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Domain theory and integration

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Abstract

We present a domain-theoretic framework for measure theory and integration of bounded real-valued functions with respect to bounded Borel measures on compact metric spaces. The set of normalised Borel measures of the metric space can be embedded into the maximal elements of the normalised probabilistic power domain of its upper space. Any bounded Borel measure on the compact metric space can then be obtained as the least upper bound of an ω-chain of linear combinations of point valuations (simple valuations) on the upper space, thus providing a constructive framework for these measures. We use this setting to define a new notion of integral of a bounded real-valued function with respect to a bounded Borel measure on a compact metric space. By using an ω-chain of simple valuations, whose lub is the given Borel measure, we can then obtain increasingly better approximations to the value of the integral, similar to the way the Riemann integral is obtained in calculus by using step functions. We show that all the basic results in the theory of Riemann integration can be extended in this more general setting. Furthermore, with this new notion of integration, the value of the integral, when it exists, coincides with the Lebesgue integral of the function. An immediate area for application is in the theory of iterated function systems with probabilities on compact metric spaces, where we obtain a simple approximating sequence for the integral of a real-valued almost everywhere continuous function with respect to the invariant measure.

1. Introduction

The theory of Riemann integration of real-valued functions was developed by Cauchy, Riemann, Stieltjes and Darboux, amongst other mathematicians of the 19th century. With its simple, elegant and constructive nature, it soon became, as it is today, a solid basis of calculus; it is now used in all branches of science. The theory, however, has its limitations in the following main areas, listed here not in any particular order of significance:

(i) It only works for integration of functions defined in $\mathbb{R}^n$.

(ii) It can only deal with integration of functions with respect to the Lebesgue measure, i.e. the usual measure, on $\mathbb{R}^n$.

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(iii) Unbounded functions have to be treated separately.

(iv) The theory lacks certain convergence properties. For example, the pointwise limit of a uniformly bounded sequence of Riemann integrable functions may fail to be Riemann integrable.

(v) A function with a "large" set of discontinuity, i.e. with nonzero Lebesgue measure, does not have a Riemann integral.

In the early years of this century, Lebesgue and Borel, amongst others, laid the foundation of a new theory of integration. With its further development, the new theory, the so-called Lebesgue integration, has become the basis of measure theory and functional analysis. A special case of the Lebesgue integral, the so-called Lebesgue–Stieltjes integral, has also played a fundamental role in probability theory. The underlying basis of the Lebesgue theory is in sharp contrast to that of the Riemann theory. Whereas in the theory of Riemann integration, the *domain* of the function is partitioned and the integral of the function is approximated by the lower and upper Darboux sums induced by the partition, in the theory of Lebesgue integration, the *range* of the function is partitioned to produce simple functions which approximate the function, and the integral is defined as the limit of the integrals of these simple functions. The latter framework makes it possible to define the integral of measurable functions on abstract measurable spaces, in particular on topological spaces equipped with Borel measures. Lebesgue integration also enjoys very general convergence properties, giving rise to the complete L^p-spaces. Moreover, when the Riemann integral of a function exists, so does its Lebesgue integral and the two values coincide, i.e. Lebesgue integration includes Riemann integration. Nevertheless, despite these desired features, Lebesgue integration is quite involved and much less constructive than Riemann integration. Consequently, Riemann integration remains the preferred theory wherever it is adequate in practice, in particular in advanced calculus and in the theory of differential equations.

A number of theories have been developed to generalise the Riemann integral while trying to retain its constructive quality. The most well-known and successful is of course the Riemann–Stieltjes integral. In more recent times, McShane [22] has developed a *Riemann–type* integral, which includes for example the Lebesgue–Stieltjes integral, but it unfortunately falls short of the constructive features of the Riemann integral.

A new idea in measure theory on second countable locally compact Hausdorff spaces was presented in [9]. It was shown that the set of normalised Borel measures on such a space can be embedded into the maximal elements of the probabilistic power domain of its upper space. The image of the embedding consists of all normalised valuations on the upper space which are supported in the set of maximal elements of the upper space, i.e. the singletons of the space. This upper space is an ω-continuous dcpo (directed complete partial order), and it follows that its probabilistic power domain is also an ω-continuous dcpo with a basis consisting of linear combinations of point valuations (simple valuations) on the upper space. The important consequence is that any bounded Borel measure on the space can be approximated by simple valuations on the upper space, and we have a constructive framework for measure theory on locally compact second countable Hausdorff spaces.

In this paper, we use the above domain-theoretic framework to present a novel approach in the theory of integration of bounded functions with respect to a bounded Borel measure on a compact metric space. Instead of approximating the *function* with simple functions as is done in the Lebesgue theory, we approximate the normalised *measure* with normalised simple valuations on the upper space; this provides us with generalised lower and upper Darboux sums, which we use to define the integral. The ordinary theory of Riemann integration, as well as the Riemann–Stieltjes integration, is precisely a particular case of this approach, since any partition of, say, the closed unit interval in fact provides a simple normalised valuation on the upper space of the interval which gives an approximation to the Lebesgue measure.

We therefore work in the *normalised probabilistic power domain* of the upper space and develop a new theory of integration, called R-integration, with the following results.

- R-integration satisfies all the elementary properties required for a theory of integration.
- For integration with respect to the Lebesgue measure on compact real intervals, R-integration and Riemann integration are equivalent.
- All the basic results in the theory of ordinary Riemann integration can be generalised to R-integration. In particular, a function is R-integrable with respect to a bounded Borel measure on a compact metric space iff it is continuous almost everywhere.
- When the R-integral of a function (with respect to a bounded Borel measure on a compact metric space) exists so does its Lebesgue integral and the two integrals are equal.

Therefore, our theory, which includes the Lebesgue–Stieltjes integral, is a faithful and sound generalisation of Riemann integration; it overcomes the limitations (i) and (ii) mentioned above, while retaining the constructive nature of Riemann integration. In practice, we are often only interested in the integral of functions which are not too discontinuous, i.e. R-integration is sufficient at least for bounded functions.

We apply the new theory to obtain a simple approximating sequence for the integral of a real-valued almost everywhere continuous function with respect to the unique invariant measure of an iterated function system with probabilities on any compact metric space.

2. A constructive framework for measure theory

In this section, we first review the domain-theoretic framework for measure theory on locally compact second countable spaces which was established in [9]. We will also present some of the background results, in particular from [17], that we need here.

We will use the standard terminology and notations of domain theory, as for example in [18]. Given a dcpo $(D, \sqsubseteq)$ and a subset $A \subseteq D$, we let

$$\uparrow A = \{d \in D \mid \exists a \in A.\, a \sqsubseteq d\} \quad \text{and} \quad \twoheaduparrow A = \{d \in D \mid \exists a \in A.\, a \ll d\},$$

where $\ll$ is the way-below relation in D. We denote the lattice of open sets of a topological space X by ΩX. Given a mapping $f : X \to Y$ of topological spaces and a subset $a \subseteq X$, we denote the forward image of a by $f[a]$, i.e. $f[a] = \{f(x) \mid x \in a\}$. Finally, for a subset $a \subseteq X$ of a compact metric space X, the diameter of a is denoted by $|a|$.

2.1. The upper space

Recall [25] that given any Hausdorff topological space X, its *upper space* UX is the set of all nonempty compact subsets of X with the *upper topology* which has basic open sets $\Box a = \{C \in UX \mid C \subseteq a\}$ for any open set $a \in \Omega X$. The following properties are easy consequences of this definition. (See [9].) The specialisation ordering $\sqsubseteq_u$ of UX is reverse inclusion, i.e.

$$A \sqsubseteq_u B \overset{\text{def}}{\iff} \forall a \in \Omega X [A \subseteq a \Rightarrow B \subseteq a] \iff A \supseteq B.$$

Furthermore $(UX, \supseteq)$ is a bounded complete dcpo, in which the least upper bound of a directed set of elements is the intersection of these elements and the Scott topology refines the upper topology. The singleton map

$$\begin{aligned} s : X &\to UX \\ x &\mapsto \{x\} \end{aligned}$$

embeds X onto the set of maximal elements of UX.

Proposition 2.1 [9]. *Let X be a second countable locally compact Hausdorff space.*

(i) *The dcpo $(UX, \supseteq)$ is ω-continuous.*

(ii) *The Scott topology on $(UX, \supseteq)$ coincides with the upper topology.*

(iii) *The way below relation $B \ll C$ holds in $(UX, \supseteq)$ iff C is contained in the interior of B as subsets of X.*

(iv) *$(UX, \supseteq)$ can be given an effective structure. From any countable basis B of X consisting of relatively compact neighbourhoods* [1], *we can get an order basis of UX consisting of the finite unions of closures of elements of B.*

Therefore, any second countable locally compact Hausdorff space X can be embedded into its upper space UX which can be given an effective structure. We would like to have a similar embedding for the set of bounded Borel measures on X. For this, we use the *probabilistic power domain* of UX.

[1] A *relatively compact* subset of a topological space is one whose closure is compact.

2.2. The probabilistic power domain

Recall from [7, 24, 21, 16] that a *valuation* on a topological space Y is a map $\nu : \Omega Y \to [0, \infty]$ which satisfies:
(i) $\nu(a) + \nu(b) = \nu(a \cup b) + \nu(a \cap b)$.
(ii) $\nu(\emptyset) = 0$.
(iii) $a \subseteq b \Rightarrow \nu(a) \leq \nu(b)$.

A *continuous* valuation [21, 17, 16] is a valuation such that whenever $A \subseteq \Omega(Y)$ is a directed set (w.r.t $\subseteq$) of open sets of Y, then

$$\nu\left(\bigcup_{O \in A} O\right) = \sup_{O \in A} \nu(O).$$

For any $b \in Y$, the *point valuation* based at b is the valuation $\eta_b : \Omega(Y) \to [0, \infty)$ defined by

$$\eta_b(O) = \begin{cases} 1 \text{ if } b \in O, \\ 0 \text{ otherwise.} \end{cases}$$

Any finite linear combination

$$\sum_{i=1}^{n} r_i \eta_{b_i}$$

of point valuations η_{b_i} with constant coefficients $r_i \in [0, \infty)$ $(1 \leqslant i \leqslant n)$ is a continuous valuation on Y, which we call a *simple valuation.*

The *probabilistic power domain, PY*, of a topological space Y consists of the set of continuous valuations ν on Y with $\nu(Y) \leqslant 1$ and is ordered as follows:

$$\mu \sqsubseteq \nu \text{ iff for all open sets } O \text{ of } Y,\ \mu(O) \leqslant \nu(O).$$

The partial order $(PY, \sqsubseteq)$ is a dcpo with bottom in which the lub of a directed set $\langle \mu_i \rangle_{i \in I}$ is given by $\bigsqcup_i \mu_i = \nu$, where for $O \in \Omega(Y)$ we have

$$\nu(O) = \sup_{i \in I} \mu_i(O).$$

The probabilistic power domain gives rise to a functor P : **DCPO** $\to$ **DCPO** on the category of dcpo's and continuous functions [16]. Given a continuous function $f : Y \to Z$ between dcpo's Y and Z, the continuous function $Pf : PY \to PZ$ is defined by $Pf(\mu)(O) = \mu(f^{-1}(O))$. For convenience, we therefore write $Pf(\mu) = \mu \circ f^{-1}$. For later use we need the following property of this functor.

Proposition 2.2. *The functor P* : **DCPO** $\to$ **DCPO** *is locally continuous, i.e. it is Scott continuous on homsets.*

Proof. Let $\langle f_i \rangle_{i \in I}$ be a directed family of maps $f_i : Y \to Z$ in the function space $Y \to Z$. Let $f = \bigsqcup_{i \in I} f_i$. It is easy to see that for any open set $O \subseteq Z$, we have

$f^{-1}(O) = \bigcup_i f_i^{-1}(O)$. Now let $\mu \in PY$. Then, we get

$$\begin{aligned}\left(P\bigsqcup_i f_i\right)(\mu)(O) &= (Pf)(\mu)(O) = \mu(f^{-1}(O)) = \mu\left(\bigcup_i f_i^{-1}(O)\right)\\ &= \sup_i \mu(f_i^{-1}(O)) = \bigsqcup_i(\mu \circ f_i^{-1})(O)\\ &= \left(\bigsqcup_i Pf_i\right)(\mu)(O). \quad \square\end{aligned}$$

It is easy to see that, if Y is a dcpo, the map

$$\begin{aligned}\eta : Y &\to PY\\ b &\mapsto \eta_b\end{aligned}$$

is continuous. Furthermore, there is a nice characterisation of the partial order on simple valuations on a dcpo, aptly called the splitting lemma by Jones.

Proposition 2.3 [16, p. 84]. *Let Y be a dcpo. For two simple valuations*

$$\mu_1 = \sum_{b\in B} r_b\eta_b, \qquad \mu_2 = \sum_{c\in C} s_c\eta_c$$

in PY, we have: $\mu_1 \sqsubseteq \mu_2$ iff, for all $b \in B$ and all $c \in C$, there exists a nonnegative number $t_{b,c}$ such that

$$\sum_{c\in C} t_{b,c} = r_b, \qquad \sum_{b\in B} t_{b,c} \leqslant s_c$$

and $t_{b,c} \neq 0$ implies $b \sqsubseteq c$.

Proof. The "if" part is the splitting lemma [16, p. 84]. For the "only if" part, assume the condition above holds and let $O \in \Omega Y$. Put $A = O \cap B$. Then, we have

$$\begin{aligned}\mu_1(O) &= \sum_{b\in A} r_b = \sum_{b\in A}\sum_{b\sqsubseteq c} t_{b,c}\\ &\leqslant \sum_{c\in C\cap\uparrow A}\sum_{b\sqsubseteq c} t_{b,c} \leqslant \sum_{c\in C\cap\uparrow A} s_c = \mu_2(O).\end{aligned}$$

Therefore $\mu_1 \sqsubseteq \mu_2$. $\square$

If Y is a continuous dcpo, then there is an analogue of the splitting lemma for the way-below relation. First we need the following characterisation of the way-below relation.

Proposition 2.4 [19, p. 46]. *Let $\xi = \sum_{b\in B} r_b\eta_b$ be a simple valuation and μ a continuous valuation on a continuous dcpo. Then $\xi \ll \mu$ iff for all $A \subseteq B$ we have*

$$\sum_{b\in A} r_b < \mu(\uparrow A).$$

Proposition 2.5. *Let Y be a continuous dcpo. For two simple valuations*

$$\mu_1 = \sum_{b\in B} r_b\eta_b, \qquad \mu_2 = \sum_{c\in C} s_c\eta_c$$

in PY, we have $\mu_1 \ll \mu_2$ iff, for all $b \in B$ and all $c \in C$, there exists a nonnegative number $t_{b,c}$ such that

$$\sum_{c\in C} t_{b,c} = r_b, \qquad \sum_{b\in B} t_{b,c} < s_c$$

and $t_{b,c} \neq 0$ implies $b \ll c$.

Proof. The "only if" part is shown in [16, p. 87]. For the "if" part, assume the above condition holds for μ_1 and μ_2. Let $A \subseteq B$, then

$$\begin{aligned}\sum_{b\in A} r_b &= \sum_{b\in A}\sum_{b\ll c} t_{b,c}\\ &\leqslant \sum_{c\in C\cap\uparrow A}\sum_{b\ll c} t_{b,c}\\ &< \sum_{c\in C\cap\uparrow A} s_c\\ &= \mu_2(\uparrow A).\end{aligned}$$

It follows, by Proposition 2.4, that $\mu_1 \ll \mu_2$. □

The following important result was established in [16, p. 94–98] and appears in [17].

Theorem 2.6. *If Y is an (ω)-continuous dcpo then PY is also (ω)-continuous and has a basis consisting of simple valuations.*

2.3. Extending valuations to measures

We also need some results about the extensions of continuous valuations to Borel measures. Recall that a Borel measure μ on a locally compact Hausdorff space is *regular* if for all Borel subsets B of X, we have

$$\mu(B) = \inf\{\mu(O) \mid B \subseteq O,\ O \text{ open}\} = \sup\{\mu(K) \mid B \supseteq K,\ K \text{ compact}\}.$$

Any bounded measure on a σ-compact and locally compact Hausdorff space is regular [23, p. 50]. In particular any bounded Borel measure on a second countable locally compact Hausdorff space is regular [20, p. 344]. Furthermore, we have

Proposition 2.7 [9]. *On a locally compact second countable Hausdorff space, bounded Borel measures and continuous valuations coincide.*

We also note the following result of J. Lawson. First, recall that the lattice ΩY of open sets of a locally quasi-compact sober space Y is a continuous distributive lattice and Y is in fact isomorphic with the spectrum Spec(ΩY), consisting of nonunit prime elements of ΩY with the hull-kernel topology. The Lawson topology on ΩY induces a topology on Spec(ΩY) and hence on Y which is finer than the original topology of Y. (See [13, p. 252].)

Proposition 2.8 [21, p. 221]. *Any continuous valuation on a second countable locally quasi-compact sober space Y extends uniquely to a regular Borel measure on Y equipped with the relative Lawson topology induced from ΩY.*

For an ω-continuous bounded complete dcpo Y, the relative Lawson topology induced from ΩY coincides with the Lawson topology on Y which is compact and Hausdorff [1, Exercise 7.3.19.8]. We then obtain the following.

Corollary 2.9. *Any continuous valuation on an ω-continuous bounded complete dcpo Y extends uniquely to a regular measure on Y equipped with its compact Lawson topology.*

For ω-continuous dcpo's with bottom, which we will only be concerned with in the next sections, we can give a more direct extension result using a lemma by Saheb-Djahromi as follows.

Lemma 2.10 [24, p. 24]. *The lub of any ω-chain $\langle\mu_i\rangle_{i\geqslant 0}$ of simple valuations μ_i on a dcpo Y with $\mu_i(Y)=1$ extends uniquely to a Borel measure on Y.*

Proposition 2.11. *Any continuous valuation μ on an ω-continuous dcpo Y with bottom extends uniquely to a Borel measure on Y.*

Proof. If $\mu(Y)=0$, then the result is trivial. Otherwise, we can assume without loss of generality, i.e. by a rescaling, that $\mu(Y)=1$. By Theorem 2.6, there exists an ω-chain $\langle\mu_i\rangle_{i\geqslant 0}$ of simple valuations with $\bigsqcup_i \mu_i=\mu$. For each $i\geqslant 0$, let $\mu_i^+=\mu+(1-\mu(Y))\eta_\perp$. Then, it is easy to check that $\langle\mu_i^+\rangle_{i\geqslant 0}$ is an ω-chain of simple valuations with $\mu_i(Y)=1$ and with lub μ. It follows by Lemma 2.10 that μ extends uniquely to a Borel measure on Y.

2.4. Measure theory via domain theory

In [9], a suitable computational framework for measure theory on a locally compact Hausdorff space X has been established using the probabilistic power domain of the upper space of X. We recall the main results here. Since UX is ω-continuous so is therefore PUX.

Proposition 2.12 [9, Proposition 5.8]. *For any open set $a\in\Omega X$, the singleton map $s: X\to UX$ induces a G_δ subset $s[a]\subseteq UX$.*

Corollary 2.13. *Any Borel subset $B \subseteq X$ induces a Borel subset $s[B] \subseteq UX$.*

For $\mu \in PUX$, let μ^* be its unique extension to a Borel measure on UX given by Proposition 2.11 above. Let $S(X) \subseteq PUX$ denote the set of valuations which are *supported* on the Borel set $s[X]$ of maximal elements of UX, i.e. $S(X) = \{\mu \in PUX \mid \mu^*(UX - s[X]) = 0\}$. We have $\mu \in S(X)$ iff $\mu(\Box a) = \mu^*(s[a])$ for all $a \in \Omega X$. Furthermore, $S(X)$ will be a sub-dcpo of PUX. Let $M(X)$ be the set of Borel measures μ on X which are bounded by one ($\mu(X) \leqslant 1$). Define a partial order on $M(X)$ by $\mu \sqsubseteq \nu$ iff $\mu(O) \leqslant \nu(O)$ for all open sets $O \in \Omega X$. Then $M(X)$ will also be a dcpo. Let $M^1(X) \subseteq M(X)$ be the subset of normalised measures ($\mu(X) = 1$). Similarly, define $P^1UX \subseteq PUX$ and $S^1(X) \subseteq S(X)$.

Proposition 2.14 [9, Proposition 5.17]. *If $\mu \in S^1(X)$ then μ is a maximal element of PUX.*

The main result is the following.

Theorem 2.15 [9, Theorem 5.20]. *The dcpo's $M(X)$ and $S(X)$ are isomorphic via the maps $e: M(X) \to S(X)$ with $e(\mu) = \mu \circ s^{-1}$ and $j: S(X) \to M(X)$ with $j(\nu) = \nu^* \circ s$. Moreover, these maps restrict to give an isomorphism between $M^1(X)$ and $S^1(X)$.*

We can therefore identify $M(X)$ with $S(X) \subseteq PUX$. But PUX has a basis consisting of simple valuations which can be used to provide it with an effective structure. This therefore gives us a constructive framework for bounded Borel measures on X.

Important note: For convenience, we often identify μ with $e(\mu)$ and write μ instead of $e(\mu)$. Therefore, depending on the context, μ can either be a Borel measure on X or a valuation on UX which is supported on $s[X]$. We will also write the unique extension μ^* simply as μ.

Example 2.16. Let $X = [0,1]$ be the unit interval with the Lebesgue measure λ. Each partition

$$q: \; 0 = x_0 < x_1 < \cdots < x_{j-1} < x_j < x_{j+1} < \cdots < x_{N-1} < x_N = 1,$$

of $[0,1]$ gives rise to a simple valuation

$$\mu_q = \sum_{j=1}^{N} r_j \eta_{b_j},$$

where $b_j = [x_{j-1}, x_j]$ and $r_j = x_j - x_{j-1}$,.

Now consider the ω-chain $\langle \mu_{q_i} \rangle_{i \geqslant 0}$ of simple valuations which are obtained by the sequence of partitions $\langle q_i \rangle_{i \geqslant 0}$, where q_i consists of dyadic numbers $x_{ij} = j/2^i$

for $j = 0, 1, 2, 3, \ldots, 2^i$. In more detail,

$$\mu_{q_i} = \sum_{j=1}^{2^i} \frac{1}{2^i} \eta_{b_j} \quad \text{with } b_j = \left[\frac{j-1}{2^i}, \frac{j}{2^i}\right].$$

For any open interval $a \subseteq [0,1]$, the number $\mu_{q_i}(\Box a)$ is the largest distance between dyadic numbers x_{ij} which are contained in a. For an arbitrary open set, the contributions from individual connected components (intervals) add up. It is easy to see that $\mu_{q_i} \sqsubseteq \lambda$ for all $i \geqslant 0$. Furthermore, we have

Proposition 2.17. *The valuation $\mu = \bigsqcup_{i\geqslant 0} \mu_{q_i}$ is supported in $s[X]$ and $j(\mu)$ is the Lebesgue measure λ on* $[0,1]$.

Proof. Let $\langle a_k \rangle_{k \in J_n}$, $n \geqslant 1$, be the collection of all open balls in $[0,1]$ with radius at most $1/n$, and put

$$O_n = \bigcup_{k \in J_n} \Box a_k.$$

Then, $\langle O_n \rangle_{n\geqslant 1}$ is a decreasing sequence of open sets in $U[0,1]$ and $s[[0,1]] = \bigcap_{n\geqslant 1} O_n$. But for each $n \geqslant 1$, $\mu_{q_i}(O_n) = 1$ if $1/2^i < 1/n$, i.e. if i is large enough. Hence, $\mu(O_n) = \sup_i \mu_{q_i}(O_n) = 1$ for all $n \geqslant 1$. Therefore,

$$\mu(s[[0,1]]) = \inf_{n\geqslant 1} \mu(O_n) = 1,$$

showing that $\mu \in S^1([0,1])$. To show that $j(\mu)$ is the Lebesgue measure, it is sufficient by Proposition 2.7 to check that they have the same value on open sets. Since any open set in $[0,1]$ is the countable union of disjoint open (or half-open half-closed at 0 or 1) intervals, it suffices to check this on such intervals. But since the dyadic numbers are dense in $[0,1]$, it is easy to see, for example, that

$$\mu(\Box(x,y)) = \sup_i \mu_{q_i}(\Box(x,y)) = y - x = \lambda((x,y)). \qquad \Box$$

3. The normalised probabilistic power domain

In this section, we consider the subset of *normalised* valuations of the probabilistic power domain and extend and sharpen the results in Section 2.2 for this subset.

For any topological space Y, let $P^1 Y \subseteq PY$ be the set of continuous valuations μ on Y which are normalised, i.e. $\mu(Y) = 1$. Note that if Y has bottom $\bot$, then $P^1 Y$ has bottom $\eta_\bot$. Let $\mathrm{DCPO}_\bot$ denote the category of dcpo's with bottom and continuous maps. Define P^1 on morphisms as for P, i.e. for $f : Y \to Z$, put $Pf(\mu) = \mu \circ f^{-1}$. Then

$$P^1 : \mathrm{DCPO}_\bot \to \mathrm{DCPO}_\bot$$

is, like P, a locally continuous functor, which we call the *normalised probabilistic power domain* functor.

We will now extend the results of Section 2.2 to the normalised power domain. *In the rest of this paper, we denote the way-below relations in PY and P^1Y by $\ll$ and $\ll^1$ respectively.* Let Y be a dcpo with bottom. We get the analogue of Proposition 2.3.

Proposition 3.1. *For two simple valuations*

$$\mu_1 = \sum_{b\in B} r_b\eta_b, \qquad \mu_2 = \sum_{c\in C} s_c\eta_c$$

in P^1Y, we have: $\mu_1 \sqsubseteq \mu_2$ iff, for all $b \in B$ and all $c \in C$, there exists a nonnegative number $t_{b,c}$ such that

$$\sum_{c\in C} t_{b,c} = r_b, \qquad \sum_{b\in B} t_{b,c} = s_c$$

and $t_{b,c} \neq 0$ implies $b \sqsubseteq c$.

Proof. The "if" part follows from Proposition 2.3. For the "only if" part, we know from the "only if" part of that proposition that there exist nonnegative numbers $t_{b,c}$ such that

$$\sum_{c\in C} t_{b,c} = r_b, \qquad \sum_{b\in B} t_{b,c} \leqslant s_c$$

and $t_{b,c} \neq 0$ implies $b \sqsubseteq c$. If for any $c \in C$, we have $\sum_b t_{b,c} < s_c$, then we obtain a contradiction, since

$$1 = \sum_b r_b = \sum_b \sum_c t_{b,c} = \sum_c \sum_b t_{b,c} < \sum_c s_c = 1.$$

It follows that $\sum_b t_{b,c} = s_c$ for all $c \in C$. $\square$

Define the maps

$$\begin{array}{rcl} m^+ : PY & \to & PY \\ \mu & \mapsto & \mu^+ \end{array} \qquad \begin{array}{rcl} m^- : PY & \to & PY \\ \mu & \mapsto & \mu^- \end{array}$$

where

$$\mu^+(O) = \begin{cases} \mu(O) & \text{if } O \neq Y, \\ 1 & \text{otherwise} \end{cases}$$

and

$$\mu^-(O) = \begin{cases} \mu(O) & \text{if } O \neq Y, \\ \mu(Y - \{\bot\}) & \text{otherwise.} \end{cases}$$

Note that $Y - \{\bot\}$ is an open set, and hence μ^- is well defined. The map m^+ puts the missing mass of μ on the bottom $\bot$ of Y to produce a normalised valuation μ^+.

The map m^- removes any mass that may exist on $\perp$. The following properties are easy consequences of the definitions; the proofs are omitted.

Propositon 3.2. (i) *The maps m^+ and m^- are well-defined, continuous and satisfy*:

$$\begin{array}{ll} m^+ \circ m^+ = m^+ \sqsupseteq 1 & m^- \circ m^- = m^- \sqsubseteq 1 \\ m^- \circ m^+ = m^- & m^+ \circ m^- = m^+ \end{array}$$

where 1 *is the identity map on PY.*

(ii) $\nu \ll \mu \;\&\; \mu \in P^1Y \;\Rightarrow\; \nu^+ \ll^1 \mu$.

(iii) $\nu \ll^1 \mu \;\Rightarrow\; \nu^- \ll \mu^- \sqsubseteq \mu$.

Corollary 3.3. *If Y is an (ω)-continuous dcpo with bottom, then P^1Y is also an (ω)-continuous dcpo with a basis of normalised simple valuations.*

Proof. Note that P^1Y is the image of the continuous idempotent function $m^+ : PY \to PY$. But the image of any continuous idempotent function (i.e. retract) on an (ω)-continuous dcpo is another (ω)-continuous dcpo [1, Theorem 3.14]. □

To prove an analogue of Proposition 2.5 for P^1Y, we need a technical lemma. Assume in the rest of this section that Y is a continuous dcpo with bottom.

Lemma 3.4. *Suppose $\mu, \nu \in P^1Y$. Then $\nu \ll^1 \mu$ implies $\nu(Y - \{\perp\}) < 1$.*

Proof. Assume $\nu \ll^1 \mu$. For $n \geqslant 1$, let $\mu_n = (1/n)\eta_\perp + (1 - 1/n)\mu$. We have $\mu_n(Y) = 1$ and $\mu_n(O) = (1 - 1/n)\mu(O)$ for any $n \geqslant 1$ and any open set $O \neq Y$. It follows that $\langle \mu_n \rangle_{n \geqslant 1}$ is an increasing chain with lub μ. Therefore, $\nu \sqsubseteq \mu_n$ for some $n \geqslant 1$. We conclude that

$$\nu(Y - \{\perp\}) \leqslant \mu_n(Y - \{\perp\}) = (1 - 1/n)\mu(Y - \{\perp\}) \leqslant 1 - 1/n < 1$$

as required. □

Proposition 3.5. *For two simple valuations*

$$\mu_1 = \sum_{b \in B} r_b \eta_b, \qquad \mu_2 = \sum_{c \in C} s_c \eta_c$$

in P^1Y, we have $\mu_1 \ll^1 \mu_2$ iff $\perp \in B$ with $r_\perp \neq 0$, and, for all $b \in B$ and all $c \in C$, there exists a nonnegative number $t_{b,c}$ with $t_{\perp,c} \neq 0$ such that

$$\sum_{c \in C} t_{b,c} = r_b, \qquad \sum_{b \in B} t_{b,c} = s_c$$

and $t_{b,c} \neq 0$ implies $b \ll c$.

Proof. For the "if" part, note that the conditions above imply, by Proposition 2.5, that

$$\mu_1^- = \sum_{b \in B - \{\perp\}} r_b \eta_b \ll \sum_{c \in C} s_c \eta_c = \mu_2.$$

Now Proposition 3.2(ii) implies $\mu_1 = (\mu_1^-)^+ \ll^1 \mu_2$. For the "only if" part, first note that by Lemma 3.4 we must have $\perp \in B$ with $r_\perp \neq 0$. Therefore by Proposition 3.2(iii)

$$\mu_1^- = \sum_{b \in B - \{\perp\}} r_b \eta_b \ll \sum_{c \in C} s_c \eta_c = \mu_2.$$

Hence, by Proposition 2.5, there exists, for each $b \in B - \{\perp\}$ and $c \in C$, a nonnegative $t_{b,c}$ with

$$r_b = \sum_{c \in C} t_{b,c} \quad (b \neq \perp), \qquad s_c > \sum_{b \neq \perp} t_{b,c}$$

such that $t_{b,c} \neq 0 \Rightarrow b \ll c$. Put

$$t_{\perp,c} = s_c - \sum_{b \neq \perp} t_{b,c}.$$

Then, it is easy to check that

$$\sum_{c \in C} t_{\perp,c} = \sum_{c \in C} (s_c - \sum_{b \neq \perp} t_{b,c}) = 1 - \sum_{b \neq \perp, c \in C} t_{b,c} = 1 - \sum_{b \neq \perp} r_b = r_\perp.$$

Furthermore we clearly have

$$\sum_{b \in B} t_{b,c} = t_{\perp,c} + \sum_{b \neq \perp} t_{b,c} = s_c$$

as required. □

4. The generalised Riemann integral

We will now use the results of the previous sections to define the generalised Riemann integral. In this section and in the rest of the paper, let $f : X \to \mathbb{R}$ be a bounded real-valued function on a compact metric space (X, d) and let μ be a bounded Borel measure on X. Let $m = \inf f[X]$ and $M = \sup f[X]$. Without loss of generality, i.e. by a rescaling, we can assume that μ is normalised. By Theorem 2.15, μ corresponds to a unique valuation $e(\mu) = \mu \circ s^{-1} \in S^1(X) \subseteq P^1 UX$, which is supported in $s[X]$ and is, by Proposition 2.14, a maximal element of $P^1 UX$. Recall that we write μ instead of $e(\mu)$.

4.1. The Lower and upper R-integrals

We will define the generalised Riemann integral by using *generalised Darboux sums* as follows.

Definition 4.1. For any simple valuation $\nu = \sum_{b \in B} r_b \eta_b \in PUX$, the *lower sum of f with respect to ν* is

$$S_X^\ell(f, \nu) = \sum_{b \in B} r_b \inf f[b].$$

Similarly, the *upper sum of f with respect to ν* is

$$S_X^{\mathrm{u}}(f,\nu) = \sum_{b\in B} r_b \sup f[b].$$

Note that, since f is bounded, the lower sum and the upper sum are well-defined real numbers. When it is clear from the context, we drop the subscript X and simply write $S^{\ell}(f,\nu)$ and $S^{\mathrm{u}}(f,\nu)$. Clearly, we always have $S^{\ell}(f,\nu)\leqslant S^{\mathrm{u}}(f,\nu)$.

Propositon 4.2. *Let $\mu_1,\mu_2 \in P^1UX$ be simple valuations with $\mu_1 \sqsubseteq \mu_2$, then*

$$S^{\ell}(f,\mu_1)\sqsubseteq S^{\ell}(f,\mu_2) \quad \text{and} \quad S^{\mathrm{u}}(f,\mu_2)\sqsubseteq S^{\mathrm{u}}(f,\mu_1).$$

Proof. Assume

$$\mu_1 = \sum_{b\in B} r_b\eta_b \quad \text{and} \quad \mu_2 = \sum_{c\in C} s_c\eta_c.$$

Let $t_{b,c}$ be the nonnegative numbers given by Proposition 3.1. Then,

$$\begin{aligned} S^{\ell}(f,\mu_1) &= \sum_b r_b \inf f[b] = \sum_b\sum_c t_{b,c} \inf f[b] \\ &\leqslant \sum_b\sum_c t_{b,c}\inf f[c] = \sum_c\sum_b t_{b,c} \inf f[c] = \sum_c s_c \inf f[c] \\ &= S^{\ell}(f,\mu_2). \end{aligned}$$

$$\begin{aligned} S^{\mathrm{u}}(f,\mu_1) &= \sum_b r_b \sup f[b] = \sum_b\sum_c t_{b,c} \sup f[b] \\ &\geqslant \sum_b\sum_c t_{b,c}\sup f[c] = \sum_c\sum_b t_{b,c} \sup f[c] = \sum_c s_c \sup f[c] \\ &= S^{\mathrm{u}}(f,\mu_2). \qquad \square \end{aligned}$$

Note that, in the above proof, it is essential that the simple valuations are normalised, i.e. that we work in P^1UX, to deduce that the upper sum decreases. The latter would not hold in general for simple valuations in PUX.

Corollary 4.3. *If $\mu_1,\mu_2 \in P^1UX$ are simple valuations with $\mu_1,\mu_2 \ll^1 \mu$, then $S^{\ell}(f,\mu_1)\leqslant S^{\mathrm{u}}(f,\mu_2)$.*

Proof. Since the set of normalised simple valuations way-below μ in P^1UX is directed, there exists a normalised simple valuation $\mu_3 \in P^1UX$ such that $\mu_1,\mu_2 \sqsubseteq \mu_3 \ll^1 \mu$. By Proposition 4.2, we therefore have

$$S^{\ell}(f,\mu_1)\leqslant S^{\ell}(f,\mu_3)\leqslant S^{\mathrm{u}}(f,\mu_3)\leqslant S^{\mathrm{u}}(f,\mu_2). \qquad \square$$

Therefore, if we consider the directed set of simple valuations way-below μ in P^1UX, then every lower sum is bounded by every upper sum. This is similar to the situation which arises for Darboux sums in Riemann integration.

Definition 4.4. The *lower R-integral* of f with respect to μ on X is

$$\mathbf{R}\underline{\int}_X f\,\mathrm{d}\mu = \sup_{v \ll^1 \mu} S_X^{\ell}(f,v).$$

Similarly, the *upper R-integral* of f with respect to μ on X is

$$\mathbf{R}\overline{\int}_X f\,\mathrm{d}\mu = \inf_{v \ll^1 \mu} S_X^{\mathrm{u}}(f,v).$$

Clearly, $\mathbf{R}\underline{\int}_X f\,\mathrm{d}\mu \leqslant \mathbf{R}\overline{\int}_X f\,\mathrm{d}\mu$.

Definition 4.5. We say f is *R-integrable* with respect to μ on X, and write $f \in R_X(\mu)$ if its lower and upper integrals coincide. If f is R-integrable, then the R-integral of f is defined as

$$\mathbf{R}\int_X f\,\mathrm{d}\mu = \mathbf{R}\underline{\int}_X f\,\mathrm{d}\mu = \mathbf{R}\overline{\int}_X f\,\mathrm{d}\mu.$$

When there is no confusion, we simply write $R(\mu)$ instead of $R_X(\mu)$ and $\int f\,\mathrm{d}\mu$ instead of $\mathbf{R}\int_X f\,\mathrm{d}\mu$ (similarly for the lower and upper R-integrals). The following characterisation of R-integrability, similar to the Lebesgue condition for the ordinary Riemann integral, is an immediate consequence of the definition.

Proposition 4.6 (The R-condition). *We have $f \in R(\mu)$ iff for all $\varepsilon > 0$ there exists a simple valuation $v \in P^1 UX$ with $v \ll^1 \mu$ such that*

$$S^{\mathrm{u}}(f,v) - S^{\ell}(f,v) < \varepsilon.$$

There is also an equivalent characterisation of the R-integral in terms of *generalised Riemann sums* with its well-known parallel in ordinary Riemann integration.

Definition 4.7. For a simple valuation $v = \sum_{b\in B} r_b \eta_b \in PUX$ and for a choice of $\xi_b \in b$ for each $b \in B$, the sum $\sum_{b\in B} r_b f(\xi_b)$ is called a *generalised Riemann sum* for f with respect to v and is denoted by $S_\xi(f,v)$.

Note that we always have

$$S^{\ell}(f,v) \leqslant S_\xi(f,v) \leqslant S^{\mathrm{u}}(f,v)$$

for any generalised Riemann sum $S_\xi(f,v)$. We therefore easily obtain:

Proposition 4.8. *We have $f \in R(\mu)$ with R-integral value K iff for all $\varepsilon > 0$ there exists a simple valuation*

$$v = \sum_{b\in B} r_b \eta_b \in P^1 UX$$

with $\nu \ll^1 \mu$ *such that*

$$|K - S_\xi(f,\nu)| < \varepsilon$$

for all generalised Riemann sums $S_\xi(f,\nu)$ *of* f *with respect to* ν.

Having defined the notion of R-integrability with respect to simple valuations way-below μ in P^1UX, we can now deduce the following results.

Proposition 4.9. *If* f *is R-integrable and* $\mu = \bigsqcup_{i\geqslant 0}\mu_i$, *where* $\langle\mu_i\rangle_{i\geqslant 0}$ *is an* ω*-chain in* PUX, *then*

$$\int f\,\mathrm{d}\mu = \lim_{i\to\infty} S^\ell(f,\mu_i) = \lim_{i\to\infty} S^{\mathrm{u}}(f,\mu_i) = \lim_{i\to\infty} S_{\xi_i}(f,\mu_i),$$

where $S_{\xi_i}(f,\mu_i)$ *is any generalised Riemann sum of* f *with respect to* μ_i.

Proof. Let $\varepsilon > 0$ be given. Let the simple valuation $\nu \in P^1UX$ with $\nu \ll^1 \mu$ be such that $S^{\mathrm{u}}(f,\nu) - S^\ell(f,\nu) < \varepsilon$. Since $\mu = \bigsqcup_i \mu_i^+$, there exists $N\geqslant 0$ such that $\nu \sqsubseteq \mu_i^+$ and $\mu_i(UX) > 1-\varepsilon$ for all $i\geqslant N$. Therefore, for all $i\geqslant N$, $\mu_i^+ = \mu_i + r_i\eta_X$ with $r_i = 1-\mu_i(UX) < \varepsilon$. For $i\geqslant N$, we therefore have

$$S^\ell(f,\mu_i^+) - S^\ell(f,\mu_i) = r_i \inf f[X] = r_i m,$$

$$S^{\mathrm{u}}(f,\mu_i^+) - S^{\mathrm{u}}(f,\mu_i) = r_i \sup f[X] = r_i M$$

and also the inequalities

$$S^\ell(f,\nu)\leqslant S^\ell(f,\mu_i^+)\leqslant S^{\mathrm{u}}(f,\mu_i^+)\leqslant S^{\mathrm{u}}(f,\nu),$$

$$S^\ell(f,\nu)\leqslant \int f\,\mathrm{d}\mu \leqslant S^{\mathrm{u}}(f,\nu).$$

It follows that $|S^\ell(f,\mu_i^+) - \int f\,\mathrm{d}\mu| < \varepsilon$ and $|S^{\mathrm{u}}(f,\mu_i^+) - \int f\,\mathrm{d}\mu| < \varepsilon$ for $i\geqslant N$. We conclude that for $i\geqslant N$,

$$|S^\ell(f,\mu_i) - \int f\,\mathrm{d}\mu| < (1+|m|)\varepsilon,$$

$$|S^{\mathrm{u}}(f,\mu_i) - \int f\,\mathrm{d}\mu| < (1+|M|)\varepsilon,$$

and the result follows. □

Corollary 4.10. *If* f *is R-integrable and* $\mu = \bigsqcup_{i\geqslant 0}\mu_i$, *where* $\langle\mu_i\rangle_{i\geqslant 0}$ *is an* ω*-chain in* P^1UX, *then* $S^\ell(f,\mu_i)$ *increases to* $\int f\,\mathrm{d}\mu$ *and* $S^{\mathrm{u}}(f,\mu_i)$ *decreases to* $\int f\,\mathrm{d}\mu$.

4.2. Elementary properties of the R-integral

We now show some simple properties of the R-integral.

Proposition 4.11. (i) *If* $f, g \in R(\mu)$ *then* $f + g \in R(\mu)$ *and* $\int (f+g)\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu + \int g\,\mathrm{d}\mu$.

(ii) *If* $f \in R(\mu)$ *and* $c \in \mathbb{R}$, *then* $cf \in R(\mu)$ *and* $\int cf\,\mathrm{d}\mu = c\int f\,\mathrm{d}\mu$.

(iii) *More generally, if* $f, g \in R(\mu)$ *so is their product* $h : X \to \mathbb{R}$ *with* $h(x) = f(x)g(x)$.

Proof. We will only prove (i). For any nonempty compact subset $b \subseteq X$ we have

$$\sup(f+g)[b] \leqslant \sup f[b] + \sup g[b],$$

$$\inf(f+g)[b] \geqslant \inf f[b] + \inf g[b].$$

Hence, for any simple valuation $\nu \ll^1 \mu$, we have

$$S^{\mathrm{u}}(f+g,\nu) \leqslant S^{\mathrm{u}}(f,\nu) + S^{\mathrm{u}}(g,\nu),$$

$$S^{\ell}(f+g,\nu) \geqslant S^{\ell}(f,\nu) + S^{\ell}(g,\nu).$$

Let $\varepsilon > 0$ be given. There exist simple valuations $\nu_1, \nu_2 \ll^1 \mu$ with

$$S^{\mathrm{u}}(f,\nu_1) < \int f\,\mathrm{d}\mu + \varepsilon/2, \qquad S^{\mathrm{u}}(g,\nu_2) < \int g\,\mathrm{d}\mu + \varepsilon/2.$$

Let the simple valuation ν be such that $\nu_1, \nu_2 \sqsubseteq \nu \ll^1 \mu$. Then $S^{\mathrm{u}}(f,\nu) \leqslant S^{\mathrm{u}}(f,\nu_1)$ and $S^{\mathrm{u}}(g,\nu) \leqslant S^{\mathrm{u}}(g,\nu_2)$, and we have

$$\begin{aligned} S^{\mathrm{u}}(f+g,\nu) &\leqslant S^{\mathrm{u}}(f,\nu) + S^{\mathrm{u}}(g,\nu) \leqslant S^{\mathrm{u}}(f,\nu_1) + S^{\mathrm{u}}(g,\nu_2) \\ &\leqslant \int f\,\mathrm{d}\mu + \int g\,\mathrm{d}\mu + \varepsilon. \end{aligned}$$

Therefore, $\bar{\int}(f+g)\,\mathrm{d}\mu \leqslant \int f\,\mathrm{d}\mu + \int g\,\mathrm{d}\mu$. Similarly, $\int f\,\mathrm{d}\mu + \int g\,\mathrm{d}\mu \leqslant \underline{\int}(f+g)\,\mathrm{d}\mu$, and the result follows. $\square$

Given the function $f : X \to \mathbb{R}$ as before, define two functions $f^+, f^- : X \to \mathbb{R}$ by

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geqslant 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f^-(x) = \begin{cases} -f(x) & \text{if } f(x) \leqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.12. *If* $f \in R(\mu)$, *then* $f^+, f^- \in R(\mu)$ *and* $\int f\,\mathrm{d}\mu = \int f^+\,\mathrm{d}\mu - \int f^-\,\mathrm{d}\mu$.

Proof. For any nonempty compact set $b \subseteq X$, we have

$$\sup f^+[b] - \inf f^+[b] \leqslant \sup f[b] - \inf f[b]$$

and therefore

$$S^{\mathrm{u}}(f^+,\mu) - S^{\ell}(f^+,\mu) \leqslant S^{\mathrm{u}}(f,\mu) - S^{\ell}(f,\mu).$$

By the R-condition (Proposition 4.6), $f^+ \in R(\mu)$. Similarly, $f^- \in R(\mu)$. Since $f = f^+ - f^-$, by Proposition 4.11, we get $\int f\,\mathrm{d}\mu = \int f^+\,\mathrm{d}\mu - \int f^-\,\mathrm{d}\mu$. $\square$

The following properties are easily shown.

Proposition 4.13. (i) *If f is nonnegative and $f \in R(\mu)$ then $\int f \,\mathrm{d}\mu \geqslant 0$.*
(ii) *$f \in R(\mu) \Rightarrow |f| \in R(\mu)$ and*

$$\left|\int f \,\mathrm{d}\mu\right| \leqslant \int |f| \,\mathrm{d}\mu. \quad \square$$

We will make frequent use of the following result in the next sections. As before, let μ be a normalised Borel measure on the compact metric space X.

Proposition 4.14. *Let $\langle \mu_i \rangle_{i \in I}$ be a directed set of simple valuations*

$$\mu_i = \sum_{b \in B_i} r_{i,b} \eta_b$$

in $P^1 UX$ with lub μ. Then for all $\varepsilon > 0$ and all $\delta > 0$, there exists $i \in I$ with

$$\sum_{b \in B_i, |b| \geqslant \delta} r_{i,b} < \varepsilon,$$

where $|b|$ is the diameter of the compact set $b \subseteq X$.

Proof. Let $\varepsilon > 0$ and $\delta > 0$ be given. Let $\langle a_k \rangle_{k \in J_n}$, $n \geqslant 1$, be the collection of all open balls in X with radius at most $1/n$, and put

$$O_n = \bigcup_{k \in J_n} \Box a_k.$$

Then, for all $n \geqslant 1$, we have $s[X] \subseteq s[O_n]$ and hence $\mu(O_n) = \mu(s[X]) = 1$, since μ is supported in $s[X]$. Now choose $n > 2/\delta$ so that $1/n < \delta/2$. Since $\sup_{i \in I} \mu_i(O_n) = \mu(O_n) = 1$, there exists $i \in I$ with $\mu_i(O_n) > 1 - \varepsilon$. It follows that

$$\sum_{b \in B_i, |b| \geqslant \delta} r_{i,b} < \varepsilon$$

as required. $\square$

5. R-integration and Riemann integration

In this section, we show that Riemann integration is equivalent to R-integration on compact real intervals with respect to the Lebesgue measure.

Let $X = [0, 1]$ be the unit real interval with the Lebesgue measure λ. Recall from Example 2.16 that any partition

$$q: \quad 0 = x_0 < x_1 < \cdots < x_{j-1} < x_j < x_{j+1} < \cdots < x_{N-1} < x_N = 1$$

of $[0, 1]$ gives rise to a simple valuation

$$\mu_q = \sum_{j=1}^{N} r_j \eta_{b_j},$$

where $b_j = [x_{j-1}, x_j]$ and $r_j = x_j - x_{j-1}$. Given any bounded function $f : [0,1] \to \mathbb{R}$, the lower and upper Darboux sums of f with respect to the partition q in the ordinary Riemann integration of f are precisely the lower and upper sums $S^{\ell}(f,\lambda)$ and $S^{\mathrm{u}}(f,\lambda)$ of R-integration. We will of course use this fact to show that f is Riemann integrable iff $f \in R(\lambda)$, and that when the two integrals exist, then they are equal, i.e.

$$\mathbf{R}\int_{[0,1]} f \, \mathrm{d}\lambda = \int_0^1 f(x)\,\mathrm{d}x.$$

However, it can be easily shown that μ_q is *not* way-below λ if $N > 1$. In fact, for $i \geqslant 1$, let $v_i = (1/i)\eta_{[0,1]} + (1 - 1/i)\mu_{q_i}$, where $\langle \mu_{q_i} \rangle_{i \geqslant 1}$ is the ω-chain given in Proposition 2.17. Then, we have $\bigsqcup_{i \geqslant 1} v_i = \lambda$, but there is no $i \geqslant 1$ with $\mu_q \sqsubseteq v_i$, showing that μ_q is not way-below λ. But, the following lemma shows that it is possible to obtain a valuation close to μ_q which *is* way-below λ.

Lemma 5.1. *Let q be any partition of* $[0,1]$ *inducing the simple valuation μ_q as above, and let* $0 < \varepsilon < 1$. *Then,* $v = \varepsilon\eta_{[0,1]} + (1-\varepsilon)\mu_q \ll^1 \lambda$.

Proof. By Proposition 3.2(ii), it suffices to prove that $(1-\varepsilon)\mu_q \ll \lambda$. Choose a real number δ such that $\varepsilon > \delta > 0$. For each $j = 1,\ldots,N$, let $b_j' = [x_{j-1} + \frac{1}{2}\delta r_j, x_j - \frac{1}{2}\delta r_j]$. Since $b_j' \subseteq (x_{j-1}, x_j) \subseteq b_j$, it follows that $b_j \ll b_j'$ holds in UX. Let $\mu' = (1-\delta)\sum_{j=1}^{N} r_j \eta_{b_j'}$. By Proposition 2.5, with $t_{jj} = (1-\varepsilon)r_j$ and $t_{jj'} = 0$ for $j \neq j'$, we have $(1-\varepsilon)\mu_q \ll \mu'$. Since the length of b_j' is $(1-\delta)r_j$ and the intervals b_j', $j = 1,\ldots,N$, are disjoint, it easily follows that $\mu' \sqsubseteq \lambda$. Thus, $(1-\varepsilon)\mu_q \ll \mu' \sqsubseteq \lambda$ as required. □

Theorem 5.2. *A bounded real-valued function on a compact real interval is Riemann integrable iff it is R-integrable with respect to the Lebesgue measure. Furthermore, the two integrals are equal when they exist.*

Proof. Assume without loss of generality that the compact real interval is in fact the unit interval. For the "if" part, assume $f : [0,1] \to \mathbb{R}$ is R-integrable. Let $\varepsilon > 0$ be given. By the R-condition, there exists $v \ll^1 \lambda$ with $S^{\mathrm{u}}(f,v) - S^{\ell}(f,v) < \varepsilon$. Now take, for example, the ω-chain $\langle \mu_{q_i} \rangle_{i \geqslant 0}$ of simple valuations of Proposition 2.17 whose lub is λ. Since $\mu_{q_i} \in P^1U[0,1]$ for all $i \geqslant 0$, there is $i \geqslant 0$ with $v \sqsubseteq \mu_{q_i}$. Hence

$$S^{\mathrm{u}}(f,\mu_{q_i}) - S^{\ell}(f,\mu_{q_i}) \leqslant S^{\mathrm{u}}(f,v) - S^{\ell}(f,v) < \varepsilon$$

and therefore the Riemann condition is satisfied and f is Riemann integrable. Since, by Corollary 4.10, the R-integral is the supremum of $S^{\ell}(f,\mu_{q_i})$ and the latter is also the Riemann integral of f, we conclude that the two integrals are equal when they exist. For the "only if " part, assume f is Riemann integrable and let $q : 0 = x_0 < x_1 < \cdots < x_N = 1$ be a partition of $[0,1]$ with $S^{\mathrm{u}}(f,\mu_q) - S^{\ell}(f,\mu_q) < \varepsilon$. By Lemma 5.1,

$\nu = \varepsilon\eta_{[0,1]} + (1-\varepsilon)\mu_q$ is way-below λ. We also have

$$\begin{aligned} S^{\mathrm{u}}(f,\nu) - S^{\ell}(f,\nu) &\leqslant (M-m)\varepsilon + (1-\varepsilon)\varepsilon \\ &< (M-m+1)\varepsilon. \end{aligned}$$

It follows that f satisfies the R-condition, and is therefore R-integrable. □

We conclude that ordinary Riemann integration is a particular instance of R-integration.

6. Further properties of R-integration

In this section, we will show that all the basic results for ordinary Riemann integration on a compact interval in $\mathbb{R}$ can be extended to R-integration. Assume as before that μ is a normalised Borel measure on the compact metric space X. We first show that continuous functions are R-integrable.

Theorem 6.1. *Any continuous function $f: X \to \mathbb{R}$ is R-integrable with respect to μ.*

Proof. Let $\varepsilon > 0$ be given. By the uniform continuity of f on the compact set X, there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $|f(x) - f(y)| < \varepsilon/2$. By Proposition 4.14, there exists a simple valuation $\nu = \sum_{b \in B} r_b \eta_b$ with $\nu \ll^1 \mu$ such that

$$\sum_{|b| \geqslant \delta} r_b \leqslant \frac{\varepsilon}{2(M-m+1)}.$$

Therefore,

$$\begin{aligned} S^{\mathrm{u}}(f,\nu) - S^{\ell}(f,\nu) &= \sum_{b \in B} r_b(\sup f[b] - \inf f[b]) \\ &= \sum_{b \in B, |b| < \delta} r_b(\sup f[b] - \inf f[b]) \\ &\quad + \sum_{b \in B, |b| \geqslant \delta} r_b(\sup f[b] - \inf f[b]) \\ &< \frac{\varepsilon}{2} + \frac{(M-m)\varepsilon}{2(M-m+1)} \\ &< \varepsilon. \end{aligned}$$

It follows by the R-condition that $f \in R(\mu)$. □

Next we will prove that a bounded function is R-integrable with respect to a Borel measure if and only if its set of discontinuities has measure zero. This will generalise the well-known Lebesgue criterion for ordinary Riemann integration of a bounded function on a compact real interval.

We need some definitions and properties relating to the oscillation of a bounded function $f : X \to \mathbb{R}$ which generalise those of a bounded function on a compact real interval as presented, for example, in [2]. For convenience and consistency, we will use the terminology in that work.

Definition 6.2. Let $T \subseteq X$. The number

$$\Omega_f(T) = \sup\{f(x) - f(y) \mid x, y \in T\}$$

is called the *oscillation* of f on T. For $x \in X$, the number

$$\omega_f(x) = \lim_{h \to 0^+} \Omega_f(B(x,h)),$$

where $B(x,h) \subseteq X$ is the open ball of radius $h > 0$ at x, is the *oscillation* of f at x. For each $r > 0$, let

$$D_r = \{x \in X \mid \omega_f(x) \geqslant 1/r\}. \qquad \square$$

The following properties then are straightforward generalisations of those in [2, p.170–171].

Proposition 6.3. (i) *f is continuous at $x \in X$ iff $\omega_f(x) = 0$.*

(ii) *If $\omega_f(x) < \varepsilon$ for all $x \in X$, then there exists $\delta > 0$ such that for all compact subsets $b \subseteq X$ with $|b| < \delta$ we have $\Omega_f(b) < \varepsilon$.*

(iii) *For any $r > 0$ the set D_r is closed.*

If D is the set of discontinuities of f, then using Definition 6.2 and Proposition 6.3(i), we can write $D = \bigcup_{n \geqslant 1} D_n$ where $D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots$ is an increasing chain of closed sets. Hence, D is an F_σ, and therefore a Borel, set. Recall that we assume μ to be a normalised measure on the compact metric space X.

Lemma 6.4. *Let $d \subseteq X$ be compact, and let $v = \sum_{b \in B} r_b \eta_b$ be a simple valuation in $P^1 UX$.*

(i) *If $v \sqsubseteq \mu$ then*

$$\sum_{b \cap d \neq \emptyset} r_b \geqslant \mu(d).$$

(ii) *If $v \ll^1 \mu$, then*

$$\sum_{b^\circ \cap d \neq \emptyset} r_b \geqslant \mu(d),$$

where b° is the interior of b.

Proof. (i) We have

$$\begin{aligned}
\sum_{b\cap d\neq\emptyset} r_b &= 1-\sum_{b\cap d=\emptyset} r_b && \text{since } \nu(UX)=1\\
&= 1-\sum_{b\subseteq X-d} r_b\\
&= 1-\sum_{b\in\Box(X-d)} r_b\\
&= 1-\nu(\Box(X-d))\\
&\geqslant 1-\mu(\Box(X-d)) && \text{since } \nu\sqsubseteq\mu\\
&= 1-(\mu(s[X])-\mu(s[d])) && \text{since } \mu \text{ is supported in } s[X]\\
&= \mu(d).
\end{aligned}$$

(ii) By the interpolative property of $\ll^1$, there exists a normalised simple valuation $\gamma=\sum_{c\in C} s_c\eta_c$ such that $\nu\ll^1\gamma\ll^1\mu$. Let $t_{b,c}$ be given as in Proposition 3.5. Then

$$\begin{aligned}
\mu(d) &\leqslant \sum_{c\cap d\neq\emptyset} s_c && \text{by part (i)}\\
&= \sum_{c\cap d\neq\emptyset}\sum_{b\ll c} t_{b,c}\\
&\leqslant \sum_{b^\circ\cap d\neq\emptyset}\sum_{c\in C} t_{b,c}\\
&= \sum_{b^\circ\cap d\neq\emptyset} r_b. && \Box
\end{aligned}$$

Theorem 6.5. *A bounded real-valued function on a compact metric space is R-integrable with respect to a bounded Borel measure iff its set of discontinuities has measure zero.*

Proof. *Necessity.* Let D be the set of discontinuities of $f: X\to\mathbb{R}$ and suppose $\mu(D)>0$. Since $D=\bigcup_{n\geqslant 1} D_n$, we must have $\mu(D_n)>0$ for some $n\geqslant 1$. Fix such n and let $\nu=\sum_{b\in B} r_b\eta_b$ be any simple valuation with $\nu\ll^1\mu$. Then

$$\begin{aligned}
S^{u}(f,\nu)-S^{\ell}(f,\nu) &= \sum_b r_b(\sup f[b]-\inf f[b])\\
&\geqslant \sum_{b^\circ\cap D_n\neq\emptyset} r_b(\sup f[b]-\inf f[b])\\
&\geqslant \sum_{b^\circ\cap D_n\neq\emptyset} r_b/n \quad \text{by definition of } D_n\\
&\geqslant \mu(D_n)/n>0. \quad \text{by Lemma 6.4(ii)}.
\end{aligned}$$

Therefore, f does not satisfy the R-condition and is not R-integrable.

Sufficiency. Assume $\mu(D)=0$. It follows that $\mu(D_n)=0$ for all $n\geqslant 1$. Fix $n\geqslant 1$. Since μ is regular, there exists an open set $v\in\Omega X$ with $D_n\subseteq v$ and $\mu(v)<1/n$.

Choose an open set $w \in \Omega X$ which contains D_n and whose closure is contained in v. Let $\delta_1 > 0$ be the minimum distance between $X - v$ and the closure of w. For $x \in X - w$ we have $\omega_f(x) < 1/n$. Therefore, by Proposition 6.3(ii) applied to $X - w$, there exists $\delta_2 > 0$ such that for any compact subset $c \subseteq X - w$ with $|c| \leqslant \delta_2$ we have

$$\Omega_f(c) < 1/n. \tag{1}$$

Let $0 < \delta < \min(\delta_1, \delta_2)$. By Proposition 4.14, there exists a simple valuation $\gamma = \sum_{b \in B} r_b \eta_b$ with $\gamma \ll^1 \mu$ such that

$$\sum_{|b| \geqslant \delta} r_b < 1/n. \tag{2}$$

Observe that if $|b| < \delta$, then b is contained in at least one of the sets v or $X - w$. We also have

$$\gamma(\Box v) \leqslant \mu(\Box v) = \mu(v) < 1/n \tag{3}$$

since μ is supported in $s[X]$. Therefore,

$$\begin{aligned} S^{u}(f,\gamma) - S^{\ell}(f,\gamma) &= \sum_{b \in B} r_b(\sup f[b] - \inf f[b]) \\ &\leqslant \sum_{|b| \geqslant \delta} \cdots \quad + \sum_{|b| \leqslant \delta, b \subseteq v} \cdots \quad + \sum_{|b| \leqslant \delta, b \subseteq X - w} \cdots \\ &\leqslant \frac{M - m}{n} + \frac{M - m}{n} + \sum_{|b| \leqslant \delta, b \subseteq X - w} \frac{r_b}{n} \quad \text{by (2), (3) and (1)} \\ &\leqslant \frac{2(M - m) + 1}{n}. \end{aligned}$$

Since $n \geqslant 1$ is arbitrary, $f \in R(\mu)$ by the R-condition. □

If μ is a bounded Borel measure on X and $C \subseteq X$ is a closed subset, then the restriction $\mu_{\restriction C}$ is a bounded Borel measure on C. For convenience, we write $f \in R_C(\mu)$ and $\int_C f \, d\mu$, if f is R-integrable on C with respect to this restriction.

Corollary 6.6. *If $f \in R_X(\mu)$ then $f \in R_C(\mu)$ for all closed subsets $C \subseteq X$.*

Proposition 6.7. *If $f \in R_X(\mu)$ and $C, D \subseteq X$ are closed subsets with $C \cap D = \emptyset$, then*

$$\int_{C \cup D} f \, d\mu = \int_C f \, d\mu + \int_D f \, d\mu.$$

Proof. By Corollary 6.6, we know that the three integrals exist. Let $\langle v_i \rangle_{i \geqslant 0}$ be an ω-chain of simple valuations in PUC with lub $\mu_{\restriction C}$. Similarly, let $\langle \gamma_i \rangle_{i \geqslant 0}$ be an ω-chain of simple valuations in PUD with lub $\mu_{\restriction D}$. Then, it is straightforward to check that

$\langle \nu_i + \gamma_i \rangle_{i \geqslant 0}$ is an ω-chain of simple valuations in $PU(C \cup D)$ with lub $\mu_{\upharpoonright C \cup D}$. For each $i \geqslant 0$, we have

$$S^{\ell}_{C \cup D}(f, \nu_i + \gamma_i) = S^{\ell}_{C}(f, \nu_i) + S^{\ell}_{D}(f, \gamma_i).$$

Therefore,

$$\begin{aligned}
\int_{C \cup D} f \, \mathrm{d}\mu &= \lim_{i \to \infty} S^{\ell}_{C \cup D}(f, \nu_i + \gamma_i) \\
&= \lim_{i \to \infty} S^{\ell}_{C}(f, \nu_i) + S^{\ell}_{D}(f, \gamma_i) \\
&= \lim_{i \to \infty} S^{\ell}_{C}(f, \nu_i) + \lim_{i \to \infty} S^{\ell}_{D}(f, \gamma_i) \\
&= \int_C f \, \mathrm{d}\mu + \int_D f \, \mathrm{d}\mu. \qquad \square
\end{aligned}$$

Next, we consider the R-integrability of the uniform limit of a sequence of R-integrable functions.

Theorem 6.8. *If the sequence $\langle f_n \rangle_{n \geqslant 0}$ of R-integrable functions $f_n : X \to \mathbb{R}$ is uniformly convergent to $f : X \to \mathbb{R}$, then f is R-integrable and $\int f \, \mathrm{d}\mu = \lim_{i \to \infty} \int f_i \, \mathrm{d}\mu$.*

Proof. Let $\varepsilon > 0$ be given. We show that f satisfies the R-condition. Let $N \geqslant 0$ be such that $|f_n(x) - f(x)| < \varepsilon/3$ for all $n \geqslant N$ and all $x \in X$. Then for all simple valuations $\nu \in P^1 UX$ we have $|S^{\mathrm{u}}(f - f_N, \nu)| < \varepsilon/3$ and $|S^{\ell}(f - f_N, \nu)| < \varepsilon/3$. Since f_N is R-integrable, there exists a simple valuation $\nu \ll^1 \mu$ with $S^{\mathrm{u}}(f_N, \nu) - S^{\ell}(f_N, \nu) < \varepsilon/3$. Therefore,

$$\begin{aligned}
S^{\mathrm{u}}(f, \nu) - S^{\ell}(f, \nu) &\leqslant S^{\mathrm{u}}(f - f_N, \nu) + S^{\mathrm{u}}(f_N, \nu) - S^{\ell}(f - f_N, \nu) - S^{\ell}(f_N, \nu) \\
&\leqslant |S^{\mathrm{u}}(f - f_N, \nu)| + |S^{\ell}(f - f_N, \nu)| + S^{\mathrm{u}}(f_N, \nu) - S^{\ell}(f_N, \nu) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

and hence, $f \in R(\mu)$. Furthermore, for all $n \geqslant N$, we have

$$\begin{aligned}
\left| \int f \, \mathrm{d}\mu - \int f_n \, \mathrm{d}\mu \right| &= \left| \int f - f_n \, \mathrm{d}\mu \right| \\
&\leqslant \int |f - f_n| \, \mathrm{d}\mu \\
&\leqslant \varepsilon. \qquad \square
\end{aligned}$$

In the next section, we will also show the generalisation of Arzelà's theorem for R-integration.

7. R-integration and Lebesgue integration

It is well known that when the ordinary Riemann integral of a bounded function on a compact real interval exists, so does its Lebesgue integral and the two integrals coincide. In this section, we will show that this result extends to R-integration.

In order to show that an R-integrable function is Lebesgue integrable, we construct an increasing sequence of simple measurable functions which tend to our function. We do this by considering the set of *deflations* on UX.

Recall [18] that a *deflation* on a dcpo Y is a continuous map

$$d : Y \to Y$$

which is below the identity $d \sqsubseteq 1_Y$ and its image $\mathrm{im}(d)$ is finite. If $b \in B = \mathrm{im}(d)$, then $D = d^{-1}(b)$ satisfies the following properties:

(i) $x \sqsubseteq y \sqsubseteq z \;\&\; x, z \in D \;\Rightarrow\; y \in D$.

(ii) For any directed set $\langle x_i \rangle_{i \in I}$ with $\bigsqcup_i x_i \in D$, we have $x_i \in D$ for some $i \in I$.

(iii) For any directed set $\langle x_i \rangle_{i \in I}$ with $x_i \in D$ for all $i \in I$, we have $\bigsqcup_i x_i \in D$.

It follows [24] that D is a *crescent*, i.e. $D = v - w$ for some open sets $v, w \in \Omega Y$. Now consider the map $Pd : PY \to PY$, induced by the probabilistic power domain functor P on the deflation $d \sqsubseteq 1_Y$. We have $Pd(\mu) = \mu \circ d^{-1} \sqsubseteq \mu$, since $d^{-1}(O) \subseteq O$ for all $O \in \Omega Y$. Consider the unique extension of $\mu \in PY$ to the ring generated by the open sets, i.e. put $\mu(D) = \mu(v) - \mu(v \cap w)$ for each crescent $D = v - w$. Then it is easily seen that

$$\mu \circ d^{-1} = \sum_{b \in B} r_b \eta_b$$

with $r_b = \mu(d^{-1}(b))$. Hence, for each deflation d and each continuous valuation μ on Y, we obtain a simple valuation $\mu \circ d^{-1}$ below μ. Note that if μ is normalised so is $\mu \circ d^{-1}$. Recall also that if Y is the retract of an SFP domain, then the set of deflations way-below the identity map is directed and has the identity as its lub [18, p.88]. We can now deduce:

Proposition 7.1. *Any continuous valuation on a retract of an SFP domain is the lub of an ω-chain of simple valuations induced from deflations below the identity map.*

Proof. Let Y be a retract of an SFP domain and μ be a continuous valuation on Y. By the above remark, there exists an ω-chain $\langle d_i \rangle_{i \geqslant 0}$ of deflations on Y with $1_Y = \bigsqcup_i d_i$. By the local continuity of P (Proposition 2.2), we have $1_{PY} = \bigsqcup_i Pd_i$ and therefore

$$\mu = \bigsqcup_i Pd_i(\mu) = \bigsqcup_i \mu \circ d_i^{-1}$$

as required. $\square$

We are now in a position to prove the main result in this section. For clarity we denote the Lebesgue integral of a real-valued function $f : X \to \mathbb{R}$ with respect to the Borel measure μ by $\mathbf{L}\int_X f\,\mathrm{d}\mu$ and the R-integral by $\mathbf{R}\int_X f\,\mathrm{d}\mu$. We also drop the subscript X.

Theorem 7.2. *If a bounded real-valued function f is R-integrable with respect to a Borel measure μ on a compact metric space X, then it is also Lebesgue integrable and the two integrals coincide.*

Proof. Since UX is an ω-continuous bounded complete dcpo with bottom, there exists by Proposition 7.1 an ω-chain $\langle d_i \rangle_{i\geqslant 0}$ of deflations $d_i : UX \to UX$ with $\bigsqcup_i \mu\circ d_i^{-1} = \mu$ and each $i\geqslant 0$ induces a simple valuation

$$\mu_i = \mu \circ d_i^{-1} = \sum_{b\in B_i} r_{i,b}\eta_b,$$

where $B_i = \operatorname{im} d_i$ and $r_{i,b} = \mu(d_i^{-1}(b))$. For each $i\geqslant 0$, define two functions

$$\begin{aligned} f_i^- : X &\to \mathbb{R} \\ x &\mapsto \inf f[s^{-1}(d_i^{-1}(d_i(s(x))))] \end{aligned} \qquad \begin{aligned} f_i^+ : X &\to \mathbb{R} \\ x &\mapsto \sup f[s^{-1}(d_i^{-1}(d_i(s(x))))] \end{aligned}$$

where $s : X \to UX$ is, as before, the singleton map. Since for each $i\geqslant 0$ and $x \in X$, we have $d_i^{-1}(d_i(s(x))) = v - w$ for some open sets $v, w \in \Omega UX$, it follows easily that $s^{-1}(d_i^{-1}(d_i(s(x)))) = s^{-1}(v) - s^{-1}(w)$ is a crescent of X. Moreover, as the image of d_i is finite, X is partitioned to a finite number of such crescents. Therefore, f_i^- and f_i^+ are simple measurable functions. Is is easy to see that for each $x \in X$ we have

$$m \leqslant \cdots \leqslant f_i^-(x) \leqslant f_{i+1}^-(x) \leqslant \cdots \leqslant f(x) \leqslant \cdots \leqslant f_{i+1}^+(x) \leqslant f_i^+(x) \leqslant \cdots \leqslant M,$$

where m and M are, as before, the infimum and the supremum of f on X. Let

$$\begin{aligned} f^- : X &\to \mathbb{R} \\ x &\mapsto \lim_{i\to\infty} f_i^-(x) \end{aligned} \qquad \begin{aligned} f_i^+ : X &\to \mathbb{R} \\ x &\mapsto \lim_{i\to\infty} f_i^+(x). \end{aligned}$$

Then $f^-(x) \leqslant f(x) \leqslant f^+(x)$ for all $x \in X$. By the monotone convergence theorem, f^- and f^- are Lebesgue integrable. We will calculate their Lebesgue integrals.

For each $b \in B_i$, let

$$\alpha_{i,b} = \sup f[s^{-1}(d_i^{-1}(b))], \qquad \beta_{i,b} = \inf f[s^{-1}(d_i^{-1}(b))].$$

Since $d_i^{-1}(b) \subseteq \uparrow b$, we have $s^{-1}(d_i^{-1}(b)) \subseteq b$. Hence, for all $i\geqslant 0$ and $b \in B_i$,

$$\inf f[b] \leqslant \beta_{i,b} \leqslant \alpha_{i,b} \leqslant \sup f[b].$$

We can now obtain the following estimates for the Lebesgue integrals of f_i^- and f_i^+:

$$\begin{aligned}\mathbf{L}\int f_i^+ \,\mathrm{d}\mu &= \sum_{b\in B_i} r_{i,b}\alpha_{i,b}\\ &\leqslant \sum_{b\in B_i} r_{i,b}\sup f[b]\\ &= S^{\mathrm{u}}(f,\mu_i)\end{aligned}$$

and

$$\begin{aligned}\mathbf{L}\int f_i^- \,\mathrm{d}\mu &= \sum_{b\in B_i} r_{i,b}\beta_{i,b}\\ &\geqslant \sum_{b\in B_i} r_{i,b}\inf f[b]\\ &= S^{\ell}(f,\mu_i).\end{aligned}$$

Since $f_i^- \leqslant f_i^+$ implies $\mathbf{L}\int f_i^- \,\mathrm{d}\mu \leqslant \mathbf{L}\int f_i^+ \,\mathrm{d}\mu$, we obtain

$$S^{\ell}(f,\mu_i) \leqslant \mathbf{L}\int f_i^- \,\mathrm{d}\mu \leqslant L\int f_i^+ \,\mathrm{d}\mu \leqslant S^{\mathrm{u}}(f,\mu_i).$$

As f is assumed to be R-integrable, we know by Corollary 4.10 that $S^{\ell}(f,\mu_i)$ increases to $\mathbf{R}\int f\,\mathrm{d}\mu$ and $S^{\mathrm{u}}(f,\mu_i)$ decreases to $\mathbf{R}\int f\,\mathrm{d}\mu$. Therefore,

$$\mathbf{L}\int f_i^- \,\mathrm{d}\mu \to \mathbf{R}\int f\,\mathrm{d}\mu \qquad \text{and} \qquad \mathbf{L}\int f_i^+ \,\mathrm{d}\mu \to \mathbf{R}\int f\,\mathrm{d}\mu$$

as $i \to \infty$. By the monotone convergence theorem, we have

$$\mathbf{L}\int f^- \,\mathrm{d}\mu = \lim_{i\to\infty} \mathbf{L}\int f_i^- \,\mathrm{d}\mu = \mathbf{R}\int f\,\mathrm{d}\mu,$$
$$\mathbf{L}\int f^+ \,\mathrm{d}\mu = \lim_{i\to\infty} \mathbf{L}\int f_i^+ \,\mathrm{d}\mu = \mathbf{R}\int f\,\mathrm{d}\mu.$$

It now follows that $\mathbf{L}\int(f^+ - f^-)\,\mathrm{d}\mu = 0$ which implies that $f^+ = f^-$ almost everywhere. Therefore $f = f^- = f^+$ almost everywhere. We conclude that f is Lebesgue integrable and

$$\mathbf{L}\int f\,\mathrm{d}\mu = \mathbf{L}\int f^- \,\mathrm{d}\mu = \mathbf{L}\int f^+ \,\mathrm{d}\mu = \mathbf{R}\int f\,\mathrm{d}\mu$$

as required. □

We can now also obtain the generalisation of Arzelà's theorem for R-integration.

Corollary 7.3. *Suppose the sequence $\langle f_n \rangle_{n\geqslant 0}$ of real-valued and uniformly bounded functions on X is pointwise convergent to an* R*-integrable function f. Then, we have*

$$\mathbf{R}\int f\,\mathrm{d}\mu = \lim_{n\to\infty} \mathbf{R}\int f_n \,\mathrm{d}\mu.$$

Proof. This follows immediately by applying Theorem 7.2 and using the Lebesgue dominated convergence theorem. □

8. Applications to fractals

An immediate area of application for R-integration is in the theory of iterated function systems (IFSs) with probabilities. Recall [15, 3] that an *IFS with probabilities*, $\{X; f_1, \ldots, f_N; p_1, \ldots, p_N\}$, is given by a finite number of contracting maps $f_i : X \to X$ $(1 \leqslant i \leqslant N)$ on a compact metric space X, such that each f_i is assigned a probability weight p_i with $0 < p_i < 1$ and

$$\sum_{i=1}^{N} p_i = 1.$$

An IFS with probabilities gives rise to a unique invariant Borel measure on X. If $X \subseteq \mathbb{R}^n$, then the support of this measure is usually a fractal, i.e. it has fine, complicated and nonsmooth local structure, some form of self-similarity and, usually, a nonintegral *Hausdorff dimension*. Conversely, given any image regarded as a compact set in the plane, one uses a self-tiling of the image and Barnsley's collage theorem to find an IFS with contracting affine transformations, whose attractor approximates the image. The theory has many applications including in statistical physics [14, 6, 10], neural nets [5, 8] and image compression [3, 4].

It was shown in [9, Theorem 6.2], that the unique invariant measure μ of an IFS with probabilities as above is the fixed point of the map

$$\begin{aligned} T : P^1UX &\to P^1UX \\ \mu &\mapsto T(\mu) \end{aligned}$$

defined by $T(\mu)(O) = \sum_{i=1}^{N} p_i \mu(f_i^{-1}(O))$. This fixed point can be written as $\bigsqcup_{m \geqslant 0} \mu_m$ where $\mu_0 = \eta_X$ and for $m \geqslant 1$,

$$\mu_m = T^m(\eta_X) = \sum_{i_1, i_2, \ldots, i_m = 1}^{N} p_{i_1} p_{i_2} \cdots p_{i_m} \eta_{f_{i_1} f_{i_2} \cdots f_{i_m}(X)}.$$

Therefore, the unique invariant measure of the IFS with probabilities is the lub of an ω-chain of simple valuations in P^1UX. This provides a better algorithm for fractal image decompression using measures [11], compared to the algorithms presented in [4].

Suppose now we have a bounded function $f : X \to \mathbb{R}$ whose set of discontinuities has μ-measure zero, then we know that its Lebesgue integral with respect to μ coincides with its R-integral with respect to μ. Fix $x \in X$ and, for each $m \geqslant 1$, consider the generalised Riemann sum

$$S_x(f, \mu_m) = \sum_{i_1, i_2, \ldots, i_m = 1}^{N} p_{i_1} p_{i_2} \cdots p_{i_m} f(f_{i_1} f_{i_2} \cdots f_{i_m}(x)).$$

From Proposition 4.9, we immediately obtain:

Theorem 8.1. *For an IFS with probabilities and a bounded real-valued function f which is continuous almost everywhere with respect to the invariant measure μ of the IFS, we have*

$$\mathbf{L}\int f\ \mathrm{d}\mu = \mathbf{R}\int f\ \mathrm{d}\mu = \lim_{m\to\infty} S_x(f,\mu_m),$$

for any $x \in X$. □

If f satisfies a Lipschitz condition, then, for any $\varepsilon > 0$, we can obtain a finite algorithm to estimate $\int f\ \mathrm{d}\mu$ up to ε accuracy [11]. Another method for computing the integral is by Elton's ergodic theorem [12]: The time-average of f with respect to the nondeterministic dynamical system $f_1, f_2, \ldots, f_N : X \to X$, where at each stage in the orbit of a point the map f_i is selected with probability p_i, tends, with probability one, to its space-average, i.e. to its integral. However, in this case, the convergence is only with probability one and there is no estimate for the rate of convergence. Therefore, the above theorem provides a better way of computing the integral.

Example 8.2. Finally, we consider a concrete example. Let $C = \{1,2,\ldots,N\}^{\omega}$ be the Cantor space with the following metric

$$d(x,y) = \sum_{n=0}^{\infty} \frac{\delta(x_n, y_n)}{2^n}$$

where the *Kronecker delta* is given by

$$\delta(k,l) = \begin{cases} 0 \text{ if } k = l, \\ 1 \text{ otherwise.} \end{cases}$$

This metric is equivalent to the Cantor (product) topology, and is frequently used in mathematics and theoretical physics. Let $\{C; f_1, \ldots, f_N; p_1, \ldots, p_N\}$ be an IFS with probabilities on C, with

$$\begin{aligned} f_k : C &\to C \\ x &\mapsto kx, \end{aligned}$$

where kx is the concatenation of k and x. Its unique invariant measure μ is defined on the closed–open subset

$$[i_1 i_2 \ldots i_m] = \{x \in C \mid x_j = i_j,\ 1 \leqslant j \leqslant m\}$$

by

$$\mu([i_1 i_2 \ldots i_m]) = p_{i_1} p_{i_2} \cdots p_{i_m}.$$

In fact, we have

$$\mu_m = T^m(\eta_X) = \sum_{i_1,i_2,\ldots,i_m=1}^{N} p_{i_1} p_{i_2} \cdots p_{i_m} \eta_{[i_1 i_2 \ldots i_m]}.$$

Let

$$\begin{aligned} f : C &\to \mathbb{R} \\ x &\mapsto d(x, 1^\omega) \end{aligned}$$

be the function which gives the distance of the point x to the point 1^ω. This function is continuous and therefore its Lebesgue integral with respect to μ coincides with its R-integral with respect to μ. The integral in fact represents the average distance in C from 1^ω with respect to the invariant measure. The R-integral is easily obtained using Theorem 8.1 above with $x = 1^\omega$. In fact a straightforward calculation shows that

$$S_{1^\omega}(f, \mu_m) = 2\left(1 - \frac{1}{2^m}\right)(1 - p_1) \to 2(1 - p_1)$$

as $m \to \infty$. Therefore, $\mathbf{L}\int f \, \mathrm{d}\mu = \mathbf{R}\int f \, \mathrm{d}\mu = 2(1 - p_1)$.

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References

[1] S. Abramsky and A. Jung, Domain theory, in: S. Abramsky, D.M. Gabbay and T.S.E. Maibaum, eds., *Handbook of Logic in Computer Science, Vol. 3* (Clarendon Press, Oxford, 1994).

[2] T.M. Apostol, *Mathematical Analysis* (Addison-Wesley, Reading, MA, 1974).

[3] M.F. Barnsley, *Fractals Everywhere* (Academic Press, New York, 1988).

[4] M.F. Barnsley and L. P. Hurd. *Fractal Image Compression* (A K Peters, Wellesley, MA, 1993).

[5] U. Behn, J.L. van Hemmen, A. Lange, R. Kühn, and V.A. Zagrebnov, Multifractality in forgetful memories, *Physica D* **68** (1993) 401–415.

[6] U. Behn and V. Zagrebnov, One dimensional Markovian-field Ising model: physical properties and characteristics of the discrete stochastic mapping, *J. Phys. A: Math. Gen.* **21** (1998) 2151–2165.

[7] G. Birkhoff, *Lattice Theory* (Amer Mathematical Soc., Providence, RI, 1967).

[8] P.C. Bressloff and J. Stark, Neural networks, learning automata and iterated function systems, in: A.J. Crilly, R.A. Earnshaw and H. Jones, eds., *Fractals and Chaos* (Springer, Berlin, 1991) 145–164.

[9] A. Edalat, Dynamical systems, measures and fractals via domain theory (extended abstract), in: G.L. Burn, S.J. Gay and M.D. Ryan, eds., *Theory and Formal Methods 1993* (Springer, Berlin, 1993); full paper to appear in *Inform. and Comput.*

[10] A. Edalat, Domain of computation of a random field in statistical physics in: *Proc. 2nd Imperial College, Department of Computing, Theory and Formal Methods Workshop*, 1994.

[11] A. Edalat, Power domains and iterated function systems, Technical Report Doc 94/13, Department of Computing, Imperial College, 1994; submitted to *Inform. and Comput.*

[12] J. Elton, An ergodic theorem for iterated maps, *Ergodic Theory Dynamical Systems* **7** (1987) 481–487. 1987.

[13] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove and D.S. Scott, *A Compendium of Continuous Lattices* (Springer, Berlin, 1980).

[14] G. Györgyi and P. Rujoriginally, Strange attractors in disordered systems, *J. Phys. C: Solid State Phys.* **17** (1984) 4207–4212.

[15] J.E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* **30** (1981) 713–747.

[16] C. Jones, Probabilistic non-determinism, Ph.D. Thesis, University of Edinburgh, 1989.

[17] C. Jones and G. Plotkin, A probabilistic powerdomain of evaluations, in: *Logic in Computer Science* (IEEE Computer Soc.Press;Silver Spring,MD, 1989) 186–195.

[18] A. Jung, *Cartesian Closed Categories of Domains*, Vol. 66 of *CWI Tract.* (Centrum voor Wiskunde en Informatica, Amsterdam, 1989).

[19] O. Kirch, Bereiche und bewertungen, Master's Thesis, Technische Hochschule Darmstadt, 1993.

[20] S. Lang, *Real Analysis* (Addison-Wesley, Reading, MA, 1969).

[21] J.D. Lawson, Valuations on continuous lattices. in: Rudolf-Eberhard Hoffman, ed., *Continuous Lattices and Related Topics*, Vol. 27 of *Mathematik Arbeitspapiere* (Universität Bremen, 1982).

[22] E.J. McShane, A Riemann-type integral that includes Lebesgue–Stieltjes, Bochner and stochastic integrals, Mem. *Amer. Math. Soc.* **88** (1969).

[23] W. Rudin, *Real and Complex Analysis* (McGraw-Hill, New York, 1966).

[24] N. Saheb-Djahromi, Cpo's of measures for non-determinism, *Theoret. Comput. Sci.* **12** (1980) 19–37.

[25] M.B. Smyth, Powerdomains and predicate transformers: a topological view, in: J. Diaz, ed., *Automata, Languages and Programming*, Lecture Notes in Computer Science, Vol. 154 (Springer, Berlin, 1983) 662–675.