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Rapid Oscillation, Nonextendability, and the Existence of Periodic Solutions to Second Order Nonlinear Ordinary Differential Equations

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Consider the second order scalar ordinary differential equation

$$x''(t) + f(t, x(t)) = 0 \qquad (' = d/dt),$$

where $f(t, x)$ is ω-periodic in t and $f(t, 0) = 0$ for all t. The usual existence results for periodic solutions employing degree theory or other fixed point arguments are generally unhelpful in this case, since the periodic solution that they predict may well be the trivial solution.

Jacobowitz has recently succeeded in applying the Poincaré–Birkhoff "twist" theorem to demonstrate that this equation has infinitely many (nontrivial) periodic solutions when f satisfies a suitable "strong nonlinearity" condition with respect to x. Essential to his method of proof, however, is the condition $xf(t, x) > 0$ for all t ($x \neq 0$). In this note we show how this hypothesis may be relaxed, by modifying a technique used by the author when considering the problem of the global existence of solutions which occurs with the removal of the sign condition.

1. Introduction

Consider the second order scalar ordinary differential equation

$$x''(t) + f(t, x(t)) = 0, \tag{1}$$

where $f(t, x)$ is continuous, periodic in t with least period $\omega > 0$, and $f(t, 0) \equiv 0$.

The standard existence theory of periodic solutions involving degree theory or other fixed point arguments will generally be unhelpful in this instance, since the periodic solution that they predict may well be the trivial solution.

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In studying certain periodic boundary value problems by the method of the calculus of variations, Nehari [9] obtained as a consequence the existence of nontrivial periodic solutions of (1) under certain assumptions on the function $f(t, x)$, among which were a sign restriction, oddness as a function of x, for each t, and a condition of nonlinearity.

Recently, Jacobowitz [7] has succeeded in applying the Poincaré–Birkhoff "twist" theorem to obtain the existence of infinitely many periodic solutions under considerably less restrictive conditions than those of Nehari. His main result is the following:

THEOREM [7]. *Let $f(t, x)$ satisfy the following conditions.*

(1) *f is periodic with period ω in t and is in $C^1(\mathscr{R}^2, \mathscr{R}^1)$.*

(2) *$xf(t, x) > 0$ for $x \neq 0$.*

(3) *$\lim_{|x|\to\infty} x^{-1}f(t, x) = \infty$, uniformly in t.*

(4) *$x^{-1}f(t, x)$ is bounded for $|x| < 1$.*

Then (1) *has an infinite number of periodic solutions of period ω; furthermore, for each sufficiently large integer N, there exists a solution with precisely $2N$ zeros in* $[0, \omega)$.

Condition (3) is a kind of "strong nonlinearity" assumption, and Jacobowitz exploits the fact that for functions satisfying (2) and (3), Eq. (1) has solutions which oscillate very rapidly.

The aim of the present paper is to relax the sign condition (2) to obtain a large class of functions which still have nontrivial periodic solutions. This is largely motivated by the belief that it is essentially the strong nonlinearity of f that guarantees the existence of such solutions.

2. Statement of Results

DEFINITION. We shall say that the continuous function $f(t, x)$ is SN (strongly nonlinear) if

(a) for each t, either $xf(t, x) \geqslant 0$ for all x or $xf(t, x) \leqslant 0$ for all x;

(b) the set Z of values of t for which $f(t, x) \equiv 0$ is an isolated set and for each $t_i \in Z$, there are left and right neighborhoods of t_i on which $f(t, x)$ is monotone t for each x;

(c) for any compact t-set K, disjoint from Z, $\lim_{|x|\to\infty} |x|^{-1} \times |f(t, x)| = \infty$, uniformly on K.

Our main result is

THEOREM 1. *Let $f(t, x)$ be periodic in t with period $\omega > 0$, and satisfy the following conditions.*

(i) *$f(t, x)$ is of locally bounded variation in t, uniformly on compact x-sets.*

(ii) *$f(t, x)$ is locally Lipschitz in x, uniformly on compact t-sets.*

(iii) *$f(t, x)$ is SN, and there exist t_i, x_i, with $(-1)^i x_i f(t_i, x_i) > 0$, $i = 1, 2$.*

(iv) *For each compact t-set K, disjoint from Z (the set of t for which $f(t, x) = 0$ for all x) there is a function $g_K(x)$ such that $|f(t, x)| \geqslant g_K(x)$ for all $t \in K$ and $\int_0^{\pm\infty} [1 + G_K(t, u)]^{-1/2} du < \infty$ where $G_K(t, x) = \int_0^x g_K(t, y) dy$.*

Then (1) *possesses a nontrivial periodic solution of period ω.*

In order to make the basic idea of the proof more transparent, we shall concentrate on the following special case.

THEOREM 1'. *Let $q(t)$ be continous, periodic of period ω, of locally bounded variation, with isolated zeros and at least one change of sign on $(0, \omega)$. Assume that g is piecewise monotone in a neighborhood of each of its zeros. Let $g(x)$ be locally Lipschitz with $xg(x) \geqslant 0$ and such that $\lim_{|x|\to\infty} (g(x)/x) = \infty$ and $\int_0^{\pm\infty} [1 + G(s)]^{-1/2} ds < \infty$ $(G(x) = \int_0^x g(u) du)$. Then for $f(t, x) = q(t)g(x)$,* (1) *has a nontrivial periodic solution of period ω.*

Since we are allowing $f(t, x)$ to change sign as we vary t, we shall have to cope with the possibility of solutions of (1) not being continuable; however, it turns out that in handling this problem we obtain a technique that allows us to show that a certain mapping, related to the Poincare map, has a fixed point.

3. NOTATION

We are dealing with the equation

$$x''(t) + q(t) g(x(t)) = 0. \tag{1'}$$

We shall use letters u, v to denote the points in the plane of initial conditions for (1'); thus the solution of (1') with initial conditions

$$x(t_0) = u, \quad x'(t_0) = v$$

will be denoted by $x(t_0, (u, v); t)$ on its maximal interval of existence (its existence and uniqueness is guaranteed by the hypotheses of §2, which are assumed throughout).

The derivative $x'(t_0, (u, v); t)$ will be denoted by $y(t_0, (u, v); t)$ and the vector $(x(t_0, (u, v); t), y(t_0, (u, v); t))$ by $z(t_0, (u, v); t)$. For $(a, b) \in \mathscr{R}^2$, define $\Omega_a{}^b$ to be the subset of points p in the (u, v)-plane for which $x(a, p; t)$ is continuable to the interval spanned by a, b. The zero vector always belongs to $\Omega_a{}^b$ and by the continuous dependence of solutions on initial conditions, it follows that $\Omega_a{}^b$ is a nonempty, open subset of $\mathscr{R}^2$.

If $p \in \Omega_0{}^\omega$, with $p = (u(p), v(p))$, define $\phi(p)$ to be $| z(0, p; \omega)| - | p |$ and $\psi(p)$ to be $\arg z(0, p; \omega) - \arg p$. Thus $\psi(p) + | p |$, $\psi(p) + \arg p$ are the polar components of the Poincaré (first return) map and ϕ is continuous on $\Omega_0{}^\omega$ while ψ is a continuous map from $\Omega_0{}^\omega \backslash \{(0, 0)\}$ to $[0, 2\pi)$, which we identify with the unit circle S^1.

By a vertical line in the (u, v)-plane, we shall mean a line parallel to the v-axis.

We shall use $B(0; r)$ to denote the unit ball of radius r in $\mathscr{R}^2$, and throughout, modulus signs will be used for the Euclidean norm in the appropriate dimension. The complement of a set S is denoted by S^c and its closure by cl S. ∂S will be used for the boundary of S.

4. Rapid Oscillation of Solutions

When $q(t)$ is nonnegative, but not identically zero, on an interval I, solutions of (1′) may be made to have an arbitrarily large number of zeros on I by choosing the initial values with sufficiently large norm. This is one of the ideas behind the following

Lemma 1. *Let $q(t) \geqslant 0$ on $I = (a, b)$, $q(t) \not\equiv 0$ on I, and let M be a positive number, n be a natural number. Then there are numbers $r = r(M, n)$, $R = R(M, n)$, $0 < r < R$, such that if $\Gamma \subset \mathscr{R}^2$ is any continuum with the property that $\Gamma \cap B(0; r) \neq \varnothing \neq \Gamma \cap B^c(0; R)$ and $\Gamma \cap B^c(0; r)$ is disjoint either from the u-axis or from the v-axis, then* (a) $| z(a, p; b)| \geqslant M$ *for all* $p \in B^c(0; r) \cap \Gamma$, (b) $\arg z(a, p; b) \pmod{2\pi}$ *as a map from* $B(0; R) \cap B^c(0; r) \cap \Gamma$ *is an n-fold covering of the unit circle S^1, that is, for each $\theta \in S^1$, the inverse image of θ under the map has cardinal at least n.*

Proof. Let $\gamma \subset \mathscr{R}^2$ be a compact, continuous arc with parameter set J, such that γ is disjoint either from the u-axis or from the v-axis. Then $\gamma^* = z(a, \gamma(s); b)$, $s \in J$, is a compact, continuous arc in $\mathscr{R}^2$, disjoint from the origin, and we may define a continuous argument function arg along γ^*. In [3, Lemma 1], it was shown that for any natural number k, there exists $\rho(M, k)$ such that $| z(a, p; b)| \geqslant M$ and the number of zeros of $x(a, p; t)$ in I is at least k, whenever $| p | \geqslant \rho(M, k)$. (In the proof of that lemma it was actually assumed that q was positive on I, but an examination of the proof makes it clear that we may permit q to have isolated zeros on I). In the corollary to that lemma, it was shown that if $s, s' \in J$ such that the numbers of zeros in I of $x(a, \gamma(s); t)$, $x(a, \gamma(s'); t)$ differ by a natural number m, then $|\arg \gamma^*(s) - \arg \gamma^*(s')| \geqslant (m-1)\pi$. For each positive number λ, define $N(\lambda)$ to be the maximum of the numbers of zeros on I of $x(a, p; t)$ as p ranges over $B(0; \lambda)\backslash\{(0, 0)\}$. $N(\lambda)$ is well defined on account of the uniqueness of the zero solution of (1′) and the continuous dependence of solutions on initial conditions.

Define the increasing sequences of positive numbers r_i and natural numbers n_{2i} inductively by

$$r_0 = \rho(M, 0), \qquad n_0 = N(r_0), \qquad r_{2i+1} = \rho(M, N(r_{2i}) + 3),$$
$$r_{2i+2} = r_{2i+1} + 1, \qquad n_{2i+2} = N(r_{2i+2}), \qquad i = 0, 1, \ldots.$$

Then if γ_i is any continuous arc intersecting each of the boundary components of $A_i = \text{cl}[B(0; r_{2i+1})\backslash B(0; r_{2i})]$ and disjoint either from the u-axis or the v-axis, there exist p_{2i}, $p_{2i+1} \in \gamma_i$ such that $|p_{2i}| = r_{2i}$, $|p_{2i+1}| = r_{2i+1}$. Therefore the numbers of zeros of $x(a, p_{2i}; t)$, $x(a, p_{2i+1}; t)$ on I differ by at least 3 and it follows that for any continuous argument function on γ_i, $|\arg p_{2i+1}^* - \arg p_{2i}^*| \geqslant 2\pi$, where p_{2i}^*, p_{2i+1}^* are the images of p_{2i}, p_{2i+1}, respectively, under the map $p \rightarrow z(a, p; b)$. It follows that arg (mod 2π) restricted to γ_i is a 1-fold cover of $[0, 2\pi)$.

Next we observe that the images A_i^* of A_i, $i = 0, 1, 2, \ldots$, under the map $p \rightarrow z(a, p; b)$ are mutually disjoint compact subsets of $\mathscr{R}^2$.

Now define $r(M, n)$ to be r_0, $R(M, n)$ to be r_{2n+1}, and let $\Gamma \subset \mathscr{R}^2$ be any continuum satisfying the hypotheses of the lemma. Let $\theta \in [0, 2\pi)$ be fixed. For $k = 1, 2, \ldots$, let $\mathscr{O}_k$ be an open, connected cover of Γ by discs of radius $1/k$. Then we may find inside $\mathscr{O}_k$, a continuous arc γ_k disjoint from one of the axes and intersecting each of the boundary components of $A = \bigcup_{i=0}^{n} A_i$, and (by considering "first entry" and "last exit" points of γ_k) we choose subarcs $\gamma_{k,i}$ lying entirely within A_i and intersecting both of its boundary components, $i = 0, 1, \ldots, n-1$. Thus we may find points $p_{k,i}^* \in \gamma_{k,i}^* \subset A_i^*$ ($\gamma_{k,i}^*$ being the image of $\gamma_{k,i}$ under the map $p \rightarrow z(a, p; b)$) such that the sequence $p_{k,i}^*$ is bounded with $\arg p_{k,i}^*$ (mod 2π) $= \theta$, and hence a subsequence which we again label $p_{k,i}^*$ converges to $p_i^* \in A_i^*$. Clearly $\arg p_i^*$ (mod 2π) $= \theta$, and the p_i^* are all different, $i = 0, 1, \ldots, n-1$. Since $p_{k,i}$ also converges, to p_i, say, and the distance from $p_{k,i}$ to Γ is less than $1/k$, it follows that $p_i \in \Gamma$ and $p_i^* = z(a, p; b)$. The lemma now follows.

Remarks. The conditions on q guarantee the extendability of solutions across I [5].

If $f(t, x)$ is SN, there is no difficulty in obtaining the corresponding result for (1) on an interval I which is disjoint from Z (see definition) and on which $xf(t, x) \geqslant 0$ for all x; the condition which enables us to prove [3, Lemma 1] being essentially (c) of the definition of SN.

5. Some Properties of the Continuability Sets of Initial Points

We establish some simple results concerning the structure of the $\Omega_a{}^b$.

Lemma 2. *Let $q(t) < 0$ on $I = (a, b)$. Then $\Omega_a{}^b$ is open, and if K is any compact subset of the reals and Π_u is the projection map from the (u, v)-plane on to the u-axis, then $\Pi_u^{-1}(K) \cap \Omega_a{}^b$ is a bounded, nonempty set.*

Proof. The openness of $\Omega_a{}^b$ has already been noted in Sect. 2. That $\Pi_u^{-1}(K) \cap \Omega_a{}^b$ is bounded is easily deduced from the condition that $[1 + G(x)]^{-1/2}$ is integrable on the real line and the proof of [2, Theorem 1] concerning the noncontinuability of solutions of (1′). We need only comment that the possibility of q vanishing at one or both of the end-points of I is of no consequence and since solutions of (1′) are, for sufficiently large initial conditions, eventually monotone in I along with their derivatives (with the same direction of monotonicity), the possibility that $g(x)$ vanishes for certain small values of x causes no problem, on account of the condition that $\lim_{|x|\to\infty} (g(x)/x) = \infty$. We refer to [2] for details. To show that $\Pi_u^{-1}(K) \cap \Omega_a{}^b$ is nonempty, we only have to consider the case that K is a single point, and this was dealt with in [3, Lemma 2].

Remarks. Again, the extension to Eq. (1) for an interval I, disjoint from Z, on which $xf(t, x) \leqslant 0$ for all x, is easy to establish, using (iv).

A symmetrical statement holds for $\Omega_b{}^a$.

Generally speaking, there seems no reason to expect that the sets $\Omega_a{}^b$ will have any particular structure such as connectedness or that their boundaries will consist of continuous arcs, when the interval spanning a, b contains values of t for which $q(t) < 0$. However it is possible to find continuous arcs of infinite length inside these sets and this is the point of the next lemma. Before stating this, we develop some notation in connection with the zeros of q. Our hypotheses allow us to assume that the zeros of q on $[0, \omega)$ are

$$0 = t_0 < t_1 < \cdots < t_k < \omega, \qquad k \geqslant 1,$$

where $q(t) < 0$ on $(0, t_1)$, $q(t) < 0$ on (t_{j-1}, t_j), $q(t) \geqslant 0$ on (t_j, ω), for some j with $1 \leqslant j \leqslant k$. Henceforth we shall make this assumption concerning q.

Lemma 3. *There is a continuous arc $\gamma\colon (\alpha, \beta) \to \mathscr{R}^2$ with $(0, 0) \in \gamma \subset \Omega_0{}^\omega$, such that if $\gamma(s) = (\gamma_1(s), \gamma_2(s))$, then $\lim_{s\to\alpha} \gamma_1(s) = \lim_{s\to\alpha} \gamma_2(s) = \pm\infty$,*

$\lim_{s\to\beta} \gamma_1(s) = \lim_{s\to\beta} \gamma_2(s) = \pm\infty$, *and* $|\, z(0, \gamma(s); t_j)|$ *and* $|\, z(0, \gamma(s); \omega)|$ *are uniformly bounded for* $s \in (\alpha, \beta)$.

Proof. $q(t) < 0$ on (t_{j-1}, t_j), so by the previous lemma there exists an interval (c, d), $c < 0 < d$ such that $\{0\} \times (c, d) \subset \Omega_{t_j}^{t_{j-1}}$ and $(0, c), (0, d) \in \partial\Omega_{t_j}^{t_{j-1}}$. Since $(0, 0) \in \Omega_{t_j}^0$ and $\Omega_{t_j}^0$ is an open subset of $\Omega_{t_j}^{t_{j-1}}$, it follows that there exist α, β with $c \leqslant \alpha < 0 < \beta \leqslant d$ for which $\{0\} \times (\alpha, \beta) \subset \Omega_{t_j}^0$ and $(0, \alpha)$, $(0, \beta) \in \partial\Omega_{t_j}^0$. Now define $\gamma(s)$ to be $z(t_j, (0, s); 0)$, $s \in (\alpha, \beta)$. Then γ is a continuous arc in $\Omega_0^{t_j} = \Omega_0^{\omega}$ with the appropriate asymptotic behavior following from a consideration of $\lim_{t\to 0+} z(t_j, p; t)$ for any $p \in \partial\Omega_{t_j}^0$. We also have that $|\, z(0, \gamma(s); t_j)| = |(0, s)| \leqslant \max -\alpha, \beta$, and since $q(t) \geqslant 0$ on (t_j, ω), $z(t_j, p; t)$ is continuable to $[t_j, \omega]$ for all $p \in \mathscr{R}^2$, hence $z(t_j, p; \omega)$ is continuous on $\mathscr{R}^2$ and therefore

$$|\, z(0, \gamma(s); \omega)| \leqslant \max \{|\, z(t_j, p; \omega)|: \; |\, p\,| < \max(\alpha, \beta)\};$$

that is, $|\, z(0, \gamma(s); \omega)|$ is uniformly bounded for $s \in (\alpha, \beta)$. This completes the proof of the lemma.

Remark. As with each of the previous lemmas, there is a corresponding result for Eq. (1).

6. A Topological Lemma

Ultimately, we are going to demonstrate the existence of a periodic solution by exhibiting it as a fixed point of a certain map (the Poincaré map). In preparation for this, we need the following

Lemma 4. *Let Ω be an open, connected subset of $\mathscr{R}^2$ with the property that for each vertical line L lying between (and including) two fixed vertical lines L_1, L_2, $L \cap \Omega$ is a nonempty bounded set. Let $\{\Gamma_i\}_{i=0}^m$ be a collection of continua contained in Ω such that for $i = 1, 2,..., m$, the Γ_i are mutually disjoint, and $L_1 \cap \Gamma_0 \neq \varnothing \neq L_2 \cap \Gamma_0$, whereas for each i, $1 \leqslant i \leqslant m$, at least one of $L_1 \cap \Gamma_i$, $L_2 \cap \Gamma_i$ is empty. Then there exists $p \in \Gamma_0$, $q \in \partial\Omega$ and an arc γ from p to q with $\gamma \subset \mathrm{cl.}\, \Omega \cap S(L_1, L_2)$, where $S(L_1, L_2)$ is the infinite closed strip of $\mathscr{R}^2$ contained between L_1 and L_2, such that γ is disjoint from $\bigcup_{i=1}^m \Gamma_i$.*

Before giving the proof, we mention that in the special case that Ω is a bounded, open rectangle, this result is essentially contained in [8, proof of Lemma 4].

Proof. Since the Γ_i are continua contained in the open set Ω, we may, for ϵ a sufficiently small positive number, cover each of them by a finite collection of closed discs of radius ϵ lying entirely within Ω and we may assume that each of these covering sets $\tilde{\Gamma}_i$ satisfy the same hypotheses as the Γ_i. Each of the $\tilde{\Gamma}_i$ has an exterior boundary $\tilde{\gamma}_i$ which is a simple closed curve (composed of finitely many circular arcs), Γ_i being contained inside the interior domain of $\tilde{\gamma}_i$. If one of L_1, L_2 intersects the interior of $\tilde{\Gamma}_i$ (they cannot both do so) we modify $\tilde{\Gamma}_i$ as follows: suppose that $L_1 \cap \text{int } \tilde{\Gamma}_i \neq \varnothing$; then we construct a square C_i, disjoint from the interior of $S(L_1, L_2)$ and one of whose sides is a segment of L_1 containing $L_1 \cap \text{int } \tilde{\Gamma}_i$. Then we replace $\tilde{\Gamma}_i$ by $C_i \cup \tilde{\Gamma}_i$. Defining Γ_i^* to be $\tilde{\Gamma}_i$, modified if necessary, we define γ_i^* to be its exterior boundary. Then Γ_i^* satisfies the hypotheses of the original Γ_i, except possibly for the condition of being pairwise disjoint. We denote the interior domain of γ_i^* by $\mathscr{I}(\gamma_i^*)$ and note that $\Gamma_i \subset \mathscr{I}(\gamma_i^*)$. For $p \in \Gamma_0$, we shall say that Γ_i is a "barrier" for p if every continuous arc contained in cl. $\Omega \cap S(L_1, L_2)$, leading from p to $\partial\Omega$, intersects Γ_i. We claim that if Γ_i is a barrier for p, then $p \in \gamma_i^* \cup \mathscr{I}(\gamma_i^*)$. For suppose not, then p is in the exterior domain of γ_i^*. Let γ be any continuous arc in cl $\Omega \cap S(L_1, L_2)$, leading from p to some $q \in \partial\Omega$ (the existence of such arcs follows from the hypotheses). Since γ intersects Γ_i, it will intersect γ_i^* at a "first" point p_1 and at a "last" point q_1. Now one (at least) of the two subarcs of γ_i^* joining p_1 and q_1 lies in $S(L_1, L_2)$; this was the point of modifying $\tilde{\gamma}_i$ if necessary. (The argument for this is as follows. Suppose both arcs from p_1 to q_1 have first exit points from $S(L_1, L_2)$. This could only occur in the situation where γ_i^* is a "modified" $\tilde{\gamma}_i$. These exit points both lie on L_1, say, and since they are first exit points, one of the two arcs must enter the interior of the square C_i associated with the construction of $\tilde{\Gamma}_i$. But this is impossible.) Either this subarc intersects $\partial\Omega$ or together with the subarcs of γ joining p with p_1 and q with q_1 forms a continuous arc from p to q, lying in cl $\Omega \cap S(L_1, L_2)$ and disjoint from Γ_i; in either case a contradiction, which verifies the claim. If $p \in \Gamma_0 \cap \text{int } S(L_1, L_2)$, then it is clear from the construction of Γ_i^* that $p \in \gamma_i^* \cup \mathscr{I}(\gamma_i^*)$ implies $p \in \tilde{\gamma}_i \cup \mathscr{I}(\tilde{\gamma}_i)$. It follows that if every point of Γ_0 had some Γ_i as a barrier, the condition that the $\tilde{\Gamma}_i$ are pairwise disjoint would be violated. Therefore there is a point $p_0 \in \Gamma_0$ for which none of the Γ_i is a barrier. Now we may repeatedly apply the construction used in verifying the claim above to obtain an arc which leads from p_0 to some $q_0 \in \partial\Omega$, lying in cl $\Omega \cap S(L_1, L_2)$ and which misses each of the Γ_i. This proves the lemma.

7. Proof of the Theorem

First we observe that if $p \in \Omega_0^{\omega}$ with $x(0, p; t_j) = 0$, then $(0, y(0, p; t_j)) \in \Omega_{t_j}^0 \subset \Omega_{t_j}^{t_j - 1}$ and by Lemma 2, $L(0) \cap \Omega_{t_j}^{t_j-1}$ is bounded, where we are using the notation $L(c)$ for the vertical line $u = c$. It follows that there exists A_1 such that

$$p \in \Omega_0^{\omega} \quad \text{with} \quad x(0, p; t_j) = 0 \quad \text{implies} \quad |y(0, p; t_j)| \leqslant A_1. \tag{2}$$

Using Lemma 3, we obtain an arc $\gamma: (\alpha, \beta) \to \mathscr{R}^2$ with $\gamma \in \Omega_0^\omega$, having the property that $(0, 0) \in \gamma$ and

$$\lim_{s\to\alpha} \gamma_1(s) = \lim_{s\to\alpha} \gamma_2(s) = \pm\infty,$$

$$\lim_{s\to\beta} \gamma_1(s) = \lim_{s\to\beta} \gamma_2(s) = \pm\infty$$

$(\gamma(s) = (\gamma_1(s), \gamma_2(s)))$, and there exists A_2 such that

$$| z(0, \gamma(s); \omega)| \leqslant A_2 \qquad \text{for} \quad s \in (\alpha, \beta). \tag{3}$$

For definiteness, we shall suppose that the limits above are both $+\infty$.

Let Ω be the connected component of Ω_0^ω which contains γ. Since $q(t) \geqslant 0$ on $[t_j, \omega]$, the maps $p \to z(t_j, p; \omega)$, $p \to z(\omega, p; t_j)$ are homeomorphisms of $\mathscr{R}^2$. Therefore we may define

$$A_3 = \max_{|p| \leqslant A_1} | z(t_j, p; \omega)| \tag{4}$$

and let L_1 denote the vertical line $L(A_3)$. $L_1 \cap \Omega \neq \varnothing$ (since it contains $L_1 \cap \gamma$) and $L_1 \cap \Omega \subset L_1 \cap \Omega_0^\omega \subset L_1 \cap \Omega_0^{t_1}$, which is bounded by Lemma 2. Let

$$A_4 = \sup\{| p | : p \in L_1 \cap \Omega\}. \tag{5}$$

Note that $A_4 > A_3$. Let

$$A_5 = \max_{|p| \leqslant A_4} | z(\omega, p; t_j)| \tag{6}$$

and apply Lemma 1 with $I = (t_j, \omega)$, $M = 2A_5$, $n = 2$, to obtain the corresponding values $r = r(M, n)$, $R = R(M, n)$. Now define A_6 by

$$A_6 = \max_{|p| \leqslant R} | z(t_j, p; \omega)| \tag{7}$$

and denote $L(2A_6)$ by L_2. If $S(L_1, L_2)$ is the closed infinite strip between L_1 and L_2 it follows from Lemma 2, since $\Omega \subset \Omega_0^{t_1}$, that $\Omega(L_1, L_2) = \Omega \cap S(L_1, L_2)$ is bounded, and we define

$$A_7 = \sup\{| p | : p \in \Omega(L_1, L_2)\}. \tag{8}$$

For each point $q \in \partial\Omega \cap S(L_1, L_2)$, there is an open disc D_q such that $| z(0, p; \omega)| > A_7$ for $p \in D_q \cap \Omega_0^\omega$. Here we have used the noncontinuability of $z(0, q; t)$ to $[0, \omega]$, together with the continuous dependence of solutions on initial conditions, to assert the existence of D_q. Let $K = \Omega(L_1, L_2) \backslash \bigcup D_q$, where the union is over all points q in $\partial\Omega \cap S(L_1, L_2)$. Then K is a compact subset of Ω, and since $\phi(p) = | z(0, p; \omega)| - | p |$ is continuous on Ω (by the continuous dependence of solutions on initial conditions), the zero set Z of ϕ restricted to K is a compact subset of Ω.

Suppose that Z does not contain a continuum intersecting both L_1 and L_2. Since Z is a compact subset of the open set Ω, we may choose a finite covering of Z by closed discs contained in Ω such that the connected components of the covering are continua Γ_i, $i = 1, 2,..., m$, and for each i, either $L_1 \cap \Gamma_i$ or $L_2 \cap \Gamma_i$ is empty (we refer to [8] for the precise details of this assertion). Clearly the Γ_i are mutually disjoint. On the other hand, if we define Γ_0 to be any component of $S(L_1, L_2) \cap \gamma$, then Γ_0 is a continuum with $\Gamma_0 \subset \Omega$ and $\Gamma_0 \cap L_1 \neq \varnothing \neq \Gamma_0 \cap L_2$. Now we apply Lemma 4 to obtain a continuous arc γ^* from $p \in \gamma$ to $q \in \partial\Omega$ with $\gamma^* \subset \text{cl}\, \Omega \cap S(L_1, L_2)$, γ^* disjoint from $\bigcup_{i=1}^m \Gamma_i$ and hence from Z.

However, γ^* intersects $\bigcup D_q$ for the "first time" at p', say, where $p' \in \partial D_{q'}$ $q' \in \partial\Omega \cap S(L_1, L_2)$ and the subarc $\tilde{\gamma}$ of γ^* leading from p to p' is in K. We have by the construction of γ that $x(0, p; t_j) = 0$, so that by (2), $|y(0, p; t_j)| \leqslant A_1$, hence by (4), $|z(0, p; \omega)| \leqslant A_3 \leqslant |p|$. Now $|z(0, p'; \omega)| \geqslant A_7 \geqslant |p'|$, by (8). It follows that there is a $\tilde{p} \in \tilde{\gamma}$ with $|z(0, \tilde{p}; \omega)| = |\tilde{p}|$. But then $\tilde{p} \in \gamma^* \cap Z$, which is a contradiction. We conclude that Z contains a continuum Γ intersecting both L_1 and L_2. Let $\bar{\Gamma}$, Γ^* be the images of Γ under the maps $p \to z(0, p; t_j)$, $p \to z(0, p; \omega)$, respectively. Let $p_1 \in \Gamma \cap L_1$, $p_2 \in \Gamma \cap L_2$. By (5), $|z(0, p_1; \omega)| = |p_1| \leqslant A_4$ which implies by (6) that $|z(0, p_1; t_j)| \leqslant A_5 < M$. Since the definition of $r = r(M, n)$ implies that $|z(t_j, p; \omega)| \geqslant M$ whenever $|p| \geqslant r$, it follows that $|z(0, p_1; t_j)| < r$. On the other hand, $|z(0, p_2, \omega)| = |p_2| \geqslant 2A_6 > A_6$ which, on account of (7), implies that $z(0, p_2; t_j) > R$. Furthermore, for any $p \in \Gamma$, $|z(0, p; \omega)| = |p| \geqslant A_4$, by (5), and $A_4 > A_3$, so it follows by (4) that $|z(0, p; t_j)| > A_1$. By (2), this means that $\bar{\Gamma}$ is disjoint from the u-axis.

We conclude that $\bar{\Gamma}$ satisfies the hypotheses of Lemma 1 with $M = 2A_5$, $n = 2$. Therefore $\arg_{z \in \Gamma^*} z$ covers $[0, 2\pi)$ at least twice. However, $|\arg p| < \pi/2 \pmod{2\pi}$ for all $p \in \Gamma$. From this we may deduce (for example by using a suitable continuous arc close to Γ, along the lines of the proof of Lemma 1) that there exists $p_0 \in \Gamma$ such that $\arg z(0, p_0; \omega) = \arg p_0 \pmod{2\pi}$.

Since $|z(0, p_0; \omega)| = |p_0|$ and $|p_0| \geqslant A_3 > 0$, it follows that $x(0, p_0; t)$ is a nontrivial periodic solution of (1) and Theorem 1 is proved.

In conclusion, we give the following

COROLLARY. *Let $q(t)$ be a continuous periodic function with period $\omega > 0$, with only isolated zeros, such that q is somewhere positive. Let $\alpha > 1$. Then there are infinitely many periodic solutions of the equation*

$$x''(t) + q(t) \mid x(t)|^\alpha \operatorname{sgn}(x(t)) = 0,$$

and if $\int_0^\omega q(t)\, dt \geqslant 0$, they all oscillate (have arbitrarily large zeros).

Proof. The existence of periodic solutions follows from Theorem 1 in the case that q has changes of sign and from the result of Jacobowitz in the case that $q(t) \geqslant 0$ for all t. In this latter case, the oscillatory nature of these solutions is a result of Atkinson [1] whilst, more generally, if $\int_0^\omega q(t)\,dt \geqslant 0$, it follows from [4].

If $\int_0^\omega q(t)\,dt < 0$, the above equation will have nonoscillatory solutions [4]; however, it is clear from the construction that the periodic solutions given by Theorem 1 will oscillate. We do not know if in this case there can exist nonoscillatory periodic solutions.

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