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Distance Graphs with Finite Chromatic Number[1]

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The distance graph $G(D)$ with distance set $D=\{d_1, d_2, \ldots\}$ has the set Z of integers as vertex set, with two vertices $i, j \in Z$ adjacent if and only if $|i-j| \in D$. We prove that the chromatic number of $G(D)$ is finite whenever $\inf\{d_{i+1}/d_i\} > 1$ and that every growth speed smaller than this admits a distance set D with infinite-chromatic $G(D)$.

1. INTRODUCTION

Graphs defined in terms of distances have lots of interesting properties. For instance, one of the famous open problems asks about the chromatic number of the unit distance graph in the Euclidean plane, with still the same "currently best" lower and upper bounds of 4 and 7 after half a century.

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In this short paper we study graphs defined on the set Z of integers. Let $D=\{d_1, d_2, ...\}$ be any finite or infinite set of natural numbers, termed the *distance set*. The *distance graph* $G(D)$ has vertex set Z and edge set

$$E=\{(i, j) \mid i<j, j-i \in D\}.$$

The systematic study of distance graphs was initiated by Eggleton *et al.* [2] in the mid-1980s. Recently, the subject has attracted considerable attention; see e.g. [4] for recent results and many related references. Notably, the chromatic number of $G(D)$ is now determined in [5] for all *three-element* distance sets D.

The aim of the present paper is to prove that if the sequence of distances in D grows sufficiently fast—more precisely, if $\inf\{d_{i+1}/d_i\}>1$—then the chromatic number of $G(D)$ is *finite*. This yields an essential improvement of the result in [6] where it is shown that the chromatic number is finite if there is an i_0 such that $d_{i+1} \geqslant (d_i+1)(d_i+2)$ for all $i \geqslant i_0$. Moreover, our result is tight in the sense that a growth rate even slightly slower than exponential is not sufficient in general for colorability with finitely many colors. A simple construction is described in Section 3. More generally, the growth sequences resulting in finite-chromatic distance graphs can be characterized (Theorem 6). On the other hand, when restricted to finite colorability, the following problem concerning distance sets of a prescribed minimum growth rate remains open.

Problem 1. Given a real $\varepsilon>0$, determine the largest possible chromatic number of distance graphs $G(D)$ with $D=\{d_1, d_2, ...\}$ such that $d_{i+1}/d_i \geqslant 1+\varepsilon$ for all $i \geqslant 1$.

Notation. For any real number x, let $\|x\|$ denote the distance from x to the nearest integer; that is, $\|x\|=\min\{x-\lfloor x\rfloor, \lceil x\rceil-x\}$. We recall that a *proper coloring* of a graph $G=(V, E)$ *with r colors* (an *r-coloring*, for short) is an assignment $\varphi: V \rightarrow \{1, ..., r\}$ such that $\varphi(u) \neq \varphi(v)$ whenever $uv \in E$ (where V and E denote the vertex set and edge set of G, respectively). The *chromatic number* $\chi(G)$ of G is the smallest r for which G admits an r-coloring.

It will be assumed throughout that the elements of the distance set D are listed in increasing order, i.e., $d_1<d_2<\cdots$.

2. UPPER BOUNDS ON THE CHROMATIC NUMBER

In this section we prove that any sequence $d_1<d_2<\cdots$ of exponential growth yields a distance graph with *finite* chromatic number.

THEOREM 2. *Let* $D = \{d_1, d_2, \ldots\}$ *be an infinite distance set such that there exists an integer* $r \geqslant 3$ *with* $d_{i+1} \geqslant \frac{2r-2}{r-2} d_i$ *for all* $i \geqslant 1$. *Then the corresponding distance graph* $G(D)$ *has a proper coloring with r colors.*

Proof. In the first step of the proof we are searching for a real number x such that $\|xd_i\| \geqslant 1/r$ for all d_i, $i = 1, 2, \ldots$.

To find such an x, we determine a sequence of nested nonempty intervals. The first interval $I_1 = [b_1, e_1]$ is defined by

$$I_1 := \left[\frac{1}{rd_1}, \frac{r-1}{rd_1}\right].$$

The length of this interval is obviously $(r-2)/rd_1$.

CLAIM. *Let* $I_i = [b_i, e_i]$ *be an interval of length* $(r-2)/rd_i$. *Then there exists an integer* z_i *such that*

$$I_{i+1} := \left[\frac{z_i}{d_{i+1}} + \frac{1}{rd_{i+1}}, \frac{z_i}{d_{i+1}} + \frac{r-1}{rd_{i+1}}\right] \subset I_i.$$

Proof of the claim. Because of the assumptions of Theorem 2, we know that the length of I_i satisfies

$$|I_i| = \frac{r-2}{rd_i} \geqslant \frac{2r-2}{rd_{i+1}} > \frac{1}{d_{i+1}}.$$

Thus, there is obviously an integer z_i such that $z_i/d_{i+1} \in I_i$. If z_i/d_{i+1} belongs to the first half $[b_i, (e_i + b_i)/2]$ of the interval I_i, then

$$\left[\frac{z_i}{d_{i+1}} + \frac{1}{rd_{i+1}}, \frac{z_i}{d_{i+1}} + \frac{r-1}{rd_{i+1}}\right]$$

is contained in I_i; and otherwise

$$\left[\frac{z_i - 1}{d_{i+1}} + \frac{1}{rd_{i+1}}, \frac{z_i - 1}{d_{i+1}} + \frac{r-1}{rd_{i+1}}\right] \subset I_i,$$

proving the claim.

In this way we find a sequence of nonempty, nested, closed intervals, which has a limit. Let us denote this limit by x.

By the choice of x, $xd_i \in [z_{i-1} + \frac{1}{r}, z_{i-1} + \frac{r-1}{r}]$ for every i (where $z_0 = 0$). Hence, x has the property required above.

Next, let us color the real line using the colors $0, 1, \ldots, r-1$. Let v be a real number, and determine the unique j with $0 \leqslant j \leqslant r-1$ such that $xv - \lfloor xv \rfloor \in [\frac{j}{r}, \frac{j+1}{r})$. Color v with j; that is $f(v) = j$. In fact, the real line is partitioned into intervals of length $\frac{1}{xr}$, where each interval gets a color and the sequence of colors is $\ldots, 0, 1, \ldots, r-1, 0, 1, \ldots, r-1, \ldots,$ periodically. Such a coloring is called a *regular coloring of the real line* (see [4]).

Finally, we have to prove that this coloring induces a proper coloring for $G(D)$. Let v and w be any two adjacent vertices in $G(D)$, with $v > w$. Then there is a $d_i \in D$ such that $v - w = d_i$. Assume, for a contradiction, that $f(v) = f(w) = j$. Then there exist integers z_v and z_w such that

$$z_v + \frac{j}{r} \leqslant xv < z_v + \frac{j+1}{r}, \qquad z_w + \frac{j}{r} \leqslant xw < z_w + \frac{j+1}{r}$$

and therefore

$$z_v - z_w - \frac{1}{r} < xv - xw < z_v - z_w + \frac{1}{r}.$$

Thus, $\|xd_i\| = \|xv - xw\| < \frac{1}{r}$, contradicting the choice of x. ∎

THEOREM 3. *Let k be a natural number. If $D = \{d_1, d_2, \ldots\}$ is a distance set such that $d_{i+1} \geqslant 4^{1/k} d_i$ for all $i \geqslant 1$, then $G(D)$ is colorable with 3^k colors; and if $d_{i+1} \geqslant 3^{1/k} d_i$ for all $i \geqslant 1$, then $G(D)$ is colorable with 4^k colors.*

Proof. We prove the two variants of the assertion simultaneously. Let us partition D into k subsets

$$D_j = \{d_i \mid i \equiv j \pmod{k}\}, \qquad 0 \leqslant j < k.$$

The first assumption on the growth of the d_i implies that $d_{i+k} \geqslant 4d_i$. Thus, each $G(D_j)$ has a proper coloring with three colors, by Theorem 2. Similarly, under the second condition we obtain that $d_{i+k} \geqslant 3d_i$ and each $G(D_j)$ is properly 4-colorable. Now the theorem follows by the facts that the chromatic number is submultiplicative with respect to graph union and that $G(D) = \bigcup_{j=0}^{k-1} G(D_j)$. ∎

Denoting $q = \inf d_{i+1}/d_i$, the best choice for k under the first and second condition in Theorem 3 is $\lceil \frac{\ln 4}{\ln q} \rceil$ and $\lceil \frac{\ln 3}{\ln q} \rceil$, respectively. Thus, both estimates yield approximately $e^{(\ln 3)(\ln 4)/\ln q}$ as an upper bound on the chromatic number of $G(D)$, for q sufficiently close to 1. Writing $q = 1 + \varepsilon$, and applying the fact that $\frac{\ln(1+\varepsilon)}{\varepsilon} \to 1$ as $\varepsilon \to 0$, we obtain that $G(D)$ has a proper coloring with at most $4.586^{1/\varepsilon}$ colors whenever ε is sufficiently close to zero.

COROLLARY 4. *If* $\inf\{d_{i+1}/d_i\} > 1$, *then the chromatic number of* $G(D)$ *is finite. More generally, the chromatic number of* $G(D)$ *is finite whenever* $\inf\{d_{i+k}/d_i\} > 1$ *for some natural number* k.

Proof. The first assertion is an immediate consequence of Theorem 3. The second one is obtained by considering the subsets D_j as defined in the proof of Theorem 3 and applying the first assertion of this corollary to each of them, together with the submultiplicative property of the chromatic number. ▮

It will be shown in the next section that the second assertion can be reversed in some sense, yielding a characterization of growth-rate sequences for finite colorability.

3. SPARSE DISTANCE SETS WITH INFINITE CHROMATIC NUMBER

The following result shows that an exponential growth is not only sufficient but also more or less necessary for making the chromatic number of $G(D)$ finite.

THEOREM 5. *Let* $\varepsilon_1 \geqslant \varepsilon_2 \geqslant \varepsilon_3 \geqslant \cdots$ *be a sequence of positive reals tending to zero (arbitrarily slowly). There exists a distance set* $D = \{d_1, d_2, \ldots\}$ *such that* $d_{i+1} \geqslant (1+\varepsilon_i)\, d_i$ *holds for all* $i \geqslant 1$, *but* $G(D)$ *does not admit a coloring with any finite number of colors.*

Proof. We define D inductively. Let $d_1 \geqslant 1$ be chosen arbitrarily. In general, suppose that i is the largest subscript for which d_i has already been defined. The next block of distances will be defined as $d_{i+j} = j d_{i+1}$ for $j = 1, 2, \ldots, \lfloor 1/\varepsilon_i \rfloor + 1$, where $d_{i+1} = \lceil (1+\varepsilon_i)\, d_i \rceil$.

Between consecutive blocks, the prescribed growth rate is satisfied by the choice of d_{i+1}. On the other hand, inside one block—where the distances form an arithmetic progression—the smallest ratio occurs for the last two elements. This ratio is $1 + 1/\lfloor 1/\varepsilon_i \rfloor$, which is at least $1 + \varepsilon_{i+j}$ for all $j \geqslant 0$, by the assumption that the sequence $\varepsilon_1, \varepsilon_2, \ldots$ is nonincreasing.

The numbers occurring inside one block, when viewed as vertices of $G(D)$, induce a complete subgraph of the distance graph (also together with vertex 0, but this is unimportant). Thus, since $1/\varepsilon_i \to \infty$ as $i \to \infty$, $G(D)$ contains arbitrarily large complete subgraphs and therefore cannot be colorable with any finite number of colors. ▮

In a more general setting, let us investigate how fast prescribed growth implies the finite colorability of $G(D)$. To formulate the result, we need some notation.

DEFINITIONS. Let $\mathbf{q} = (q_1, q_2, \ldots)$ be an infinite sequence of reals greater than 1. We say that a distance set $D = \{d_1, d_2, \ldots\}$ is **q**-*compatible* if $d_{i+1} \geqslant q_i d_i$ for all $i \geqslant 1$. The collection of all **q**-compatible distance sets will be denoted by $\mathscr{D}(\mathbf{q})$. Define

$$\chi(\mathbf{q}) = \sup\{\chi(G(D)) \mid D \in \mathscr{D}(\mathbf{q})\}.$$

THEOREM 6. *With the above notation,* $\chi(\mathbf{q}) = \infty$ *if and only if*

$$\inf_{i \geqslant 1} \left\{ \prod_{j=0}^{k-1} q_{i+j} \right\} = 1 \tag{$*$}$$

for every natural number k.

Proof. If the infimum is greater than 1, then every $D \in \mathscr{D}(\mathbf{q})$ satisfies the second condition in Corollary 4; hence $\chi(G(D))$ is finite. This proves necessity.

To prove sufficiency, suppose that $\mathbf{q}$ satisfies $(*)$. We are going to construct a **q**-compatible distance set D such that $G(D)$ is not finitely colorable. We first define a subsequence of indices i_k $(k = 0, 1, 2, \ldots)$ as follows. Let $i_0 = 1$. If i_{k-1} has been defined, let i_k be the smallest integer such that $i_k \geqslant i_{k-1} + k$ and $\prod_{j=0}^{k-1} q_{i_k+j} \leqslant 1 + 1/k$. It follows from $(*)$ that i_k is well defined for every k.

The blocks of the distance set $D = \{d_1, d_2, \ldots\}$ to be constructed are the sets $\{d_{i_k}, d_{i_k+1}, \ldots, d_{i_k+k}\}$ for each k and also the remaining natural numbers as one-element blocks. The d_i will be defined blockwise.

Choose d_1 arbitrarily. Assuming that the distances have been defined up to d_i, the first element d_{i+1} of the next block is defined to be $d_{i+1} = \lceil q_i d_i \rceil$. (Alternatively, any larger number might be chosen for d_{i+1}.) All elements of the block are chosen to form an arithmetic progression of difference d_{i+1}; that is, $d_{i+j} = j d_{i+1}$ for all $1 \leqslant j \leqslant k+1$ inside that block. It can be seen along the lines of the proof of Theorem 5 that $D \in \mathscr{D}(\mathbf{q})$ and $\chi(G(D)) = \infty$. ∎

4. REGULAR COLORINGS AND RELATED PROBLEMS

There is an interesting conjecture which goes back to Wills [3]: For any set of positive integers $d_1, \ldots, d_n$ there exists a real number x such that $\|x d_i\| \geqslant \frac{1}{n+1}$ for each $i = 1, \ldots, n$.

So far the conjecture is proved for $n \leqslant 4$. Moreover, Cusick and Chen [1] proved that when $2n-3$ is a prime and $n \geqslant 4$, then it is possible to find a real number x with $\|x d_i\| \geqslant \frac{1}{2n-3}$ for all $1 \leqslant i \leqslant n$.

If Wills' conjecture is true, then we immediately obtain a regular coloring of the real line with $n+1$ colors, inducing a proper coloring of the distance graph $G(D)$ with $D=\{d_1, \ldots, d_n\}$. Vice versa, if we could prove that there is such a coloring for every distance graph with distance set of cardinality n, then the conjecture would be true.

Since both n and $-n$ have degree at most $|D|$ in the subgraph induced by $\{i \mid -n \leqslant i \leqslant n\}$, it is very easy to color $G(D)$ sequentially with at most $|D|+1$ colors, e.g., in increasing order of $|z|$, $z \in Z$. Now the question is whether there always exists a *regular* coloring of the real line inducing such a coloring of $G(D)$.

REFERENCES

1. Y. G. Chen and T. W. Cusick, The view-obstruction problem for n-dimensional cubes, *J. Number Theory* **74** (1999), 126–133.
2. R. B. Eggleton, P. Erdős, and D. K. Skilton, Coloring the real line, *J. Combin. Theory B* **39** (1985), 86–100.
3. J. M. Wills, Zwei Sätze über inhomogene diophantische Approximation von Irrationalzahlen, *Monatsch. Math.* **71** (1967), 263–269.
4. X. Zhu, Pattern periodic coloring of distance graphs, *J. Combin. Theory B* **73** (1998), 195–206.
5. X. Zhu, The circular chromatic number of distance graphs with distance sets of cardinality 3, in press.
6. M. Voigt, On the chromatic number of distance graphs, *J. Inform. Process. Cybernet. EIK* **28** (1992), 21–28.