COMMUNICATION

PERFECTIONNESS AND DILWORTH NUMBER

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In this note, we examine the relation between the perfectionness of a graph and its vicinal preorder.
In particular, we answer a question of C. Benzaken, P. L. Hammer and D. De Werra [2, 3] by proving that graphs with Dilworth number 3 or 4 are perfect.
All main definitions can be found in [1].

Perfectness

The chromatic number of a graph $G = (V, E)$ is denoted by $\gamma(G)$ and the clique number by $\omega(G)$.
$\bar{G}$ denotes the complementary graph of $G$.
A graph is said to be perfect iff for every induced subgraph $H$: $\gamma(H) = \omega(H)$.

Conjecture (C. Berge, cf. [1]). (1) $G$ is perfect iff the complementary graph $\bar{G}$ is perfect.
(2) (Strong Perfect Graph Conjecture) $G$ is perfect iff neither $G$ nor $\bar{G}$ contains an odd cycle without chords.

The first conjecture was proved by L. Lovász [4].
A critical non-perfect graph $G$ is a non-perfect graph such that every proper induced subgraph of $G$ is perfect.
By Lovász’s theorem, $G$ is a critical non-perfect graph iff $\bar{G}$ is a critical non-perfect graph.
Every non-perfect graph has a critical non-perfect induced subgraph.
Berge’s Strong Perfect Graph Conjecture is equivalent to the following: “The only critical non-perfect graphs are the odd cycles and the complementary graphs of odd cycles”.

The vicinal preorder and the Dilworth number of a graph

Let $G = (V, E)$ be a graph and let $N_G(x)$ be the set of neighbours of the vertex $x$. 

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We define the following relation on $V$

$$x \preceq_G y \iff N_G(x) \subseteq N_G(y) \cup \{y\}.$$ 

Clearly $\preceq_G$ is transitive and reflexive. It is called the vicinal preorder of $G$.

The Dilworth number $D(G)$ of $G$ is the maximum integer $k$ such that there exists in $G$, $k$ mutually uncomparable vertices with respect to $\preceq_G$. Clearly if $H$ is an induced subgraph of $G$, $D(H) \leq D(G)$.

In [2] and [3], C. Benzaken, P. L. Hammer and D. De Werra raise the question of the perfectness of the graphs with Dilworth number less than or equal to 4. We answer this question by the following result:

**Theorem.** If $G$ is a critical non-perfect graph, then $D(G) = |V(G)|$ [i.e. the vicinal preorder is an antichain].

**Proof.** Suppose $\exists x, y \in V(G)$, $x \neq y \mid x \preceq_G y$.

(i) Case 1: $\{x, y\} \notin E(G)$. $G - x$ is perfect. Hence $\gamma(G - x) = \omega(G - x)$. Since $N_G(x) \subseteq N_G(y)$ every coloration of $G - x$ can be extended in a coloration of $G$: we give to $x$ the color of $y$. Hence $\gamma(G) = \gamma(G - x) = \omega(G - x) \leq \omega(G)$. Therefore $G$ is perfect. A contradiction.

(ii) Case 2: $\{x, y\} \in E(G)$ and hence $\{x, y\} \notin E(\tilde{G})$. $N_G(x) \subseteq N_G(y) \cup \{y\}$. Then $N_G(y) \subseteq N_G(x)$. By (i) $\tilde{G}$ is perfect. (We consider now $\tilde{G} - y$.)

Therefore $G$ is perfect. A contradiction. \(\square\)

**Corollary.** If $D(G) \leq 4$, then $G$ is perfect.

**Proof.** If not, $G$ contains a critical non-perfect induced subgraph $H$. Now $D(H) \leq D(G) \leq 4$. By the Theorem $|V(H)| \leq 4$. A contradiction (the smallest non-perfect graph is $C_5$).

**Remark.** Similarly it is easy to show that in any statement of the form: Berge's Strong Perfect Graph Conjecture is true for graphs $G$ with $|V(G)| \leq k$, one can replace $|V(G)|$ by $D(G)$.

**References**