems*, Springer, New York, 1974.
15 W. M. Wonham, *Linear Multivariable Control: A geometric approach*, 2nd ed., Springer, New York, 1979.
16 G. D. Forney, Minimal bases of rational vector spaces with applications to multivariable linear systems, *SIAM J. Control Optim.* 13:493–520 (1975).

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