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Approximation of a Solution for a K-Positive Definite Operator Equation

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Let E be separable q-uniformly smooth Banach space, $q > 1$, and let A: $D(A) \subseteq E \to E$ be a K-positive definite operator. Let $f \in E$ be arbitrary. An iterative method is constructed which converges strongly to the unique solution of the equation $Ax = f$. Our result resolves two questions raised by C. E. Chidume and S. J. Aneke (*Appl. Anal.* **50**, 1993, 293).

1. INTRODUCTION

Let H be a separable Hilbert space and let H_1 be a dense subspace of H. An operator A with domain $D(A) \supseteq H_1$ is called *continuously H_1-invertible* if the range of A, $R(A)$, with A considered as an operator restricted to H_1 is dense in H and A has a bounded inverse on $R(A)$. Suppose H is a separable and complex Hilbert space and A is a linear unbounded operator defined on a dense domain $D(A)$ in H with the property that there exist a continuously $D(A)$-invertible closed linear operator K with $D(A) \subseteq D(K)$, and a constant $k > 0$ such that

$$\langle Au, Ku \rangle \geq k\|Ku\|^2, \qquad u \in D(A) \tag{1}$$

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then A is called a K-*positive definite* (Kpd) operator (see, for example, Petryshyn [11]). If $K = I$ (the identity operator), inequality (1) reduces to $\langle Au, u\rangle \geq k\|u\|^2$, and in this case, A is called *positive definite*. Positive definite operators have been studied by various authors (see, for example, Brézis and Browder [1–3], Browder [4], Bruck [5], Chidume [7, 8], Patterson [10], Petryshyn [11], and Reid [12]).

It is easy to see that the class of Kpd operators contains, among others, the class of positive definite operators, and also the class of invertible operators (when $K = A$) as its subclasses. Furthermore, for a proper choice of K, the ordinary differential operators of odd order, the weakly elliptic partial differential operators of odd order, and others, are members of the class of Kpd operators. Moreover, if the operators are bounded, the class of Kpd operators forms a subclass of symmetrizable operators studied by Reid [12] (see, for example, Petryshyn [11]).

In [11], Petryshyn studied the operator equation $Au = f, f \in H$, where H is a complex separable Hilbert space and A is a Kpd operator with domain in H. He proved the following theorem:

THEOREM P. *If A is a Kpd operator and $D(A) = D(K)$ then there exists a constant $\alpha > 0$ such that for all $u \in D(K)$,*

$$\|Au\| \leq \alpha\|Ku\|.$$

Furthermore, the operator A is closed, $R(A) = H$, and the equation $Au = f$, $f \in H$ has a unique solution.

In the case that K is bounded and A is closed, Browder [4] obtained a result similar to the second part of Theorem P. Recently, Chidume and Aneke [9] extended the notion of a Kpd operator to real separable Banach spaces E with strictly convex dual E^* and then proved the following theorem:

THEOREM CA1. *Let E be a real separable Banach space with a strictly convex dual E^* and let A be a Kpd operator with $D(A) = D(K)$. Assume condition* (6) *of* [9] *is satisfied. Then there exists a constant $\alpha > 0$ such that for all $x \in D(A)$,*

$$\|Ax\| \leq \alpha\|Kx\|. \tag{2}$$

Furthermore, the operator A is closed, $R(A) = E$, and the equation $Ax = h$, for each $h \in E$, has a unique solution.

Corrigendum. The Banach space X in Theorem 1 of [9] need not be strictly convex. It is intended that the dual E^* be strictly convex and this is imposed only to ensure that the duality map (defined below) is single-valued. Moreover, condition (6) of the paper should be added in the statement of the theorem.

In the special case of Theorem CA1 in which $E = L_p$ (or l_p) spaces with $p \geq 2$, and is separable, the authors constructed an iteration process which converges strongly to the unique solution, *provided A and K commute*. In fact, they proved the following theorem:

THEOREM CA2. *Suppose $E = L_p$ (or l_p), $p \geq 2$, and is separable, and suppose $A: D(A) \subseteq E \to E$ is a Kpd operator with $D(A) = D(K) = R(K)$. Define the sequence $\{x_n\}_{n=0}^{\infty}$ by*

$$x_0 \in D(A) \tag{3}$$

$$x_{n+1} = x_n + t_n K^{-1} \gamma_n, \qquad n \geq 0 \tag{4}$$

$$t_n = \frac{\langle B\gamma_n, j(K\gamma_n)\rangle}{(p-1)\|B\gamma_n\|^2}, \qquad n \geq 0, \tag{5}$$

where $B = KAK^{-1}$ and $\gamma_n = f - Ax_n$, $f \in R(K)$. If A and K commute and condition (6) *of* [9] *is satisfied, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of $Ax = f$ in E.*

Remark 1. The j in Theorem CA2 is the *normalized duality map* (defined below). The authors remarked that it is of interest to obtain a convergence result similar to Theorem CA2 in L_p (or l_p) spaces for $1 < p < 2$.

It is our purpose in this note to prove that if E is a separable q-uniformly smooth real Banach space ($q > 1$), under weaker hypotheses than in Theorem CA2, the iteration process (3)–(5) converges strongly to the unique solution of the equation $Au = f$, $f \in E$. *This class of Banach spaces includes the L_p (or l_p) spaces for $1 < p < +\infty$. Moreover, the commutativity assumption imposed in Theorem* CA2 *will not be necessary in our theorem.*

Remark 2. We remark here that the iteration method (3)–(5) for solving the operator equation $Ax = f$ was motivated by the desire to extend methods which had hitherto been studied only in Hilbert spaces (see, for example, [10, 11]) to more general Banach spaces. With the main theorem of this paper and the iteration process (8)–(10) (below), it is now possible to extend these iteration processes to Banach spaces much more general than Hilbert spaces (see also Remark 3 (below)).

2. PRELIMINARIES

Let E be a real Banach space and let $q > 1$. The *generalized duality mapping* J_q from E to 2^{E^*} is defined by

$$J_q(x) = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^q \text{ and } \|f^*\| = \|x\|^{q-1} \right\},$$

where E^* denotes the dual space of E and $\langle .,. \rangle$ denotes the generalized duality pairing. In particular, $J_2(x)$ is the usual normalized duality map. It is known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$ for $x \neq 0$, and that J_q is single-valued if E^* is strictly convex (see, for example, [13]). If $E = H$ is a Hilbert space, J_2 becomes the identity operator of H. In the sequel we shall denote the single-valued generalized duality map by j_q.

Let E be a Banach space. The *modulus of smoothness* of E is the function

$$\rho_E : [0, \infty) \to [0, \infty)$$

defined by

$$\rho_E(t) = \sup\left\{\tfrac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \le 1, \|y\| \le t\right\}.$$

The Banach space E is called *uniformly smooth* if

$$\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0.$$

E is called *q-uniformly smooth* if there exists a constant $c > 0$ such that

$$\rho_E(t) \le ct^q, \qquad q > 1.$$

Hilbert spaces, L_p (or l_p) spaces, $1 < p < +\infty$, and the Sobolev spaces $W^{m,p}$, $1 < p < +\infty$, are all q-uniformly smooth. In his study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [13] proved the following theorem:

THEOREM X. *Let E be a uniformly smooth Banach space. Then E is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in E$,*

$$\|x + y\|^q \le \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q. \tag{6}$$

Following Chidume and Anake [9] we have the following definition:

DEFINITION. Let E be a real Banach space and let A be a linear unbounded operator defined on a dense domain, $D(A)$, in E. The operator A is called *K-positive definite* (Kpd) if there exist a continuously $D(A)$-invertible closed linear operator K with $D(A) \subseteq D(K)$, and a constant $c > 0$ such that for $j(Ku) \in J(Ku)$,

$$\langle Au, j(Ku)\rangle \ge c\|Ku\|^2, \qquad u \in D(A). \tag{7}$$

Observe that for Hilbert spaces, inequality (7) reduces to that of Petryshyn [11] and the above definition agrees with that given in [11].

Main Results

We prove the following results:

LEMMA. *Let E be a separable Banach space and let $q > 1$. If $A : D(A) \subseteq E \to E$ is Kpd, then*

$$\langle Au, j_q(Ku)\rangle \geq c\|Ku\|^q$$

for all $u \in D(A)$, $j_q(Ku) \in J_q(Ku)$, where c is the constant appearing in inequality (7).

Proof. This is obvious from inequality (7) and the properties of J_q and K.

THEOREM. *Let E be a real q-uniformly smooth separable Banach space, and A: $D(A) \subseteq E \to E$ be a Kpd operator with $D(A) = D(K)$. Suppose*

$$\langle Ax, j_q(Ky)\rangle = \langle Kx, j_q(Ay)\rangle$$

for all $x, y \in D(A)$. For arbitrary $f \in E$, define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively from an arbitrary $x_0 \in D(A)$ by

$$x_{n+1} = x_n + t_n \gamma_n, \qquad n \geq 0, \tag{8}$$

$$\gamma_n = K^{-1}f - K^{-1}Ax_n, \qquad n \geq 0, \tag{9}$$

$$t_n = \left[\frac{\langle A\gamma_n, j_q(K\gamma_n)\rangle}{c_q\|A\gamma_n\|^q}\right]^{(q-1)^{-1}}, \qquad n \geq 0. \tag{10}$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of $Ax = f$.

Proof. The existence of a unique solution to $Ax = f$ follows from Theorem CA1, after observing that the strict convexity assumption on E^* was needed only to ensure that J is single-valued. Using Eqs. (8)–(10), and the linearity of A and K we obtain

$$K\gamma_{n+1} = K\gamma_n - t_n A\gamma_n.$$

Using inequality (6) we obtain the computations

$$\begin{aligned}\|K\gamma_{n+1}\|^q &= \|K\gamma_n - t_n A\gamma_n\|^q \\ &\leq \|K\gamma_n\|^q - qt_n\langle A\gamma_n, j_q(K\gamma_n)\rangle + c_q t_n^q\|A\gamma_n\|^q \\ &= \|K\gamma_n\|^q - q\frac{\langle A\gamma_n, j_q(K\gamma_n)\rangle^{q(q-1)^{-1}}}{\left[c_q\|A\gamma_n\|^q\right]^{(q-1)^{-1}}} + \frac{\langle A\gamma_n, j_q(K\gamma_n)\rangle^{q(q-1)^{-1}}}{\left[c_q\|A\gamma_n\|^q\right]^{(q-1)^{-1}}}.\end{aligned}$$

Thus

$$\|K\gamma_{n+1}\|^q \le \|K\gamma_n\|^q - (q-1)\frac{\langle A\gamma_n, j_q(K\gamma_n)\rangle^{q(q-1)^{-1}}}{\left[c_q\|A\gamma_n\|^q\right]^{(q-1)^{-1}}}. \tag{11}$$

It follows from inequality (11) and the inequality of the lemma that $\{\|K\gamma_n\|\}_{n=0}^{\infty}$ is monotone decreasing and consequently converges to some real number $\delta_q \ge 0$. Moreover, inequality of the above Lemma and inequality (11) imply that

$$\lim_{n\to\infty}\frac{\langle A\gamma_n, j_q(K\gamma_n)\rangle^{q(q-1)^{-1}}}{\left[c_q\|A\gamma_n\|^q\right]^{(q-1)^{-1}}} = 0. \tag{12}$$

Using inequality of the Lemma and inequality (2) of Theorem CA1 we obtain

$$\frac{\langle A\gamma_n, j_q(K\gamma_n)\rangle^{q(q-1)^{-1}}}{\|A\gamma_n\|^{q(q-1)^{-1}}} \ge \left(\frac{c}{\alpha}\right)^{q(q-1)^{-1}}\|K\gamma_n\|^q. \tag{13}$$

Since K is continuously $D(A)$-invertible, there exists $\beta > 0$ such that

$$\|Kx\| \ge \beta\|x\| \qquad \forall x \in D(K).$$

Thus,

$$\frac{\langle A\gamma_n, j_q(K\gamma_n)\rangle^{q(q-1)^{-1}}}{\|A\gamma_n\|^{q(q-1)^{-1}}} \ge \left(\frac{c}{\alpha}\right)^{q(q-1)^{-1}}\beta^q\|\gamma_n\|^q. \tag{14}$$

This inequality and Eq. (12) yield that $\gamma_n \to 0$ as $n \to \infty$, i.e., $Ax_n \to f$ as $n \to \infty$. Since A has a bounded inverse, this implies $x_n \to A^{-1}f$, the unique solution of $Ax = f$. The proof is complete.

Remark 3. The iteration process of our theorem has been studied by various authors in the special case in which E is a Hilbert space (see, for example, [10]).

L_p (or l_p) spaces $p \ge 2$ are 2-uniformly smooth and satisfy inequality (6) with $c_q = p - 1$. If we set $q = 2$ and $c_q = p - 1$ in our theorem, then the conditions of our theorem reduce exactly to the conditions of Theorem 2 of [9]. Thus, it is obvious that our theorem extends Theorem 2 of [9] from L_p (or l_p) spaces, $p \ge 2$, to the more general class of Banach spaces considered here and without the commutativity assumption on A and K imposed in [9].

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REFERENCES

1. H. Brézis and F. E. Browder, Singular Hammerstein equation and maximal monotone operators, *Bull. Amer. Math. Soc.* **82** (1976), 623–625.
2. H. Brézis and F. E. Browder, Existence theorems for nonlinear integral equations of Hammerstein type, *Bull. Amer. Math. Soc.* **81** (1975), 73–78.
3. H. Brézis and F. E. Browder, Some new results about Hammerstein equations, *Bull. Amer. Math. Soc.* **80** (1974), 567–572.
4. F. E. Browder, Functional analysis and partial differential equations, *Math. Ann.* **138** (1995), 55–59.
5. R. E. Bruck, Jr., The iterative solution of the equation $f \in x + Tx$ for a monotone operator T in Hilbert space, *Bull. Amer. Math. Soc.* **79** (1973), 1258–1262.
6. C. E. Chidume, The iterative solution of the equation $f \in x + Tx$ for a monotone operator T in L_p spaces, *J. Math. Anal. Appl.* **116** (1986), 531–537.
7. C. E. Chidume, Iterative approximation of fixed points of Lipschitz strictly pseudo-contractive mappings, *Proc. Amer. Math. Soc.* **99** (1987), 283–288.
8. C. E. Chidume, Approximation of fixed points of strongly pseudocontractive mappings, *Proc. Amer. Math. Soc.* **120** (1994), 545–551.
9. C. E. Chidume and S. J. Aneke, Existence, uniqueness and approximation of a solution for a K-positive definite operator equation, *Appl. Anal.* **50** (1993), 285–294.
10. W. M. Patterson, Iterative methods for the solution of a linear operator equation in Hilbert space—A survey, *in* "Lecture Notes in Mathematics," Vol. 394, Springer-Verlag, New York/Berlin, 1974.
11. W. V. Petryshyn, Direct and iterative methods for the solution of linear operator equations in Hilbert spaces, *Trans. Amer. Math. Soc.* **105** (1962), 136–175.
12. T. Reid, Symmetrizable completely continuous linear transformation in Hilbert space, *Duke Math. J.* **18** (1951), 41–176.
13. H.-K. Xu, Inequalities in Banach spaces with applications, *Nonlinear Anal.* **16** (1991), 1127–1138.